

MODULE 3

LINEAR SYSTEMS

WITH

STOCHASTIC INPUTS

- Linear system analysis is concerned with relating the input signal, system function, and output signal of a linear system.
- In the deterministic case, this is done by specifying the input, output, and impulse response as functions, either in time or in frequency.
- In the probabilistic case, it is generally impossible to specify the input and output signals as functions. Instead, they are characterized in terms of their first- and second-order moments.
- Henceforth, lower case letters will be used for time domain signals and uppercase letters will be used for frequency domain signals.

Thus:



$x(t)$: input signal, a stochastic process.

$g(t)$: impulse response, deterministic

$G(s)$: Transfer function, deterministic

$y(t)$: Output signal, a stochastic process

- Two types of analysis are common. Both arise from classical control theory:

1. "Steady-State" or "stationary" analysis.

→ The system is LSI and BIBO stable.

→ The input is stationary.

2. "Transient" or "nonstationary" analysis.

→ The system is LSI, and may be BIBO stable.

→ The input is assumed to be one sided,
usually starting at $t=0$.

→ As compared to steady state analysis, any response due to an input for $t < 0$ is modeled by a nonzero initial state for the system at $t=0$. In other words, the system might not be initially relaxed.

Continuous - Time Steady-State Analysis

$$x(t) \rightarrow [H] \rightarrow y(t)$$

- Let H be LSI and BIBO stable with impulse response $h(t)$, frequency response $H(\omega)$, and transfer function $H(s)$.

→ If $x(t)$ is SSS, then $y(t)$ is SSS.

→ If $x(t)$ is WSS, then $y(t)$ is WSS, and $x(t)$ and $y(t)$ are jointly WSS.

→ If $x(t)$ is Gaussian, then $y(t)$ is Gaussian.

$$\begin{aligned} \rightarrow y(t) &= \int_{-\infty}^{\infty} x(\theta) h(t-\theta) d\theta \\ &= \int_{-\infty}^{\infty} x(t-\theta) h(\theta) d\theta \end{aligned}$$

$$\rightarrow Y(\omega) = X(\omega) H(\omega).$$

$$\rightarrow Y(s) = X(s) H(s).$$

- First moment of output:

$$\begin{aligned} E[y(t)] &= E\left\{\int_{-\infty}^{\infty} x(\theta) h(t-\theta) d\theta\right\} \\ &= \int_{-\infty}^{\infty} E[x(\theta)] h(t-\theta) d\theta \end{aligned}$$

→ If $x(t)$ is WSS, then

$$\begin{aligned} E[y(t)] &= E[x(t)] \int_{-\infty}^{\infty} h(t-\theta) d\theta \\ &= E[x(t)] \underbrace{\int_{-\infty}^{\infty} h(\theta) d\theta}_{H(0)} \\ &= H(0) E[x(t)]. \end{aligned}$$

a number for WSS.

- Second moment of output:

$$\begin{aligned} E[y^2(t)] &= E\left\{\int_{-\infty}^{\infty} x(t-\theta) h(\theta) d\theta \int_{-\infty}^{\infty} x(t-\lambda) h(\lambda) d\lambda\right\} \\ &= E\left\{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t-\theta) x(t-\lambda) h(\theta) h(\lambda) d\theta d\lambda\right\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\{x(t-\theta) x(t-\lambda)\} h(\theta) h(\lambda) d\theta d\lambda \end{aligned}$$



$$\dots E[y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t-\theta, t-\lambda) h(\theta) h(\lambda) d\theta d\lambda$$

→ If $x(t)$ is WSS, then

$$E[y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(\theta-\lambda) h(\theta) h(\lambda) d\theta d\lambda$$

- Output autocorrelation:

$$R_y(t_1, t_2) = E[y(t_1) y(t_2)]$$

$$= E \left\{ \int_{-\infty}^{\infty} x(t_1-\theta) h(\theta) d\theta \int_{-\infty}^{\infty} x(t_2-\lambda) h(\lambda) d\lambda \right\}$$

$$= E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1-\theta) x(t_2-\lambda) h(\theta) h(\lambda) d\theta d\lambda \right\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[x(t_1-\theta) x(t_2-\lambda)] h(\theta) h(\lambda) d\theta d\lambda$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t_1-\theta, t_2-\lambda) h(\theta) h(\lambda) d\theta d\lambda$$

