HAVLICEK

ECE 6973
KALMAN FILTERING
HW 3 SOLUTION

(1) 4.4 $\quad S_s = \dfrac{8}{-s^2+4}, \quad S_m = \dfrac{2}{-s^2+1}$

$$S_{s+m} = S_s + S_m = \frac{-10s^2+16}{(-s^2+4)(-s^2+1)} = 10\,\frac{-s^2+8/5}{(-s^2+4)(-s^2+1)}$$

$$\frac{S_s}{S^-_{s+m}} = \frac{\dfrac{8}{(s+2)\cancel{(-s+2)}}}{\sqrt{10}\,\dfrac{-s+\sqrt{8/5}}{\cancel{(-s+2)}(-s+1)}} = \frac{8}{\sqrt{10}}\;\frac{-s+1}{(s+2)(-s+\sqrt{8/5})}$$

$$= \frac{8}{\sqrt{10}}\left[\frac{K_1}{s+2} + \frac{K_2}{-s+\sqrt{8/5}}\right], \quad K_1 = \frac{3}{2+\sqrt{8/5}}$$

K_2 is not needed.

$$\text{Optimum } G(s) = \frac{1}{\sqrt{10}\,\dfrac{s+\sqrt{8/5}}{(s+2)(s+1)}}\left[\frac{8}{\sqrt{10}}\cdot\frac{3}{2+\sqrt{8/5}}\cdot\frac{1}{s+2}\right]$$

$$= \frac{12}{5}\cdot\frac{1}{2+\sqrt{8/5}}\cdot\frac{s+1}{s+\sqrt{8/5}}$$

To find mean square error, use Eq (4.3.12):

$$E(e^2) = R_s(0) - \int_0^\infty g(u)\,R_s(u)\,du$$

In this case,

$$R_s(0) = 2, \quad g(u) = \frac{12}{5}\cdot\frac{1}{2+\sqrt{8/5}}\cdot\left[\delta(u) + (1-\sqrt{8/5})\,e^{-\sqrt{8/5}\,u}\right],$$

and $R_s(u) = 2e^{-2u}$ (for $u>0$)

Substituting the above into Eq (4.3.12) yields

$$E(e^2) \approx .65$$

(Note: See solution to Prob. 4.5 for a comment on this result.)

4.5 Noncausal solution for Prob. 4.4. As before:

$$S_s = \frac{8}{-s^2+4}, \quad S_m = \frac{2}{-s^2+1}, \quad S_{s+m} = \frac{-10s^2+16}{(-s^2+4)(-s^2+1)}$$

$$\text{Optimal noncausal } G(s) = \frac{S_s}{S_{s+m}} = \frac{8}{10} \cdot \frac{-s^2+1}{-s^2+8/5}$$

Weighting function $g(u)$ is found by rewriting the above $G(s)$ as

$$G(s) = \frac{8}{10}\left[1 - \frac{-.6}{-s^2+8/5}\right]$$

The inverse 2-sided Laplace transform will be recognized as

$$g(u) = .8\,\delta(u) - \frac{.48}{2\sqrt{8/5}}\, e^{-\sqrt{8/5}\,|u|}$$

Mean square error is found from Eq (4.3.12)

This leads to: $E(e^2) \approx .63$

Note that this is only slightly less than the .65 value obtained in the causal case (Prob 4.5). This is due to the "convenient" numbers used in this example. The noise and the signal have almost the same spectral characteristics, so the resulting optimal filter is almost flat with frequency. As a matter of fact, if we were to consider a trivial flat-gain filter, the best gain works out to be 2/3 and its $E[e^2]$ is 2/3, only slightly more than .63 or .65.

4.6 $S_s = \dfrac{\omega^2+1}{\omega^4+8\omega^2+16}$; or $S_s(s) = \dfrac{-s^2+1}{s^4+8(-s^2)+16}$

Assume zero noise. Therefore $S_{s+m} = S_s$

First, factor S_{s+m}:

$$S_{s+m} = S_{s+m}^{+} \cdot S_{s+m}^{-} = \frac{s+1}{(s+2)(s+2)} \cdot \frac{-s+1}{(-s+2)(-s+2)}$$

Next, find S_s/S_{s+m}^{-} and its inverse.

$$S_s/S_{s+m}^{-} = \frac{s+1}{(s+2)(s+2)} = \left[\frac{-1}{(s+2)^2} + \frac{1}{(s+2)}\right]$$

$\mathcal{L}^{-1}\left[S_s/S_{s+m}^{-}\right] = -te^{-2t} + e^{-2t}$,

Now shift to the left one unit as shown.

Call the shifted function $g_{sh}(t)$

$$g_{sh}(t) = \begin{cases} -(t+1)e^{-2(t+1)} + e^{-2(t+1)}, & t > -1 \\ 0, & t < -1 \end{cases}$$

Now truncate the negative-time part and take the ordinary single-sided L.T. of the result. Truncated fcn. for $t > 0$ is $-e^{-2}te^{-2t}$. Its Laplace transform is $-e^{-2}\dfrac{1}{(s+2)^2}$. Therefore, Optimal predictor $G(s)$ is:

$$G(s) = \frac{-e^{-2}\frac{1}{(s+2)^2}}{\frac{(s+1)}{(s+2)^2}} = -e^{-2}\frac{1}{s+1}$$