KALMAN FILTERING HW02 SOLUTION P1

HAVLICEK

(1) 2.3a) Let the signal be $x(t)$. Time is divided into unit-length slots. There is a slot that begins at $t = t_0$, where t_0 is a uniform RV distributed $U(0,2)$.

- Consider the set of slots $[t_0+2k, t_0+2k+1]_{k\in\mathbb{Z}}$. We will call these the <u>EVEN SLOTS</u>. Note that the slot which begins at t_0 is the even slot corresponding to $k=0$.

- Now consider the other slots; they are the set $[t_0+2k+1, t_0+2k+2]_{k\in\mathbb{Z}}$. We will call these the <u>ODD SLOTS</u>.

→ on the odd slots, $x(t)=0$.

→ on the even slots, it is equally likely that $x(t)=1$ or $x(t)=0$.

(1) 2,3a), . . . Here is our strategy: we will first solve $R_X(\tau)$ for $\tau = 0$. Then we will solve $R_X(\tau)$ for the rest of the integers. Finally, we'll solve for $R_X(\tau)$ off the integers. (P2)

$\underline{\underline{\tau = 0}}$ $R_X(0) = E[x(t)x(t)]$

→ with prob. $\frac{1}{2}$, t is in an odd slot. In this case, $x(t)x(t) = 0 \cdot 0 = 0$.

→ with prob. $\frac{1}{2}$, t is in even slot.

In this case $x(t)x(t) = 0 \cdot 0 = 0$ with probability $\frac{1}{2}$ and $x(t)x(t) = 1 \cdot 1 = 1$ with probability $\frac{1}{2}$.

Thus,
$$R_X(0) = \underbrace{\tfrac{1}{2} \cdot 0 \cdot 0}_{\text{odd slot case}} + \underbrace{\tfrac{1}{2}\left[\tfrac{1}{2} \cdot 0 \cdot 0 + \tfrac{1}{2} \cdot 1 \cdot 1\right]}_{\text{even slot case}}$$

$$\underline{\underline{R_X(0) = \frac{1}{4}}}$$

① 2.3a)....

Now we consider the case where τ is an odd integer: $\underline{\underline{\tau = 2k+1}}$, $k \in \mathbb{Z}$:

→ with prob. $\frac{1}{2}$, t is in an odd slot, which implies $x(t)=0$, so that $x(t)x(t+\tau)=0$.

→ with prob. $\frac{1}{2}$, t is in an even slot. This implies that $x(t)=1$ and $x(t)=0$ are equally likely. However, in this case $t+\tau$ falls in an <u>odd slot</u>, so it is guaranteed that $x(t+\tau)=0$. So $x(t)x(t+\tau)=0$.

So for τ an odd integer, we have

$$R_x(\tau) = E[x(t)x(t+2k+1)]$$
$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = \underline{\underline{0}}.$$

→

1) 2.3a)...

Now consider the case where τ is an even integer, so $\underline{\tau = 2k}$, $k \in \mathbb{Z}$.

→ with prob. $\frac{1}{2}$, t is in an odd slot, which implies $x(t) = 0$ so $x(t)x(t+\tau) = 0$.

→ with prob $\frac{1}{2}$, t is in an even slot where $x(t) = 0$ and $x(t) = 1$ are equally likely. This implies that $t + \tau$ also falls in an even slot. Given t is in an even slot,

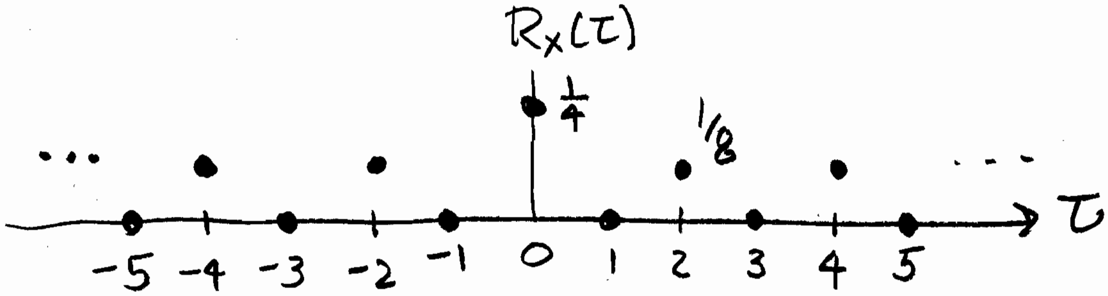
$$\rightarrow P\big(x(t)x(t+\tau) = 1\big) = P\big(x(t) = 1\big)P\big(x(t+\tau) = 1\big)$$
$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$\rightarrow P\big(x(t)x(t+\tau) = 0\big) = 1 - P\big(x(t)x(t+\tau) = 1\big)$$
$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

Overall for this case:

$$R_x(\tau) = E\big[x(t)\,x(t+\tau)\big]$$
$$= \frac{1}{2} \cdot 0 + \frac{1}{2}\Big[\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0\Big]$$
$$= \frac{1}{8} \quad (\tau \text{ an even integer}).$$

① 2.3a)... Up to this point we have solved $R_X(\tau)$ $\forall$ $\tau \in \mathbb{Z}$. Here is what we have obtained:



Next we consider noninteger τ in the interval $(0, 1)$;

$$\underline{\underline{\tau \in (0, 1)}}$$

- with prob. $\frac{1}{2}$, t is in an odd slot. This implies $x(t)=0$ so that $x(t)\,x(t+\tau)=0$.
- with prob. $\frac{1}{2}$, t falls in an even slot where $x(t)=1$ and $x(t)=0$ are equally likely.
  - → in this case, with prob $\frac{1}{2}$, $x(t)=0$ so that $x(t)\,x(t+\tau)=0$.
  - → also, with prob. $\frac{1}{2}$, $x(t)=1$.
- Thus, we have overall that