→ If $x(t)$ is WSS, then

$$R_x(t_1-\theta, t_2-\lambda) = R_x[(t_2-\lambda) - (t_1-\theta)]$$

$$= R_x[(t_1-\theta) - (t_2-\lambda)]$$

$$= R_x(t_1-t_2-\theta+\lambda)$$

$$= R_x(t_2-t_1+\theta-\lambda)$$

So,

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t_2 - t_1 + \theta - \lambda) h(\theta) h(\lambda) d\theta d\lambda.$$

→ Let $\tau = t_2 - t_1$. Then

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(\tau + \theta - \lambda) h(\theta) h(\lambda) d\theta d\lambda.$$

- Crosscorrelation of Input and Output:

$$\begin{aligned} R_{x,y}(t_1, t_2) &= E[x(t_1)x(t_2)] \\ &= E\left\{ x(t_1) \int_{-\infty}^{\infty} x(t_2 - \theta) h(\theta) d\theta \right\} \\ &= \int_{-\infty}^{\infty} E[x(t_1)x(t_2 - \theta)] h(\theta) d\theta \\ &= \int_{-\infty}^{\infty} R_x(t_1, t_2 - \theta) h(\theta) d\theta. \end{aligned}$$

→ if $x(t)$ is WSS, let $\tau = t_2 - t_1$.

Then

$$\begin{aligned} R_{x,y}(\tau) &= \int_{-\infty}^{\infty} R_x[t_2 - \theta - t_1] h(\theta) d\theta \\ &= \int_{-\infty}^{\infty} R_x(\tau - \theta) h(\theta) d\theta = \underline{\underline{R_x(\tau) * h(\tau)}} \end{aligned}$$

$$\begin{aligned}
 R_{yx}(t_1, t_2) &= E[y(t_1)x(t_2)] = E\left[\int_{-\infty}^{\infty} x(t_1 - \theta) h(\theta) d\theta | x(t_2)\right] \\
 &= \int_{-\infty}^{\infty} E[x(t_1 - \theta)x(t_2)] h(\theta) d\theta \\
 &= \int_{-\infty}^{\infty} R_x(t_1 - \theta, t_2) h(\theta) d\theta
 \end{aligned}$$

→ For WSS $x(t)$ with $\tau = t_2 - t_1$,

$$\begin{aligned}
 R_{yx}(\tau) &= \int_{-\infty}^{\infty} R_x(t_2 - t_1 + \theta) h(\theta) d\theta = \int_{-\infty}^{\infty} R_x(\tau + \theta) h(\theta) d\theta \\
 (\lambda = -\tau) &= \int_{-\infty}^{\infty} R_x(\tau - \lambda) h(-\lambda) d\lambda \\
 &= \underline{\underline{R_x(\tau) * h(-\tau)}}
 \end{aligned}$$

- Output Power Spectral Density for WSS input:

→ Using the expression for $R_y(t)$ on page 3-6,

$$\begin{aligned}
 S_y(\omega) &= \Im\{R_y(t)\} \\
 &= \int_{-\infty}^{\infty} R_y(t) e^{-j\omega t} dt \\
 &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t + \theta - \lambda) h(\theta) h(\lambda) d\theta d\lambda \right\} e^{-j\omega t} dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\theta) h(\lambda) \left\{ \int_{-\infty}^{\infty} R_x(t + \theta - \lambda) e^{-j\omega t} dt \right\} d\theta d\lambda
 \end{aligned}$$



$$\rightarrow \int_{-\infty}^{\infty} R_x[t - (\lambda - \theta)] e^{-j\omega t} dt = S_x(\omega) e^{j\omega(\theta - \lambda)}$$

$$\text{So, } S_y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\theta) h(\lambda) S_x(\omega) e^{j\omega(\theta - \lambda)} d\theta d\lambda$$

$$= S_x(\omega) \int_{-\infty}^{\infty} h(\theta) e^{j\omega\theta} d\theta \int_{-\infty}^{\infty} h(\lambda) e^{j\omega\lambda} d\lambda$$

$$= S_x(\omega) H(-\omega) H(\omega)$$

$$\left\{ \begin{array}{l} \text{if } h(t) \\ \text{real} \end{array} \right\} \Rightarrow = S_x(\omega) |H(\omega)|^2$$

→ This is called the "Wiener-Khintchine" relation.

