KALMAN FILTERING HW Ø1 SOLUTION

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(1) This problem has been widely studied. We are considering the most comman setup, where Monty <u>always</u> shows you behind a door & he <u>always</u> gives you a chance to switch.

The easiest solution to understand is full enumeration of the possible cases.

$$P(\text{prize behind door } 1) = P(\text{prize behind door } 2) = P(\text{prize behind door } 3) = 1/3.$$

Since you have no knowledge of which door holds the prize, you pick any one of the doors with equal probability, independently of where the prize actually is.

$P(\cdot)$	Prize behind	$P(\cdot)$	you pick	P(case)
1/3	1	1/3	1	1/9
		1/3	2	1/9
		1/3	3	1/9
1/3	2	1/3	1	1/9
		1/3	2	1/9
		1/3	3	1/9
1/3	3	1/3	1	1/9
		1/3	2	1/9
		1/3	3	1/9

For example,

$$P(\text{prize behind 1 and you pick 1})$$
$$= P(\text{prize behind 1}) \cdot P(\text{you pick 1})$$
$$= 1/3 \cdot 1/3 = 1/9$$

Now we analyze what happens whem Monty shows a door. This is the part that makes the problem counterintuitive.

→ If you have picked a "goat" door, then Monty shows you the other "goat" door with probability 1.

→ If you have picked the "prize" door, then Monty shows you one of the goat doors with equal probability, i.e., each of the goat doors has a ½ chance of being shown.

After constructing the table of probabilities, we will add a column to show if you win the prize with the strategy "switch" and another column to show if you win the prize with the strategy "stick", e.g. keep your initially picked door.

P(·)	Prize Behind	P(·)	You Pick	P(·)	Monty Shows	P(case)	WIN IF SWITCH?	WIN IF STICK?
1/3	1	1/3	1	0	1	0	—	—
				1/2	2	1/18	N	Y
				1/2	3	1/18	N	Y
		1/3	2	0	1	0	—	—
				0	2	0	—	—
				1	3	1/9	Y	N
		1/3	3	0	1	0	—	—
				1	2	1/9	Y	N
				0	3	0	—	—
1/3	2	1/3	1	0	1	0	—	—
				0	2	0	—	—
				1	3	1/9	Y	N
		1/3	2	1/2	1	1/18	N	Y
				0	2	0	—	—
				1/2	3	1/18	N	Y
		1/3	3	1	1	1/9	Y	N
				0	2	0	—	—
				0	3	0	—	—
1/3	3	1/3	1	0	1	0	—	—
				1	2	1/9	Y	N
				0	3	0	—	—
		1/3	2	1	1	1/9	Y	N
				0	2	0	—	—
				0	3	0	—	—
		1/3	3	1/2	1	1/18	N	Y
				1/2	2	1/18	N	Y
				0	3	0	—	—

"—" means "can't happen"

From the table on P3, we see that

- there are 6 cases where you win by switching. All six of them have probability $1/9$, so

$$P(\text{win} \mid \text{switch}) = 6 \cdot \frac{1}{9} = \frac{6}{9} = \frac{2}{3}.$$

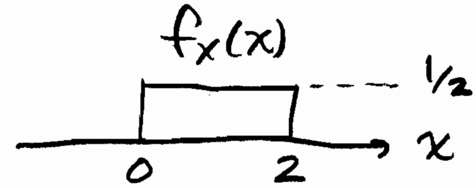
- there are six cases where you win if you stick. All six have probability $1/18$. So,

$$P(\text{win} \mid \text{stick}) = 6 \cdot \frac{1}{18} = \frac{6}{18} = \frac{1}{3}.$$

To maximize your probability of winning the prize, you should switch doors. This doubles your probability of winning compared to keeping the door you picked initially.

2) 1.12c) X is a $U(0,2)$ variable

with pdf



$$E(X) = \int_{-\infty}^{\infty} \theta f_X(\theta)\,d\theta = \int_0^2 \tfrac{1}{2}\theta\,d\theta = \tfrac{1}{4}\theta^2\Big|_{\theta=0}^{2}$$

$$= \tfrac{1}{4}[4-0] = \underline{\underline{1}}.$$

$$E(X^2) = \int_{-\infty}^{\infty} \theta^2 f_X(\theta)\,d\theta = \int_0^2 \tfrac{1}{2}\theta^2\,d\theta = \tfrac{1}{2}\tfrac{1}{3}\theta^3\Big|_{\theta=0}^{2}$$

$$= \tfrac{1}{6}\theta^3\Big|_{\theta=0}^{2} = \tfrac{1}{6}[8-0] = \tfrac{8}{6} = \underline{\underline{\tfrac{4}{3}}}$$

$$Var(X) = E\left\{[X-E(X)]^2\right\} = E(X^2) - [E(X)]^2$$

$$= \tfrac{4}{3} - 1^2 = \tfrac{4}{3} - 1 = \underline{\underline{\tfrac{1}{3}}}$$

③ 1.19) The joint pdf $f_{X,Y}(x,y)$ is given as an exhaustive table. (P6)

a) Are X and Y independent?