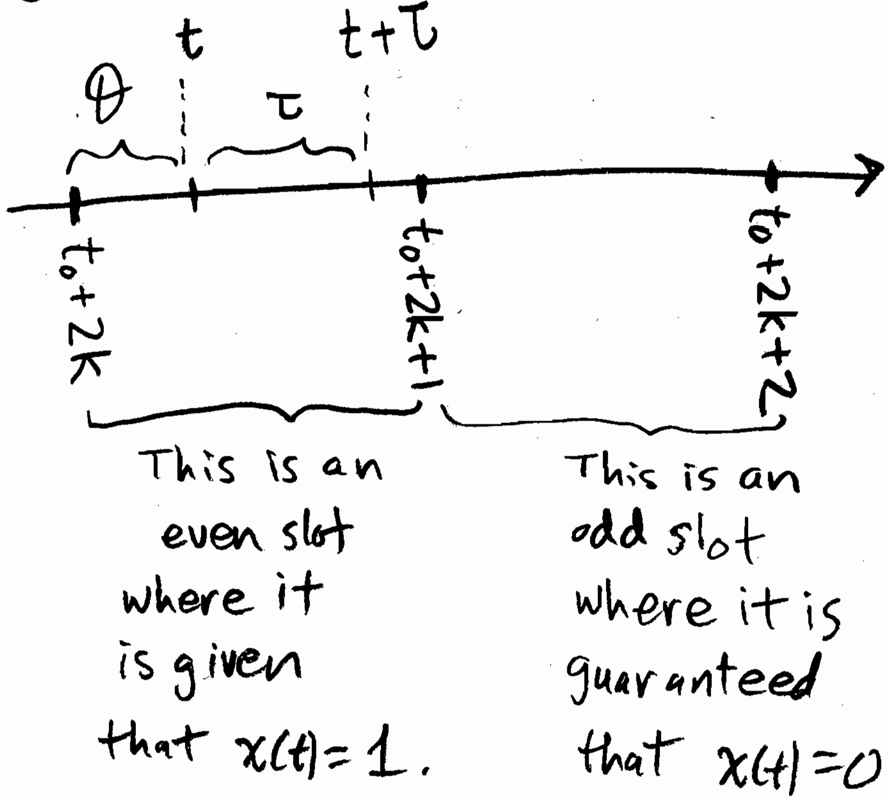
$$P(x(t)=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

1) 2.3a) ... We are considering $\tau \in (0,1)$. (P6)

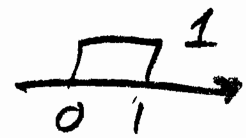
Now, if $x(t)=1$, then the event "$x(t)\,x(t+\tau)=1$" is equivalent to the event "t and $t+\tau$ both fall in the same slot."

→ It is an even slot, so it begins at t_0+2k for some $k \in \mathbb{Z}$.

→ The situation is depicted in the figure below:



→ The variable θ is the offset between the beginning of the slot (an even slot) and the point "t"...

① 2.3a)... which is an arbitrary point in the interior of the slot that is chosen at random. Thus, θ takes any value between 0 and 1 with equal probability. In other words, θ is a $U[0,1]$ variable with pdf .

- Thus, the probability that t and $t+\tau$ are in the same slot is the same as the probability that $\theta+\tau<1$, which is given by

$$P(\theta+\tau<1)=P(\theta<1-\tau)$$
$$=\int_{-\infty}^{1-\tau} f_\theta(\xi)\,d\xi=\int_0^{1-\tau} d\xi = 1-\tau.$$

- Referring back to the bottom of (P5), we have for $\tau\in(0,1)$ that

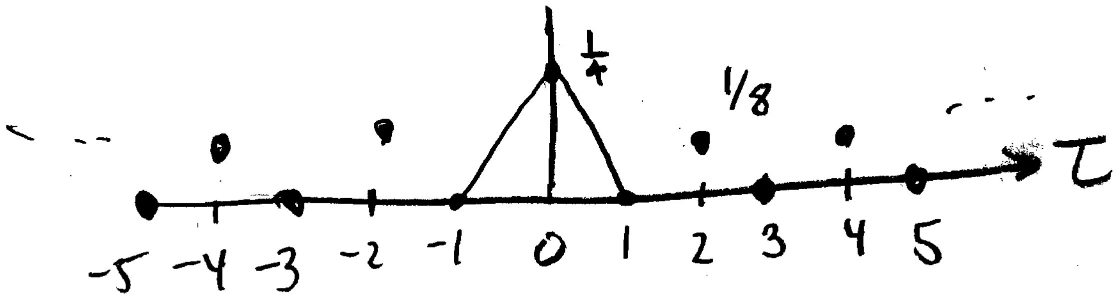
$$R_X(\tau)=P[X(t)X(t+\tau)=1]=P[X(t)=1]\,P[\theta+\tau<1]=\frac{1}{4}(1-\tau).$$

1) 2.3a) ... Since $R_x(\tau)$ must be even,

this implies that $R_x(\tau) = \frac{1}{4}(1+\tau)$ for $\tau \in (-1, 0)$.

All together so far, here is what we have for $R_x(\tau)$:

$R_x(\tau)$



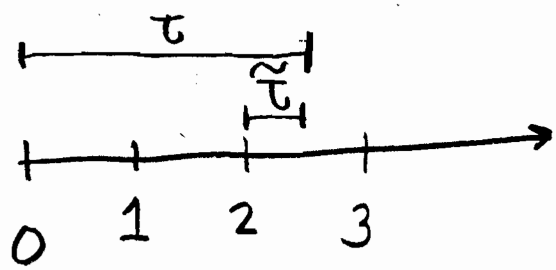
- Now consider $\tau \in (2,3)$, $\tau \in (4,5)$, or generally $\tau \in (2k, 2k+1)$ for $k \in \mathbb{N}$.