→ The last line follows from the fact that $H(\omega) = H^*(-\omega)$ when $h(t)$ is real (Transform of a real signal is conjugate symmetric).

→ With the Laplace transform, the Wiener-Kintchine relation becomes

$$S_y(s) = H(s) H(-s) S_x(\omega),$$

- Input /output Cross Power Spectrum:

→ Using the relation $R_{xy}(t) = R_x(t) * h(t)$ from PAGE 37,

$$\begin{aligned} S_{xy}(\omega) &= \mathcal{F}\{R_{xy}(t)\} \\ &= \mathcal{F}\{R_x(t) * h(t)\} \\ &= H(\omega) S_x(\omega). \end{aligned}$$

→ Also,

$$\begin{aligned} S_{yx}(\omega) &= \mathcal{F}\{R_{yx}(\omega)\} = \mathcal{F}\{R_x(t) * h(-t)\} \\ &= S_x(\omega) H(-\omega). \end{aligned}$$

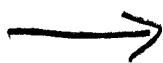
For $h(t)$ real, $S_{yx}(\omega) = H^*(\omega) S_x(\omega).$

- Second Moment of Output:

$$\begin{aligned} E[y^2(t)] &= R_y(0) = \mathcal{F}^{-1}\{S_y(\omega)\} \Big|_{t=0} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) e^{j\omega t} d\omega \Big|_{t=0} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) |H(\omega)|^2 d\omega \quad (h(t) \text{ real}). \end{aligned}$$

-Spectral Factorization:

- "Spectral factorization" is the technique of splitting a rational Laplace transform into two terms, one with poles in the right half-plane only and one with poles in the left half-plane only.
- This technique is important in the development of the Wiener filter.
- Let $f(t)$ be a square-integrable function with rational Laplace transform $F(s)$.
- If $f(t)$ is causal, so that $f(t)=0 \forall t < 0$, then $F(s)$ has poles in the left half-plane only.
- If $f(t)$ is anticausal, so that $f(t)=0 \forall t > 0$, then $F(s)$ has poles in the right half-plane only.



→ Now let $f(t)$ be an arbitrary square-integrable function with rational Laplace transform and define

$$f_+(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f_-(t) = \begin{cases} f(t), & t < 0 \\ 0, & t \geq 0 \end{cases}$$

→ Clearly,

$$f(t) = f_+(t) + f_-(t),$$

so

$$F(s) = F_+(s) + F_-(s),$$

where $F_+(s)$ has all poles in the left half-plane and $F_-(s)$ has all poles in the right half-plane.

→ By performing "backwards" partial fractions, this can also be written as the product

$$F(s) = \tilde{F}_+(s) \tilde{F}_-(s),$$

where $\tilde{F}_+(s)$ has left half-plane poles only and $\tilde{F}_-(s)$ has right half-plane poles only.

EX: $f(t) = e^{-\alpha|t|}$, $\alpha > 0$.

$$f_-(t) = \begin{cases} e^{\alpha t}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

$$f_+(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} F(s) &= \frac{2\alpha}{\alpha^2 - s^2} \\ &= \underbrace{\frac{1}{s + \alpha}}_{F_+(s)} + \underbrace{\frac{-1}{s - \alpha}}_{F_-(s)} \end{aligned}$$

$$\begin{aligned} &= \underbrace{\frac{\sqrt{2\alpha}}{s + \alpha}}_{\widetilde{F}_+(s)} \cdot \underbrace{\frac{\sqrt{2\alpha}}{s - \alpha}}_{\widetilde{F}_-(s)} \end{aligned}$$

EX: Let H be a first-order low-pass filter with frequency response

$$H(\omega) = \frac{1}{1 + jT\omega}$$

and let the input $x(t)$ be bandlimited white noise with PSD

$$S_x(\omega) = \begin{cases} A, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$\text{Then } S_y(\omega) = S_x(\omega) |H(\omega)|^2$$

$$= \begin{cases} \frac{A}{1 + T^2\omega^2}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

The second moment of the output is

$$E[y^2(t)] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{A}{1 + T^2\omega^2} d\omega$$

$$= \frac{A}{\pi T} \arctan(\omega_c T).$$

EX: Let H be as in the last example with

$$H(\omega) = \frac{1}{1+jT\omega}$$

and let $x(t)$ be a white noise with PSD

$$S_x(\omega) = A.$$

Then $S_y(\omega) = S_x(\omega) |H(\omega)|^2$

$$= \frac{A}{1+T^2\omega^2}$$

and the second moment of the output is

$$\begin{aligned} E[y^2(t)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{1+T^2\omega^2} d\omega \\ &= \lim_{B \rightarrow \infty} \frac{A}{\pi T} \arctan(B) \\ &= \frac{A}{2T} \end{aligned}$$