- They are independent iff $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

Let us consider the event $X = Y = 1$.

$f_{X,Y}(1,1) = \frac{1}{18}$, from the table.

$$f_X(1) = \sum_{k \in [1,3,5]} f_{X,Y}(1,k)$$

$$= f_{X,Y}(1,1) + f_{X,Y}(1,3) + f_{X,Y}(1,5)$$

$$= \frac{1}{18} + \frac{1}{18} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$$

$$f_Y(1) = \sum_{k \in [1,3,5]} f_{X,Y}(k,1)$$

$$= f_{X,Y}(1,1) + f_{XY}(3,1) + f_{XY}(5,1)$$

$$= \frac{1}{18} + \frac{1}{18} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$$

$$f_X(1) f_Y(1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \neq \frac{1}{18} = f_{XY}(x,y).$$

Therefore, X and Y are <u>NOT</u> independent.

③ ... 1.19 b)

$$P(Y=5) = \sum_{k\in[1,3,5]} f_{XY}(k,5)$$

$$= \frac{1}{18}+\frac{1}{6}+\frac{1}{3} = \frac{1}{18}+\frac{3}{18}+\frac{6}{18}$$

$$= \frac{10}{18} = \underline{\underline{\frac{5}{9}}}$$

c) $$P(Y=5\mid X=3) = \frac{P(Y=5 \text{ and } X=3)}{P(X=3)}$$

$$= \frac{f_{XY}(3,5)}{f_X(3)} = \frac{f_{XY}(3,5)}{\sum_{k\in[1,3,5]} f_{XY}(3,k)}$$

$$= \frac{1/6}{\frac{1}{18}+\frac{1}{18}+\frac{1}{6}} = \frac{3/18}{\frac{1}{18}+\frac{1}{18}+\frac{3}{18}} \cdot \frac{18}{18}$$

$$= \frac{3}{1+1+3} = \underline{\underline{\frac{3}{5}}}.$$

(4) 1.21 b)

Poisson variable: $P(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \geq 0$.

→ Here is a direct solution for the mean, ignoring the "hint":

$$E[X] = \sum_{k=0}^{\infty} k P(k) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k}{k!} \lambda^k = e^{-\lambda} \left[(k=0 \text{ term}) + (\text{Rest}) \right]$$

$$= e^{-\lambda} \left[\frac{0}{0!} \lambda^0 + \sum_{k=1}^{\infty} \frac{k}{k!} \lambda^k \right] = e^{-\lambda} \left[\frac{0 \cdot 1}{1} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \lambda \lambda^{k-1} \right]$$

$$= e^{-\lambda} \left[0 + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right]$$

let $m = k-1$
when $k=1 \rightarrow m=0$
when $k=\infty \rightarrow m=\infty$

$$= e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = e^{-\lambda} \lambda e^{\lambda} = \underline{\underline{\lambda}}$$

→ But the book wants you to use the the characteristic function, so let's see how that goes:

$$\Psi_X(\omega) = \text{DTFT}\{P(k)\}^* = \sum_{k=0}^{\infty} P(k) e^{+j\omega k} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{+j\omega k}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{+j\omega k}$$

$$= e^{-\lambda} \left[(k=0 \text{ term}) + (\text{Rest}) \right] \longrightarrow$$

④ 1.21 b) . . .