→ This is exactly the same as the case for $\tau \in (0,1)$ except for one difference: t and $t+\tau$ are now in different slots. Therefore, given that $x(t) = 1$, here it is only <u>half</u> as likely that $x(t+\tau) = 1$ compared to the case where $\tau \in (0,1)$.

→

① 2.3a)... Let $\tilde{\tau} = \tau \bmod 2$.

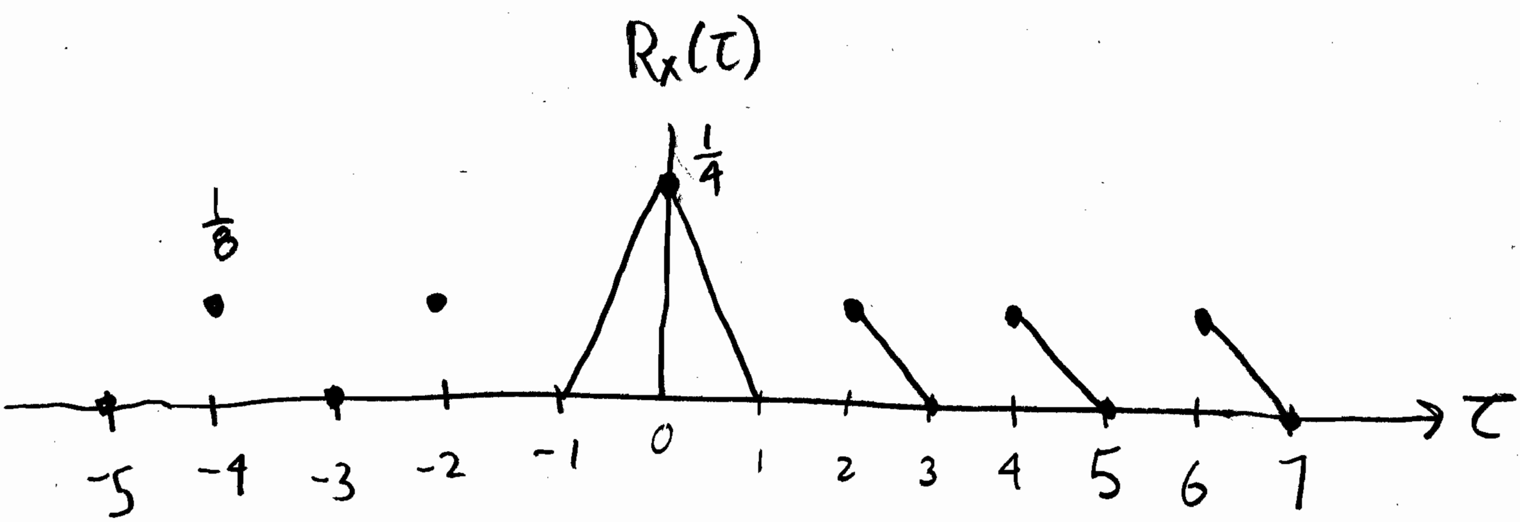
Then, e.g., if $\tau = 2\frac{1}{3}$, then $\tilde{\tau} = \frac{1}{3}$:



For $\tau \in (2k, 2k+1)$, $k \in \mathbb{N}$, we have

$$R_X(\tau) = \frac{1}{2}\left[\frac{1}{4}(1-\tilde{\tau})\right]$$
$$= \frac{1}{8}(1-\tilde{\tau})$$
$$= -\frac{1}{8}\tilde{\tau} + \frac{1}{8}.$$

Adding this to what we already had, here's what we've got so far for $R_X(\tau)$:



① 2.3a) ... Now consider $\tau \in (1,2)$ or $\tau \in (3,4)$, or more generally

$\tau \in (2k-1, 2k)$ for $k \in \mathbb{N}$

→ with prob. $\frac{1}{2}$, t falls in an odd slot, implying $x(t)=0$ and $x(t)x(t+\tau)=0$.

→ with prob. $\frac{1}{2}$, t falls in an even slot, where $x(t)=1$ and $x(t)=0$ are equally likely.

→ Given that the slot is even,

- with prob. $\frac{1}{2}$ $x(t)=0$ implying that $x(t)x(t+\tau)=0$.
- with prob $\frac{1}{2}$, $x(t)=1$.

→ Overall, $P[x(t)=1] = \frac{1}{4}$.

→ So $P[x(t)x(t+\tau)=1] = P\{$[t is in an even slot] and $[x(t)=1]$ and [$t+\tau$ is in an even slot] and $[x(t+\tau)=1]\}$

→

① 2.3a)...

$= P[t \text{ is in an even slot}]\, P[x(t)=1 \mid t \text{ is in an even slot}]$

$\times P[t+\tau \text{ is in an even slot} \mid t \text{ is in an even slot}]$

$\times P[x(t+\tau)=1 \mid t+\tau \text{ is in an even slot and } t \text{ is in an even slot}]$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot P[t+\tau \text{ is in an even slot} \mid t \text{ is in an even slot}] \cdot \frac{1}{2}$

$= \frac{1}{8} \underbrace{P[t+\tau \text{ is in an even slot} \mid t \text{ is in an even slot}]}_{\text{call this probability } \psi.}$

→ We have an arbitrary t that lies in an even slot. Then $\exists\, m \in \mathbb{Z}$ s.t.

$$t_0 + 2m \le t \le t_0 + 2m + 1.$$

Let $\tilde{t} = t - 2m$,

so that $t_0 \le \tilde{t} \le t_0 + 1$.