DISCUSSION: True white noise is not physically realizable, because $R_x(t) = \delta(t)$

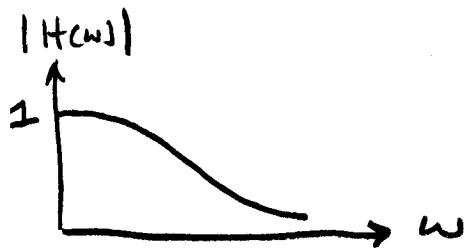
$$\Rightarrow R_x(0) \rightarrow \infty$$

$$\Rightarrow \text{Var}(X(t)) \rightarrow \infty \text{ for all } t.$$

What the last two examples show is that if $\omega_c T$ is large, so that the noise bandwidth is much larger than the filter passband, there is little error in modelling the bandlimited noise as white.

- Noise Equivalent Bandwidth:

→ Let H be a filter with frequency response $H(\omega)$ normalized to have a peak value of 1:

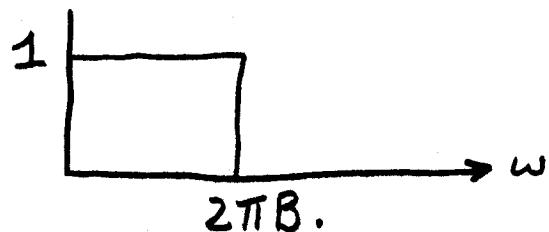


→ Let the input be a white noise with "amplitude" A so that $S_x(\omega) = A$.

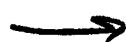
→ Then the second moment of the output is

$$E[y^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A H(s) H(-s) ds.$$

→ Now consider an ideal low-pass filter with frequency response



where the input is $X(t)$ from above and the output is $\tilde{Y}(t)$.



→ Question: What value of B will make
 $E[y^2(t)] = E[\tilde{y}^2(t)]$?

→ Answer:

$$E[\tilde{y}^2(t)] = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} A \, dw$$

$$= 2AB = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} A H(s) H(-s) \, ds$$

$$B = \frac{1}{4\pi j} \int_{-j\infty}^{j\infty} H(s) H(-s) \, ds. \quad (\text{Hertz})$$

⇒ B is called the "noise equivalent bandwidth" of the filter H .

Note: $\frac{1}{2\pi j} \int_{-\infty}^{\infty} H(s) H(-s) \, ds$ is like a inverse Laplace transform evaluated at $t=0$.

- "Coloring" or "Shaping" Filter:

→ The need arises frequently to generate a correlated noise process by passing a white noise through a linear filter.

EX: You have a filter, and you want to find the output in response to a correlated stochastic process with a specific PSD.

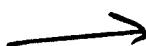
You have a uniform congruential random deviate generator available, and this can simulate a uniform white noise.

You want to design a filter H that will input the white noise and output a noise having your required correlation structure.

→ Suppose the input white noise $X(t)$ has a unity power spectrum:

$$S_x(\omega) = 1.$$

→ Suppose $S_y(\omega)$ is the desired PSD.



→ To design $H(s)$, observe that

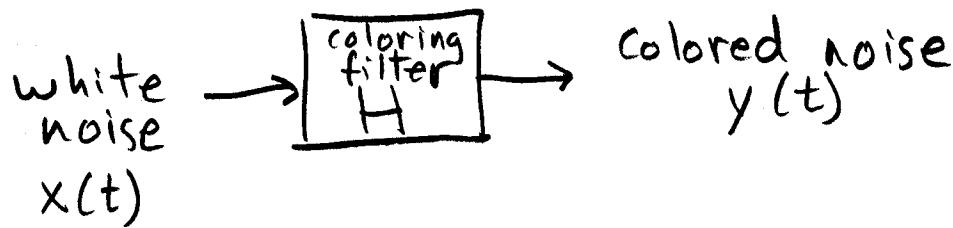
$$S_y(\omega) = S_x(\omega) |H(\omega)|^2 = 1 \cdot |H(\omega)|^2.$$

Assuming that the desired transfer function $H(s)$ is rational and minimum phase, we have

$$S_y(s) = H(s) H(-s),$$

which can be solved for $H(s)$.