P9

$$\Psi_X(\omega) = \dots = e^{-\lambda}\left[\frac{\lambda^0}{0!}e^{-j\omega 0} + \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{+j\omega k}\right]$$

$$= e^{-\lambda}\left[\frac{1}{1}\cdot 1 + \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{+j\omega k}\right]$$

$$= e^{-\lambda}\left[1 + \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{+j\omega k}\right] = e^{-\lambda} + e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{+j\omega k}$$

$$\frac{\partial}{\partial\omega}\Psi_X(\omega)\Big|_{\omega=0} = \frac{\partial}{\partial\omega}\left[e^{-\lambda} + e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{+j\omega k}\right]_{\omega=0}$$

$$= \left[0 + e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\frac{\partial}{\partial\omega}e^{+j\omega k}\right]_{\omega=0}$$

$$= \left[e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\left(+jke^{+j\omega k}\right)\right]_{\omega=0}$$

$$= e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}(+jk) = je^{-\lambda}\sum_{k=1}^{\infty}\frac{k}{k!}\lambda^k$$

$$= je^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!} = j\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$$

let $m = k-1$
when $k=1 \rightarrow m=0$
when $k=\infty \rightarrow m=\infty$

$$= j\lambda e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!} = j\lambda e^{-\lambda}e^{\lambda} = j\lambda$$

Now, using formula (1.8.20) on p. 25 of the text, we have

$$E(X) = \frac{1}{j}\left[\frac{\partial}{\partial\omega}\Psi_X(\omega)\right]_{\omega=0}$$

$$= \frac{1}{j}j\lambda = \underline{\underline{\lambda}}$$

5 1.24a) $f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$, $r \geq 0$ P10

$$E(R) = \int_{-\infty}^{\infty} r f_R(r)\,dr = \int_0^{\infty} \frac{r^2}{\sigma^2} e^{-r^2/2\sigma^2}\,dr$$

$$= \frac{1}{\sigma^2} \int_0^{\infty} r^2 e^{-r^2/2\sigma^2}\,dr$$

From Schaum's Mathematical Handbook, p. 98, 15.77:

$$\int_0^{\infty} x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$$

We have $m = 2$

$a = \frac{1}{2\sigma^2}$

$$E(R) = \frac{1}{\sigma^2} \frac{\Gamma(\frac{3}{2})}{2(\frac{1}{2\sigma^2})^{3/2}} = \frac{1}{\sigma^2} \frac{1}{2} \frac{1}{(\frac{1}{2})^{3/2}} \frac{1}{(\frac{1}{\sigma^2})^{3/2}} \Gamma(1+\tfrac{1}{2})$$

$$= \sigma^{-2} \cdot 2^{-1} \cdot \left(\tfrac{1}{2}\right)^{-3/2} \cdot \left(\tfrac{1}{\sigma^2}\right)^{-3/2} \cdot \tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \sigma^{-2} \cdot 2^{-1} \cdot 2^{3/2} \cdot (\sigma^2)^{3/2} \cdot \tfrac{1}{2} \cdot \sqrt{\pi}$$

$$= \sigma^{-2} \cdot 2^{1/2} \cdot \sigma^3 \cdot 2^{-1} \cdot \sqrt{\pi}$$

$$= \sigma \cdot 2^{-1/2} \cdot \sqrt{\pi} = \underline{\underline{\sigma\sqrt{\frac{\pi}{2}}}}$$

5 1.24a) ...

P11

$$E(R^2) = \int_{-\infty}^{\infty} r^2 f_R(r)\,dr = \int_0^{\infty} \frac{r^3}{\sigma^2} e^{-r^2/2\sigma^2}\,dr$$

$$= \frac{1}{\sigma^2}\int_0^{\infty} r^3 \exp\left[-\frac{1}{2\sigma^2} r^2\right]\,dr$$

Using again formula 15.77 from the previous page with $a = \frac{1}{2\sigma^2}$ and $m = 3$,

$$E(R^2) = \frac{1}{\sigma^2} 2^{-1}\left(\frac{1}{2\sigma^2}\right)^{-2} \Gamma(2)$$

$$= \sigma^{-2} 2^{-1} (2\sigma^2)^2\, 1! = 2^{-1} 2^2 \sigma^2 = 2\sigma^2.$$

So $\mathrm{VAR}(R) = E(R^2) - [E(R)]^2$

$$= 2\sigma^2 - \left[\sigma\sqrt{\frac{\pi}{2}}\right]^2$$

$$= 2\sigma^2 - \frac{\pi}{2}\sigma^2$$

$$= \underline{\underline{\left(2 - \frac{\pi}{2}\right)\sigma^2}}$$

1.32) $f_X(x) = \frac{1}{2}e^{-|x|}$, $f_Y(y) = e^{-2|y|}$ } X and Y are independent.

$Z = X+Y$. Find $f_Z(z)$.