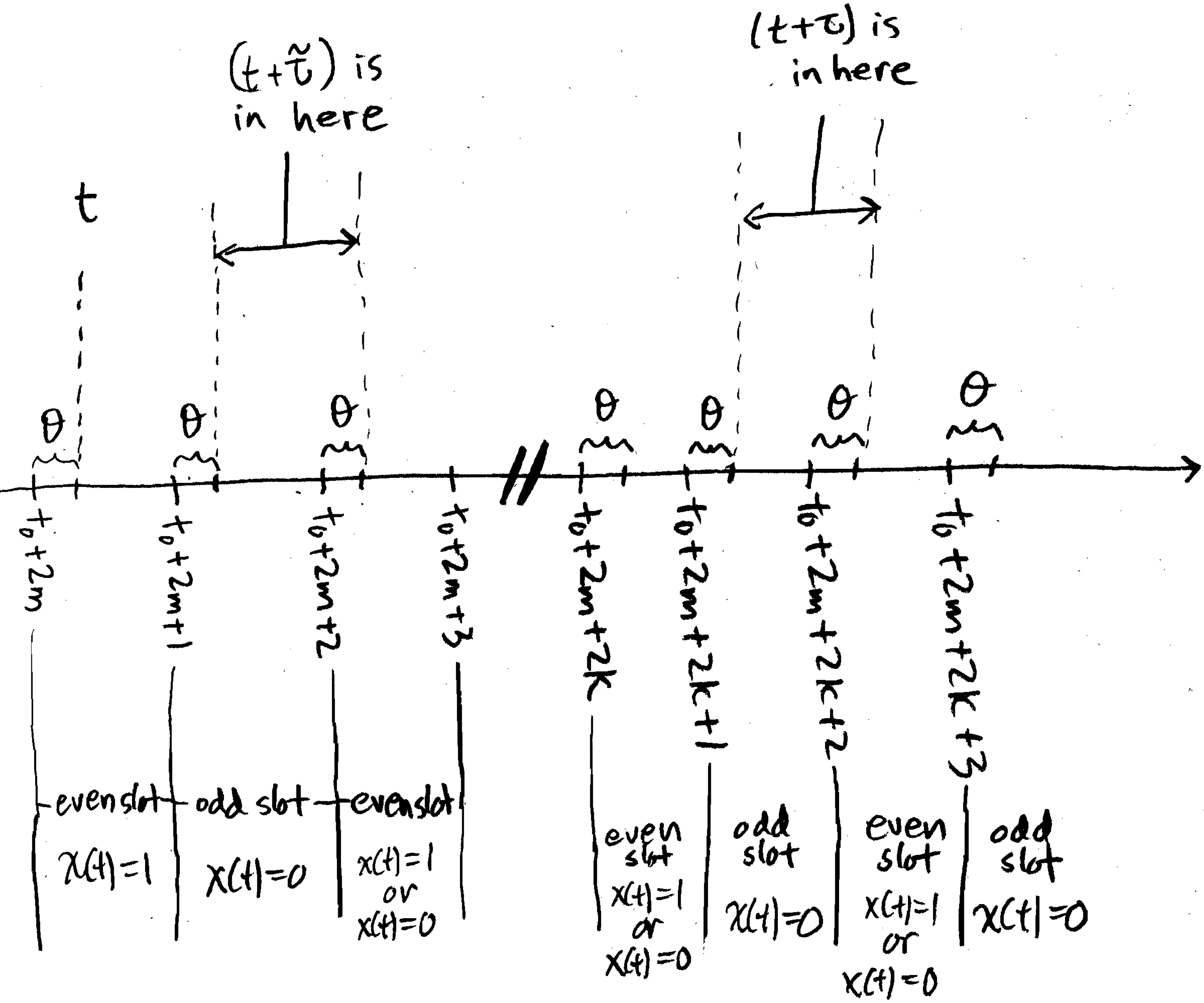
→

Also, by hypothesis (on top of (P10)),

$\exists\, k \in \mathbb{N}$ s.t. $\tau \in (2k-1, 2k)$.

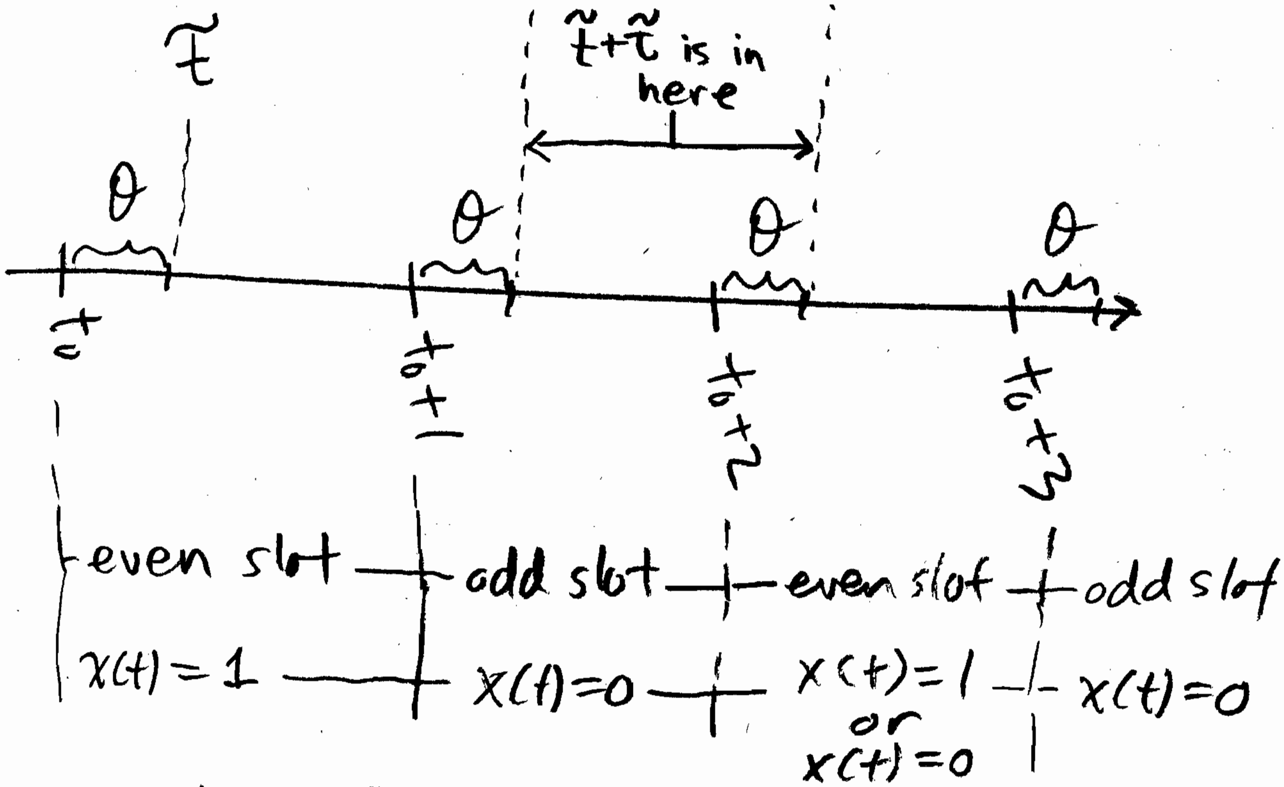
Let $\tilde{\tau} = \tau - 2k + 1$, so that $0 \leq \tilde{\tau} \leq 1$.