Then:



where $y(t)$ will have the desired correlation structure.

EX: We want a filter that will input a stationary, unity-variance white noise and output a colored noise with PSD

$$S_y(\omega) = \frac{\omega^2 + 1}{\omega^4 + 64}$$

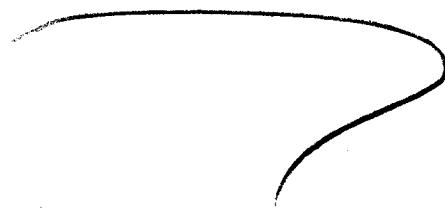
→ with $s = j\omega$,

$$S_y(s) = \frac{-s^2 + 1}{s^4 + 64}$$

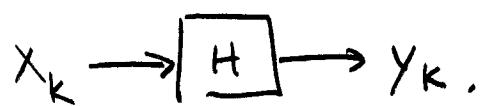
$$= \underbrace{\frac{s+1}{s^2+4s+8}}_{H(s)} \cdot \underbrace{\frac{1-s}{s^2-4s+8}}_{H(-s)}$$

⇒ The required transfer function is

$$H(s) = \frac{s+1}{s^2+4s+8}$$



Discrete Time Steady State Analysis



- Let H be LSI and BIBO stable with impulse response h_k , frequency response $H(e^{j\omega})$, and transfer function $H(z)$.
- First moment of the output:

$$\begin{aligned} E[y_k] &= E\left[\sum_{i=-\infty}^{\infty} x_i h_{k-i}\right] = E\left[\sum_{i=-\infty}^{\infty} x_{k-i} h_i\right] \\ &= \sum_i h_{k-i} E[x_i] = \sum_i h_i E[x_{k-i}] \end{aligned}$$

→ If x_k is WSS,

$$\begin{aligned} E[y_k] &= E[x_k] \sum_i h_i \\ &= E[x_k] H(0). \end{aligned}$$

$\underbrace{\hspace{2cm}}$
a number

- Second Moment of the output:

$$\begin{aligned} E[y_k^2] &= E\left\{\left[\sum_i x_i h_{k-i}\right]\left[\sum_j x_j h_{k-j}\right]\right\} \\ &= E \sum_i \sum_j x_i x_j h_{k-i} h_{k-j}, \\ &= \sum_i \sum_j E[x_i x_j] h_{k-i} h_{k-j} \\ &= \sum_i \sum_j h_{k-i} h_{k-j} R_x(i, j) \end{aligned}$$

→ If X_k is WSS then $R_x(i, j) = R_x(j-i)$, so

$$\begin{aligned} E[y_k^2] &= E \sum_i \sum_j x_{k-i} x_{k-j} h_i h_j \\ &= \sum_i \sum_j E[x_{k-i} x_{k-j}] h_i h_j \\ &= \sum_i \sum_j R_x(k-i, k-j) h_i h_j \\ &= \sum_i \sum_j R_x(i-j) h_i h_j \\ &= \sum_{i,j} R_x(j-i) h_i h_j. \quad \left(\begin{array}{l} R_x(k) \text{ is} \\ \text{even} \end{array} \right) \end{aligned}$$

- Output Auto correlation:

$$\begin{aligned} R_y(k, l) &= E[y_k y_l] \\ &= E\left\{\left[\sum_i h_i x_{k-i}\right]\left[\sum_j h_j x_{l-j}\right]\right\} \\ &= \sum_{i,j} h_i h_j E[x_{k-i} x_{l-j}] \\ &= \sum_{i,j} h_i h_j R_x(k-i, l-j). \end{aligned}$$

→ If X_k is WSS, then

$$R_x(k, l) = R_x(l-k) = R_x(m),$$

where $m = l-k$.

$$\text{So } R_y(m) = \sum_{i,j} h_i h_j R_x(l-k+i-j)$$

$$= \sum_{i,j} h_i h_j R_x(m+i-j).$$

- Input - Output Cross Correlation:

$$\begin{aligned} R_{x,y}(k,l) &= E\{x_k y_l\} \\ &= E\left[x_k \sum_i x_{l-i} h_i\right] \\ &= \sum_i E[x_k x_{l-i}] h_i \\ &= \sum_i h_i R_x(k, l-i), \end{aligned}$$

