KALMAN FILTERING HW04 SOLUTION (P1)
HAVLICEK

(1)

3.2 Use integral tables p. 142 for all parts.

(a) $$\overline{x^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} A \cdot \frac{Ts}{(1+Ts)^2} \cdot \frac{T(-s)}{(1+T(-s))^2}\, ds$$ (Overscore means "average".)

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{\sqrt{A}Ts}{T^2s^2+2Ts+1} \cdot \left(\text{same with } s \to -s\right) \cdot ds$$

Using the tables:

$n = 2$, $c_1 = \sqrt{A}T$, $c_0 = 0$; $d_2 = T^2$, $d_1 = 2T$, $d_0 = 1$

$$\therefore \overline{x^2} = \frac{c_1^2 d_0}{2 d_0 d_1 d_2} = \frac{AT^2}{2 \cdot 2T \cdot T^2} = \frac{A}{4T}$$

(b) $$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

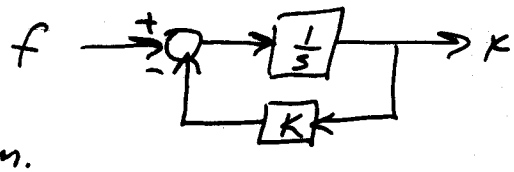
Following the same procedure as in (a):

$n = 2$; $c_1 = 0$, $c_0 = \sqrt{A}\,\omega_0^2$; $d_2 = 1$, $d_1 = 2\zeta\omega_0$, $d_0 = \omega_0^2$

$$\therefore \overline{x^2} = I_2 = \frac{c_0^2 d_2}{2 d_0 d_1 d_2} = \frac{A\omega_0^4}{2 \cdot \omega_0^2 \cdot 2\zeta\omega_0} = \frac{A\omega_0}{4\zeta}$$

3.4

First, find the transfer function for feedback system.



$$G(s) = \frac{X(s)}{F(s)} = \frac{\frac{1}{s}}{1 + \frac{1}{s}\cdot K} = \frac{1}{s+K}$$

(a) The spectral density function is then

$$S_x'(s) = \frac{2\sigma^2\beta}{-s^2+\beta^2}\cdot\frac{1}{s+K}\cdot\frac{1}{-s+K}$$

Or, after spectral factorization

$$S_x(s) = \frac{\sqrt{2\sigma^2\beta}}{(s+\beta)(s+K)}\cdot\frac{\sqrt{2\sigma^2\beta}}{(-s+\beta)(-s+K)}$$

(b) Mean square value is obtained from tables, p. 142

$n = 2$ $\quad c_1 = 0$ $\quad d_2 = 1$

$c_0 = \sqrt{2\sigma^2\beta}$ $\quad d_1 = \beta + K$

$d_0 = \beta K$

$$E[x^2] = \frac{c_0^2 d_2}{2 d_0 d_1 d_2} = \frac{2\sigma^2\beta}{2\cdot\beta K\cdot(\beta+K)} = \frac{\sigma^2}{K(\beta+K)}$$

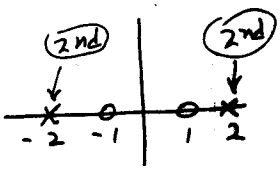
3.9 First write S in terms of complex s.

$$S(s) = \frac{-s^2+1}{s^4-8s^2+16}$$

Next, find poles and zeros of $S(s)$

zeros: ± 1

poles: ± 2 (Both 2nd order)



Now use spectral factorization:

$$S(s) = \left[\frac{s+1}{(s+2)^2}\right]\left[\frac{-s+1}{(-s+2)^2}\right]$$

The first factor in the above expression is the shaping filter; i.e.,

unity white noise $\longrightarrow \boxed{\dfrac{s+1}{(s+2)^2}} \longrightarrow$ Noise with spectral function $S(s)$

3.17

Initial value is $N(0,\sigma^2)$

$$f(t) \longrightarrow \boxed{\frac{1}{s}} \longrightarrow x(t)$$

$f(t)$ is white noise with autocorrelation fcn. $A\delta(\tau)$.

$$x(t) = x(0) + \int_0^t f(u)\,du, \quad x(0) \text{ is } N(0,\sigma^2)$$

$$x^2(t) = x^2(0) + 2x(0)\int_0^t f(u)\,du + \int_0^t\int_0^t f(u)f(v)\,du\,dv$$

$$E[x^2(t)] = E[x^2(0)] + E\left[2x(0)\int_0^t f(u)\,du\right] + \int_0^t\int_0^t E[f(u)f(v)]\,du\,dv$$

$$= \sigma^2 + 0 + \int_0^t\int_0^t R_f(u-v)\,du\,dv$$

$$= \sigma^2 + \int_0^t\int_0^t A\,\delta(u-v)\,du\,dv$$

$$= \sigma^2 + At$$

3.25

$$\gamma_{xy}^2(\omega) = \frac{|S_{xy}(j\omega)|^2}{S_x(j\omega)\cdot S_y(j\omega)} \; ; \qquad x \rightarrow \boxed{G(\omega)} \rightarrow y$$

(a) $$y(t) = \int_{-\infty}^{\infty} g(u)\,x(t-u)\,du$$

$$S_{xy}(j\omega) = \mathcal{F}[R_{xy}(\tau)] = \mathcal{F}[E[x(t)\,y(t+\tau)]]$$

$$= \mathcal{F}\left[E\left[x(t)\int_{-\infty}^{\infty} g(u)\,x(t+\tau-u)\,du\right]\right]$$

$$= \mathcal{F}\left[\int_{-\infty}^{\infty} g(u)\,E[x(t)\,x(t+\tau-u)]\,du\right]$$

$$= \mathcal{F}\left[\int_{-\infty}^{\infty} g(u)\,R_x(\tau-u)\right] = G(j\omega)\,S_x(j\omega)$$

(b) $$\gamma_{xy}^2 = \frac{|G(j\omega)|^2\,S_x(j\omega)\cdot S_x(j\omega)}{S_x(j\omega)\cdot \underbrace{S_x(j\omega)\,|G(j\omega)|^2}_{S_y(j\omega)}}$$

(Note S_x is Real.)

$$= 1$$