Here is a picture:



1) 2.3a) ...

Here is a picture showing the relationship between $\tilde{t}$ and $\tilde{\tau}$:



As defined on P11, we have

$$\psi = P(\theta + \tilde{\tau} > 1)$$

$$= P(\theta > 1 - \tilde{\tau})$$

$$= \int_{1-\tilde{\tau}}^{\infty} f_\theta(\xi)\,d\xi \qquad \{\theta \text{ is } u(0,1)\}$$

$$= \int_{1-\tilde{\tau}}^{1} d\xi = 1 - (1 - \tilde{\tau}) = \underline{\underline{\tilde{\tau}}}$$

$\longrightarrow$

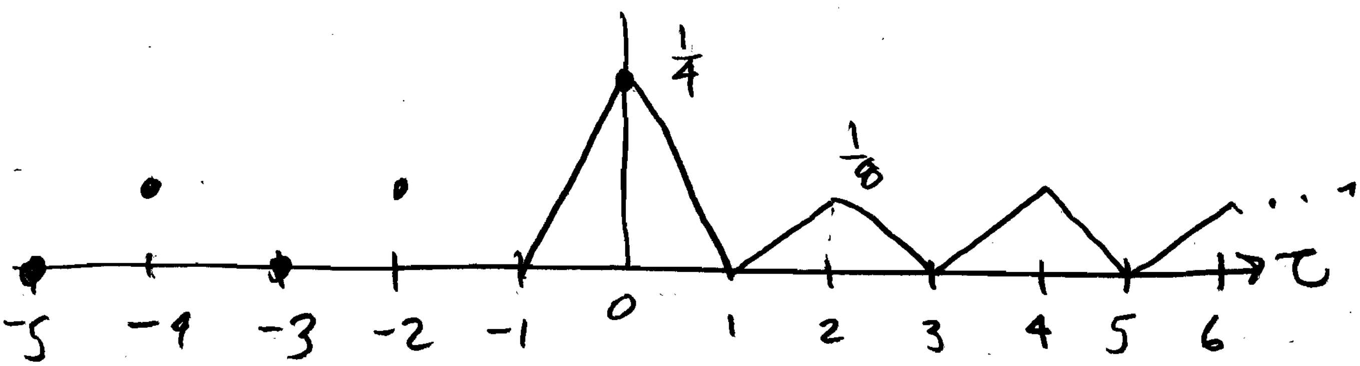
(1) 2.3a) ... So, from (P10), (P11), and (P13), we have that for

P14

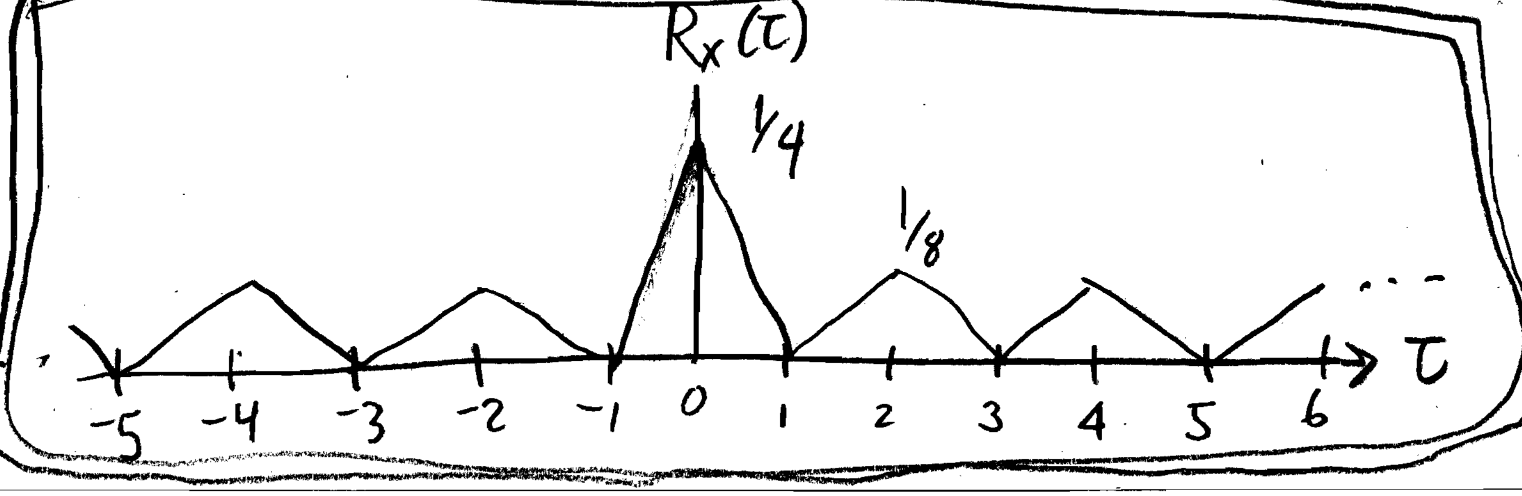
$\tau \in (2k-1, 2k)$ for $k \in \mathbb{N}$,