→ If x_k is WSS,

$$\begin{aligned} R_{x,y}(k,l) &= \sum_i h_i R_x(l-k-i) \\ (m=l-k) \quad R_{x,y}(m) &= \sum_i h_i R_x(m+i) \\ &= R_x(m) * h_m. \end{aligned}$$

$$\begin{aligned} R_{yx}(m) &= R_{xy}(-m) = \sum_i R_x(-m-i) h_i \\ &= \sum_i R_x(m+i) h_i \quad (R_x \text{ is even}) \\ (k=-i) \quad &= \sum_k R_x(m-k) h_{-k} \end{aligned}$$

$$= R_x(m) * h_{-m}.$$

- Output Power Spectrum for WSS input:

$$S_y(z) = \mathcal{Z}\{R_y(m)\}$$

$$= \mathcal{Z}\left\{\sum_{i,j} h_i h_j R_x(m+i-j)\right\} \quad \begin{matrix} \text{last result} \\ \text{on PAGE} \\ 3.22 \end{matrix}$$

$$= \sum_m \sum_{i,j} h_i h_j R_x(m+i-j) z^{-m}$$

$$= \sum_i h_i \sum_m \left[\sum_j h_j R_x(m+i-j) \right] z^{-m}$$

$$\text{Let } k = m+i : \quad \sum_i h_i \sum_k \left[\sum_j R_x(k-j) h_j \right] z^{-(k-i)}$$

$$= \sum_i h_i z^i \sum_k \{R_x(k) * h_k\} z^{-k}$$

$$= H(\frac{1}{z}) H(z) S_x(z)$$

\Rightarrow This is the discrete-time
Wiener-Kintchine relation.

- Input/Output Cross Power Spectrum for WSS input:

$$S_{xy}(z) = \mathbb{E}\{R_{xy}(m)\} = \mathbb{E}\{R_x(m) * h(m)\}$$
$$= H(z)S_x(z)$$

$$S_{yx}(z) = \mathbb{E}\{R_{yx}(m)\} = \mathbb{E}\{R_x(m) * h(-m)\}$$
$$= H(\frac{1}{z})S_x(z)$$

- First Moment of Output for WSS input:

$$E[x_k] = E\left[\sum_i h_i x_{k-i}\right]$$

$$= \sum_i h_i \underbrace{E[x_{k-i}]}_{\text{a number}}$$

$$= E[x_k] \sum_i h_i$$

$$= E[x_k] \sum_i h_i e^{-j\omega n} \Big|_{\omega=0}$$

$$= E[x_k] H(0)$$

Continuous-Time Nonstationary Analysis

- As before,

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

$x(t)$: stochastic input

$y(t)$: stochastic output

H : causal deterministic LSI system.

$H(\omega)$: Frequency response

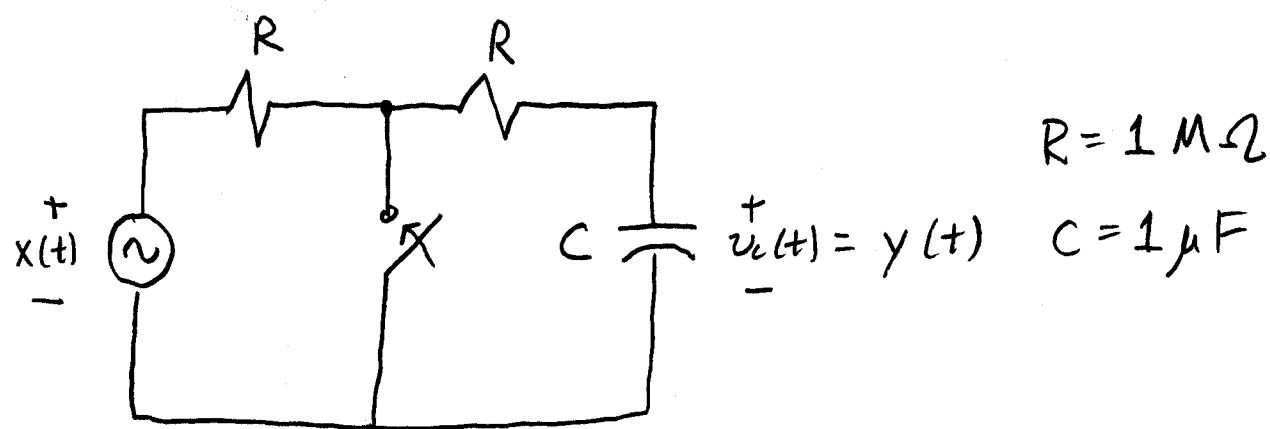
$H(s)$: Transfer function

$h(t)$: Impulse response

- We now assume that the input begins at time $t=0$.