- Since X and Y are independent,

$$f_Z(z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{f_X(x)\}\, \mathcal{F}\{f_Y(y)\} \right\},$$

where $\mathcal{F}\{g(\theta)\} = \int_{-\infty}^{\infty} g(\theta) e^{-j\omega\theta}\, d\theta$

and $\mathcal{F}^{-1}\{G(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega\theta}\, d\omega$

Note:

$$\mathcal{F}\{e^{-a|x|}\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-j\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{ax} e^{-j\omega x}\, dx + \int_{0}^{\infty} e^{-ax} e^{-j\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)x}\, dx + \int_{0}^{\infty} e^{-(a+j\omega)x}\, dx$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)x}\Big|_{x=-\infty}^{0} + \frac{-1}{a+j\omega} e^{-(a+j\omega)}\Big|_{x=0}^{\infty}$$

$$= \frac{1}{a-j\omega}[1-0] - \frac{1}{a+j\omega}[0-1]$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+j\omega}{(a+j\omega)(a-j\omega)} + \frac{a-j\omega}{(a+j\omega)(a-j\omega)}$$

$\longrightarrow$

1.32)...

P13

$$\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{a^2-(j\omega)^2} .$$

In other words, $e^{-a|x|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2-(j\omega)^2}$ (*).

Applying (*), we have

$$\mathcal{F}\{f_X(x)\} = \frac{1}{2}\ \frac{2}{1-(j\omega)^2} = \frac{1}{1-(j\omega)^2}$$

$$\mathcal{F}\{f_Y(y)\} = \frac{4}{4-(j\omega)^2}$$

$$\mathcal{F}\{f_Z(z)\} = \mathcal{F}\{f_X(x)\}\,\mathcal{F}\{f_Y(y)\}$$

$$= \frac{1}{1-(j\omega)^2} \cdot \frac{4}{4-(j\omega)^2} = \frac{4}{[1-(j\omega)^2][4-(j\omega)^2]}$$

(PFE expansion) $= \frac{A}{1-(j\omega)^2} + \frac{B}{4-(j\omega)^2}$

$$\left.\frac{4}{4-(j\omega)^2}\right|_{(j\omega)^2=1} = A = \frac{4}{4-1} = \frac{4}{3}$$

$$\left.\frac{4}{1-(j\omega)^2}\right|_{(j\omega)^2=4} = B = \frac{4}{1-4} = -\frac{4}{3}$$

$$\mathcal{F}\{f_Z(z)\} = \frac{4}{3}\ \frac{1}{1-(j\omega)^2} - \frac{4}{3}\ \frac{1}{4-(j\omega)^2} \longrightarrow$$

1.32) ...

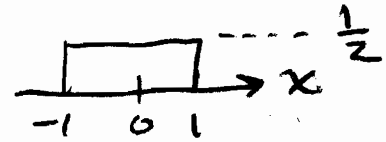
$$\mathcal{F}\{f_Z(z)\} = \cdots = \frac{2}{3}\,\frac{2}{1-(j\omega)^2} - \frac{1}{3}\,\frac{4}{4-(j\omega)^2}$$

Applying again (*) on P13, we have

$$f_Z(z) = \frac{2}{3}e^{-|z|} - \frac{1}{3}e^{-2|z|}$$

7 1.33)

X is a $U(-1,1)$ variable with pdf



Y is defined by

$$Y = g(X) = X^3 + 1, \quad \text{where } g(\theta) = \theta^3 + 1.$$

Solve for g^{-1}:

$$y = x^3 + 1$$
$$y - 1 = x^3$$
$$x = [y-1]^{1/3} = g^{-1}(y).$$

NOTE that $g(x)$ is monotonically increasing in x, and also that $f_X(x) = 0 \;\forall\, x \notin [-1,1]$.

Since $g(-1) = 0$ and $g(1) = 2$, this implies that $f_Y(y) = 0 \;\forall\, y \in [0,2]$.

Applying formula (1.14.7) from the text,

$$f_Y(y) = \left|\frac{\partial}{\partial y} g^{-1}(y)\right| f_X(g^{-1}(y))$$

$$= \begin{cases} \left|\frac{\partial}{\partial y} g^{-1}(y)\right| \frac{1}{2}, & -1 \le g^{-1}(y) \le 1 \\ 0, & \text{otherwise} \end{cases} \longrightarrow$$

7 1.33) ...

P16

$$f_Y(y) = \begin{cases} \left|\frac{\partial}{\partial y}[y-1]^{1/3}\right| \frac{1}{2}, & -1 \le [y-1]^{1/3} \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left|\frac{1}{3}(y-1)^{-2/3}\right|, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{6}(y-1)^{-2/3}, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$