$$P[x(t)x(t+\tau) = 1] = \frac{1}{8}\psi = \frac{1}{8}\tilde{\tau}.$$

- Adding this information to the graph on (P9), we have so far:



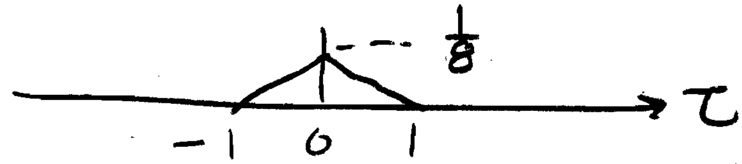
Now part (a) is completed by realizing that $R_x(\tau)$ has to be even symmetric:



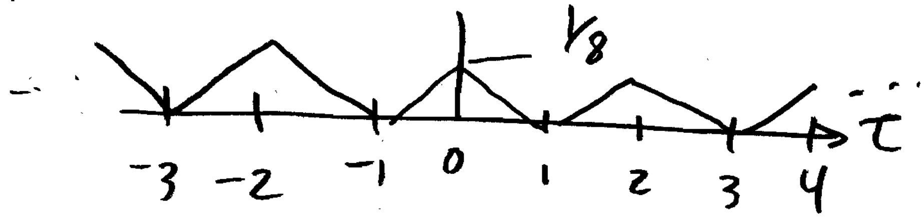
(1) 2.3b) Find the PSD,

$$S_x(\omega) = \mathcal{F}\{R_x(\tau)\}.$$

Let $a(\tau) =$



Let $b(\tau) =$



Then $R_x(\tau) = a(\tau) + b(\tau)$,

So $S_x(\omega) = A(\omega) + B(\omega)$.

Now $A(\omega)$ can be found in elementary Fourier transform tables and is given by

$$A(\omega) = \frac{1}{8}\left[\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right]^2.$$

① 2.3b)... Now $b(\tau)$ is periodic, P16
so we need to write it in a Fourier series in order to correctly find $B(\omega)$. Once we have the series, we can compute the Fourier transform term-by-term.

→ $b(\tau)$ is periodic with period $T=2$.

- In Fourier series terms, we have

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi.$$

- The series is given by

$$b(\tau) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}\tau} = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k\tau},$$

where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} b(\tau) e^{-jk\frac{2\pi}{T}\tau}\, d\tau$$

$$= \frac{1}{2}\int_{-1}^{1} b(\tau) e^{-j\pi k\tau}\, d\tau$$

$$= \frac{1}{2}\left[\int_{-1}^{0} \frac{1}{8}(\tau+1) e^{-j\pi k\tau}\, d\tau + \int_{0}^{1} \frac{1}{8}(1-\tau) e^{-j\pi k\tau}\, d\tau\right] \rightarrow$$

① 2.3b)... From here, the antiderivatives needed in the solution are different for the $k=0$ case than for the rest. P17

$\underline{\underline{k=0}}$

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} \frac{1}{8}\tau + \frac{1}{8}\, d\tau + \int_0^1 -\frac{1}{8}\tau + \frac{1}{8}\, d\tau \right]$$

$$= \frac{1}{2}\int_{-1}^{0} \frac{1}{8}\tau\, d\tau + \frac{1}{2}\int_{-1}^{0} \frac{1}{8}\, d\tau - \frac{1}{2}\int_0^1 \frac{1}{8}\tau\, d\tau + \frac{1}{2}\int_0^1 \frac{1}{8}\, d\tau$$

$$= \frac{1}{16}\int_{-1}^{0} \tau\, d\tau + \frac{1}{16}\int_{-1}^{1} d\tau - \frac{1}{16}\int_0^1 \tau\, d\tau$$

$$= \frac{1}{16}\left[\frac{1}{2}\tau^2\right]_{\tau=-1}^{0} + \frac{1}{16}\left[\tau\right]_{\tau=-1}^{1} - \frac{1}{16}\left[\frac{1}{2}\tau^2\right]_{\tau=0}^{1}$$

$$= \frac{1}{32}\left[0-1\right] + \frac{1}{16}\left[1--1\right] - \frac{1}{32}\left[1-0\right]$$

$$= -\frac{1}{32} + \frac{1}{16}\left[2\right] - \frac{1}{32} = \frac{2}{16} - \frac{2}{32}$$

$$= \frac{2}{16} - \frac{1}{16} = \frac{1}{16}.$$

$\longrightarrow$

(1) 2.3b) ... Now for the rest...