→ Any nontrivial system response at $t=0$ due to excitation prior to $t=0$ is considered to be a nonzero initial condition on $y(t)$.

Note: It is generally better to handle this situation using state space methods, but the book looks at it from the classical viewpoint in Chapter 3.



$x(t)$: WSS Gaussian white noise with unit magnitude PSD.

→ At $t=0$, the switch has been open for a long time and the circuit is at steady state.

→ The switch closes at $t=0$.

Step 1: Characterize the initial condition:

⇒ Since $x(t)$ is Gaussian, so is the initial condition $y(0^-)$.

⇒ Since the capacitor blocks DC, $y(0^-)$ is a zero-mean normal RV.

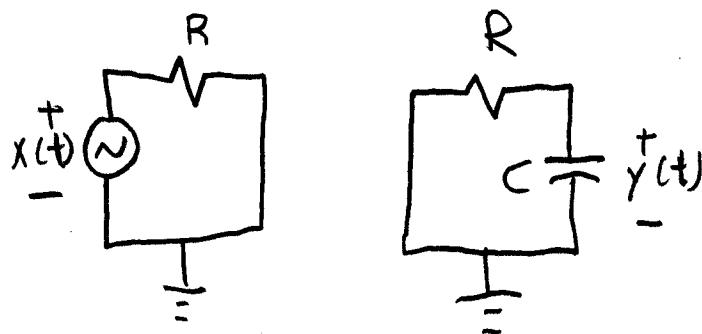
Transfer Function: $H(s) = \frac{1}{1+2s}$

$$\begin{aligned}
E[u_c^2(t)]_{t=0^-} &= \text{Var}[y(t)]_{t=0^-} = R_y(0) \\
&= \mathcal{L}^{-1}\{S_y(s)\}|_{t=0} \\
&= \mathcal{L}^{-1}\{S_x(s)H(s)H(-s)\}|_{t=0} \\
&= \mathcal{L}^{-1}\{1 \cdot H(s)H(-s)\}|_{t=0} \\
&= \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{1}{1+2s} \frac{1}{1-2s} ds \\
&= \frac{1}{2\pi j} \int_{j\infty}^{j\infty} \frac{1}{1-4s^2} e^{st} ds |_{t=0} \\
&= \frac{1}{4}.
\end{aligned}$$

→ So $y(0^-)$ is an $N(0, \frac{1}{4})$ RV.

Step 2: Characterize the Transient Response:

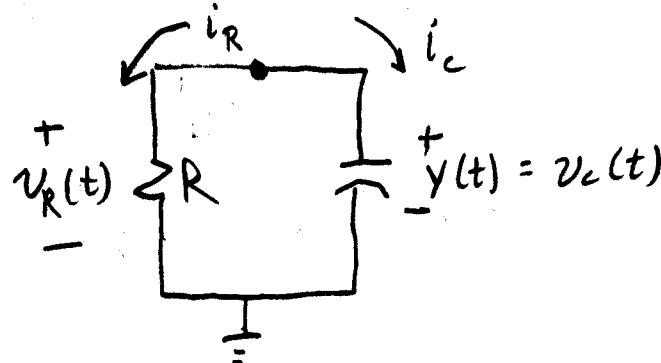
- When the switch is closed, the node between the resistors is grounded:



- From $t=0$ onward, the response $y(t)$ is due to the initial condition alone, and is not affected by the Gaussian voltage source.

\Rightarrow This is called the "unforced" transient response.

\rightarrow Redrawing and writing KCL at the top node:



$$i_C(t) = C v_C(t)$$

$$i_R(t) = \frac{1}{R} v_R(t) = \frac{1}{R} v_C(t)$$

$$\text{KCL: } i_C(t) + i_R(t) = 0$$

$$C v_C(t) + \frac{1}{R} v_C(t) = 0$$

$$C [s V_C(s) - v_C(0^-)] + \frac{1}{R} V_C(s) = 0$$

$$V_C(s) = \frac{v_C(0^-)}{s + \frac{1}{RC}}$$



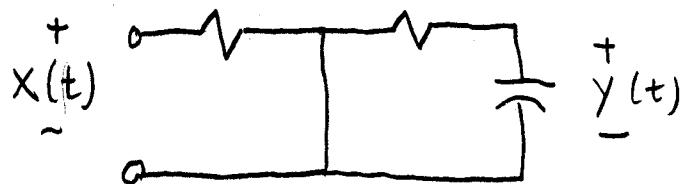
$$y(t) = v_c(t) = v_c(0^-) e^{-t/RC}$$

- The second moment of the response is then given by

$$\begin{aligned} E[y^2(t)] &= R_y(0) \\ &= E[v_c^2(0^-) e^{-2t/RC}] \\ &= E[v_c^2(0^-)] e^{-2t/RC} \\ &= \frac{1}{4} e^{-2t/RC} \end{aligned}$$

NOTE : the second moment is a function of t. That is, the unforced transient response is not WSS.

NOTE : What we have just done is analyze the transient response of the LSI system



due to the initial condition at $t=0$ alone, with the input $x(t) = 0$.

\Rightarrow Unforced transient response.

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Forced transient Response

- Now we analyze the transient response of an LSI system due to the input alone, with zero initial condition (initial relaxation).
- This is called the "Forced" transient response.
- Once again, we assume that the input starts at $t=0$.

Note: we are still assuming that the system is causal.

$\xrightarrow{\hspace{1cm}}$ "t" because H is causal.

$$\begin{aligned} y(t) &= \int_0^t x(\theta) h(t-\theta) d\theta \\ &= \int_0^t x(t-\theta) h(\theta) d\theta \end{aligned}$$

$$\begin{aligned} R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\ &= E\left[\int_0^{t_1} x(t_1-\theta) h(\theta) d\theta \int_0^{t_2} x(t_2-\lambda) h(\lambda) d\lambda\right] \\ &= \int_0^{t_1} \int_0^{t_2} h(\theta) h(\lambda) E[x(t_1-\theta)x(t_2-\lambda)] d\theta d\lambda \\ &= \int_0^{t_1} \int_0^{t_2} h(\theta) h(\lambda) R_x(t_1-\theta, t_2-\lambda) d\theta d\lambda \end{aligned}$$

→ If $x(t)$ is WSS, then

$$R_y(\tau) = \int_0^t \int_0^t h(\theta) h(\lambda) R_x(\tau + \theta - \lambda) d\theta d\lambda$$

- In the WSS case, the second moment of the output is given by

$$\begin{aligned} E[y^2(t)] &= R_y(0) \\ &= \int_0^t \int_0^t h(\theta) h(\lambda) R_x(\theta - \lambda) d\theta d\lambda \end{aligned}$$

Note: The above are just special cases of the stationary analysis we did before, under the assumption that H is causal so that $h(t) = 0 \forall t < 0$.

⇒ In general, the total response of the system is the sum of the response due to the initial condition and the response due to the input.

e.g.) Total Response = Unforced Response
+ Forced Response.

Relationship Between Response of a Discrete-Time System to Stochastic Inputs and ARMA Models

- For a continuous-time LSI system with rational transfer function, the input/output relationship is a differential equation:

$$\sum_{k=0}^n a_k \frac{d^k}{dt^k} y(t) = \sum_{l=0}^m b_l \frac{d^l}{dt^l} x(t)$$

→ Without loss of generality, we assume that $a_n = 1$.

- The transfer function is

$$H(s) = \frac{\sum_{l=0}^m b_l s^l}{\sum_{k=0}^n a_k s^k}$$

Note! For $H(s)$ to be a proper rational function, we require that $m < n$.

- Likewise, for a discrete-time LSI system with rational transfer function, the input/output relationship is a difference equation:

$$\sum_{p=0}^n a_p y_{k+p} = \sum_{l=0}^m b_l x_{k+l}$$

→ Without loss of generality, we assume that $a_n = 1$.

→ For $H(z)$ to be a proper rational function, we require that $m < n$.

- If x_k is a stochastic process with rational PSD, then y_k is also a stochastic process with rational PSD.

- The I/O relation above specifies an ARMA model for the output process y_k .

→ The sum on the left of the " $=$ " mark is the AR part of the model.

→ The sum on the right of the " $=$ " mark is the MA part of the model.