$\underline{\underline{k \neq 0}}$

$$a_k = \frac{1}{16}\left[\int_{-1}^{0} (\tau+1) e^{-j\pi k \tau} d\tau + \int_0^1 (1-\tau) e^{-j\pi k\tau} d\tau \right]$$

$$= \frac{1}{16}\left[\int_{-1}^{0} \tau e^{-j\pi k\tau} d\tau + \int_{-1}^{0} e^{-j\pi k\tau} d\tau + \int_0^1 e^{-j\pi k\tau} d\tau - \int_0^1 \tau e^{-j\pi k\tau} d\tau \right]$$

$$= \frac{1}{16}\left[\int_{-1}^{0} \tau e^{-j\pi k\tau} d\tau + \int_{-1}^{1} e^{-j\pi k\tau} d\tau - \int_0^1 \tau e^{-j\pi k\tau} d\tau \right]$$

$$= \frac{1}{16}\left\{ \left[\frac{1}{-jk\pi} e^{-j\pi k\tau}\left(\tau - \frac{1}{-j\pi k}\right)\right]_{\tau=-1}^{0} + \left[\frac{1}{-j\pi k} e^{-j\pi k\tau}\right]_{\tau=-1}^{1} - \left[\frac{1}{-j\pi k} e^{-j\pi k\tau}\left(\tau - \frac{1}{-j\pi k}\right)\right]_{\tau=0}^{1} \right\}$$

$$= \frac{-1}{j16\pi k}\left\{ \left[e^{-j\pi k\tau}\left(\tau + \frac{1}{j\pi k}\right)\right]_{\tau=-1}^{0} + \left[e^{-j\pi k\tau}\right]_{\tau=-1}^{1} - \left[e^{-j\pi k\tau}\left(\tau + \frac{1}{j\pi k}\right)\right]_{\tau=0}^{1} \right\}$$

$\longrightarrow$

(1) 2.3b) ... $\underline{\underline{k \neq 0}}$ ...

$$a_k \overset{\dots}{=} \frac{-1}{j16\pi k}\left\{\left[1\left(0+\frac{1}{j\pi k}\right) - e^{j\pi k}\left(\frac{1}{j\pi k}-1\right)\right]\right.$$

$$\left. + \left[e^{-j\pi k} - e^{j\pi k}\right] - \left[e^{-j\pi k}\left(1+\frac{1}{j\pi k}\right) - 1\left(0+\frac{1}{j\pi k}\right)\right]\right\}$$

$$= \frac{-1}{j16\pi k}\left\{\left[\frac{1}{j\pi k} - \frac{e^{j\pi k}}{j\pi k} + e^{j\pi k}\right] - 2j\left[\frac{e^{j\pi k}-e^{-j\pi k}}{2j}\right]\right.$$

$$\left. - \left[e^{-j\pi k} + \frac{e^{-j\pi k}}{j\pi k} - \frac{1}{j\pi k}\right]\right\}$$

$$= \frac{-1}{j16\pi k}\left\{\frac{1}{j\pi k} - \frac{e^{j\pi k}}{j\pi k} + e^{j\pi k} - \underbrace{2j\sin \pi k}_{\text{zero}} - e^{-j\pi k}\right.$$

$$\left. - \frac{e^{-j\pi k}}{j\pi k} + \frac{1}{j\pi k}\right\}$$

$$= \frac{-1}{j16\pi k}\left\{\frac{2}{j\pi k} - \frac{1}{j\pi k}\left[e^{j\pi k} + e^{-j\pi k}\right] + \left[e^{j\pi k} - e^{-j\pi k}\right]\right\}$$

$$= \frac{-1}{j16\pi k}\left\{\frac{2}{j\pi k} - \frac{1}{j\pi k} 2\cos \pi k + \underbrace{2j\sin \pi k}_{\text{zero}}\right\}$$

$$= \frac{-2}{-16\pi^2 k^2} - \frac{-2}{-16\pi^2 k^2}\cos \pi k$$

$\longrightarrow$

① 2.3 b) ... $\underline{k \neq 0}$

P20

$$a_k \overset{\dots}{=} \frac{1}{8\pi^2 k^2}\left[1 - \cos \pi k\right] = \begin{cases} 0, & k \text{ even} \\ \frac{2}{8\pi^2 k^2}, & k \text{ odd} \end{cases}$$

$$= \begin{cases} 0, & k \text{ even} \\ \frac{1}{4\pi^2 k^2}, & k \text{ odd} \end{cases} \quad k \neq 0$$

Putting the Fourier series together as on P16, we have

$$b(\tau) = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k \tau} = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{j\pi k\tau} + a_{-k} e^{-j\pi k\tau}\right]$$

$$= \frac{1}{16} + \sum_{k=1}^{\infty} a_k e^{j\pi k\tau} + a_{-k} e^{-j\pi k\tau} \quad \{a_k = 0 \text{ for } k \text{ even}\}$$

$$= \frac{1}{16} + \sum_{k=0}^{\infty} a_{2k+1} e^{j\pi(2k+1)\tau} + a_{-2k-1} e^{-j\pi(2k+1)\tau}$$

$$= \frac{1}{16} + \sum_{k=0}^{\infty} \frac{e^{j\pi(2k+1)\tau}}{4\pi^2(2k+1)^2} + \sum_{k=0}^{\infty} \frac{e^{-j\pi(2k+1)\tau}}{4\pi^2(2k+1)^2}$$

1) 2.3b)... so,

$$B(\omega) = \mathcal{F}\{b(\tau)\} = \mathcal{F}\left\{ \frac{1}{16} + \sum_{k=0}^{\infty} \frac{e^{j\pi(2k+1)\tau}}{4\pi^2(2k+1)^2} + \sum_{k=0}^{\infty} \frac{e^{-j\pi(2k+1)\tau}}{4\pi^2(2k+1)^2} \right\}$$

$$= \frac{2\pi}{16}\delta(\omega) + \sum_{k=0}^{\infty} \frac{2\pi}{4\pi^2(2k+1)^2}\Big[\delta(\omega - \pi(2k+1)) + \delta(\omega + \pi(2k+1))\Big]$$

$$= \frac{\pi}{8}\delta(\omega) + \sum_{k=0}^{\infty} \frac{1}{2\pi(2k+1)^2}\Big[\delta(\omega - \pi(2k+1)) + \delta(\omega + 2\pi(2k+1))\Big]$$

Whew!!

2) 2.5) $x(t)$ is Gaussian and stationary with $R_x(\tau) = 4e^{-|\tau|}$.

→ Since $\lim_{\tau \to \infty} R_x(\tau) = 0$ and $x(t)$ is stationary, it must be that $\mu = E[x(t)] = 0$.

→ So $R_x(0) = E[x^2(t)] = E\{[x(t)-\mu]^2\} = \sigma_x^2 = 4$.

→ Then $\sigma_x = 2$.

At each $t \in \mathbb{R}$, $x(t)$ is a Gaussian variable with mean $\mu = 0$, variance $\sigma_x^2 = 4$, and standard deviation $\sigma_x = 2$.

→ $P(|x(t)| > 4) = P(|x(t) - \mu| > 2\sigma)$

$= P(x(t) - \mu < 2\sigma) + P(x(t) - \mu > 2\sigma)$.

From Table 1.9 on p. 28 of the text, this is

$$2[1 - F_x(2)] = 2[1 - .97725]$$

$$= 2[.02275] = \underline{\underline{.0455}}$$

③ 2.16) $x(t)$ is stationary with

PSD $S_X(\omega) = \dfrac{6\omega^2 + 12}{(\omega^2+4)(\omega^2+1)}$.

Consider $S(\theta) = \dfrac{6\theta + 12}{(\theta+4)(\theta+1)}$. $(\theta = \omega^2)$.

Then $S(\theta)$ can be expanded in partial fractions according to

$$S(\theta) = \frac{6\theta+12}{(\theta+4)(\theta+1)} = \frac{A}{\theta+4} + \frac{B}{\theta+1}.$$

$$A = \left.\frac{6\theta+12}{\theta+1}\right|_{\theta=-4} = \frac{-24+12}{-3} = \frac{-12}{-3} = 4$$

$$B = \left.\frac{6\theta+12}{\theta+4}\right|_{\theta=-1} = \frac{-6+12}{3} = \frac{6}{3} = 2$$

So $S(\theta) = \dfrac{6\theta+12}{(\theta+4)(\theta+1)} = \dfrac{4}{\theta+4} + \dfrac{2}{\theta+1}$.

with $\theta = \omega^2$, we have

$$S_X(\omega) = \frac{4}{\omega^2+4} + \frac{2}{\omega^2+1}$$

③ 2.16)... Using (2.7.16) on p. 90 of the text, we have

$$E\left[X^2(t)\right] = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{4}{\omega^2+4} + \frac{2}{\omega^2+1}\right] d\omega$$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+4}\, d\omega + \frac{2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+1}\, d\omega$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+4}\, d\omega + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+1}\, d\omega$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]_{\omega=-\infty}^{\infty} + \frac{1}{\pi}\left[\tan^{-1}(\omega)\right]_{\omega=-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - -\frac{\pi}{2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2} - -\frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}[\pi] + \frac{1}{\pi}[\pi] = 1+1 = \underline{\underline{2}}$$

④ 2.17) $X(t)$ is stationary and

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}.$$

The process $Y(t)$ is defined by $Y(t) = aX(t) + b$, where a, b are deterministic constants.

a) Find $R_Y(\tau)$;

→ Since $X(t)$ is stationary and a, b are deterministic constants, $Y(t)$ is also stationary.

$$R_Y(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\{[aX(t)+b][aX(t+\tau)+b]\}$$

$$= E[a^2X(t)X(t+\tau) + abX(t) + abX(t+\tau) + b^2]$$

$$= a^2E[X(t)X(t+\tau)] + ab\cdot 0 + ab\cdot 0 + E[b^2]$$

$$= a^2R_X(\tau) + b^2 = \underline{\underline{a^2\sigma^2 e^{-\beta|\tau|} + b^2}}$$

→

④ 2.17) ...

b) Find $R_{XY}(\tau)$:

$$R_{XY}(\tau) = E\left[X(t)\,Y(t+\tau)\right]$$

$$= E\left\{X(t)\left[aX(t+\tau)+b\right]\right\}$$

$$= E\left[aX(t)X(t+\tau) + bX(t)\right]$$

$$= aE\left[X(t)X(t+\tau)\right] + b\underbrace{E\left[X(t)\right]}_{\text{zero}}$$

$$= aR_X(\tau)$$

$$= a\sigma^2 e^{-\beta|\tau|}$$

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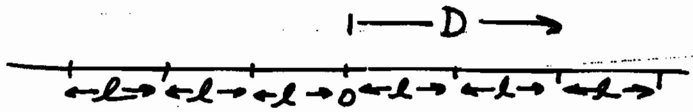
P27

2.21



Probability mass dist.

l_i	Prob
l	$1/2$
$-l$	$1/2$

$$D = l_1 + l_2 + l_3 \cdots l_N$$

$$E(D) = E(l_1) + E(l_2) + \cdots = 0$$

$$E(D^2) = E[l_1^2 + l_2^2 + \cdots + 2l_1l_2 + 2l_1l_3 + \cdots]$$

$$= l^2 + l^2 + \cdots + 0 + 0 + \cdots$$

$$= Nl^2$$

$$\therefore \operatorname{Var} D = E(D^2) - [E(D)]^2 = Nl^2$$

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2.24

X is zero-mean because $R_x(\tau) \to 0$ as $\tau \to \infty$.

X_1 and X_2 are separated by 1 unit.

Therefore, $E[X_1 X_2] = R_X(1) = 4e^{-1}$

The bivariate normal density is then

$$f_{X_1X_2} = \frac{1}{2\pi |C|^{1/2}} e^{-\frac{1}{2}(\underline{x}^T C^{-1} \underline{x})}$$

where $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 4e^{-1} \\ 4e^{-1} & 4 \end{bmatrix}$