

MODULE 8

SMOOTHING

Chapter 8:

Smoothing

- A smoother estimates the state vector at time t using measurements made before and after time t .

- Recall the discrete Kalman filter:

- process model:

$$X_{k+1} = \Phi_k X_k + W_k \quad (5.5.1)$$

$$Z_k = H_k X_k + V_k \quad (5.5.2)$$

- initialize: \hat{X}_0^- & P_0^-

- iterate:

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1} \quad (5.5.17)$$

$$\hat{X}_k = \hat{X}_k^- + K_k (Z_k - H_k \hat{X}_k^-) \quad (5.5.8) \quad (\text{Filter})$$

$$P_k = (I - K_k H_k) P_k^- \quad (5.5.22)$$

$$\hat{X}_{k+1}^- = \Phi_k \hat{X}_k \quad (5.5.23) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} (\text{predict})$$

$$\hat{P}_{k+1}^- = \Phi_k P_k \Phi_k^T + Q_k \quad (5.5.25)$$

- there are three main types of smoothing techniques:

1. Fixed-interval Smoothing:

Given measurements in a fixed time interval, we seek to estimate the state vector at all times in the same interval using all the measurements over the interval.

Algorithm:

- Given $(N+1)$ measurements in a fixed interval.
- the algorithm consists of a forward recursive pass followed by a backward pass.
- the forward pass uses Kalman filter, and saves the intermediate results \hat{x}_k^- , \hat{x}_k , P_k^- , P_k .
- the backward pass starts at time t_N of the last measurement, computing the smoothed state estimate using the results obtained by the forward pass.
- initial conditions

- recursive equations: (backward sweep)

$$\hat{x}(k|N) = \hat{x}(k|k) + A(k) [\hat{x}(k+1|N) - \hat{x}(k+1|k)] \quad (8.2.1)$$

where $A(k)$ is the smoothing gain

$$A(k) = P(k|k) \Phi^T(k+1, k) P^{-1}(k+1|k) \quad (8.2.2)$$

where $k = N-1, N-2, \dots, 0$

for $k = N-1$ (1st smoothed estimate)

$$\hat{x}(N-1|N) = \hat{x}(N-1|N-1) + A(N-1) [\hat{x}(N|N) - \hat{x}(N|N-1)]$$

$$A(N-1) = P(N-1|N-1) \Phi^T(N, N-1) P^{-1}(N|N-1)$$

for $k = N-2$

$$\hat{x}(N-2|N) = \hat{x}(N-2|N-2) + A(N-2) [\hat{x}(N-1|N) - \hat{x}(N-1|N-2)]$$

$$\& A(N-2) = P(N-2|N-2) \Phi^T(N-1, N-2) P^{-1}(N-1|N-2)$$

↳ so on ...

- the error covariance matrix for the smoothed estimates is

$$P(k|N) = P(k|k) + A(k) [P(k+1|N) - P(k+1|k)] A^T(k) \quad (8.2.3)$$

and it is not needed for the backward pass.

⊗ - which estimator would provide higher accuracy, the smoother or filter?

2. Fixed-point Smoothing:

- Given the measurements up to the current time, we seek to estimate the state vector at a fixed time in the past.

Algorithm:

- Kalman filter is used to estimate the state at the current time
- two equations are used to obtain a smoothed estimate at a fixed time in the past.

the two equations:

$$\hat{x}(k|j) = \hat{x}(k|j-1) + B(j) [\hat{x}(j|j) - \hat{x}(j|j-1)]$$

k is fixed

$$j = k+1, k+2, \dots$$

$$\Phi B(j) = \prod_{i=k}^{j-1} A(i) \quad ; \quad A(i) = P(i|i) \Phi^T(i+1|i) P^{-1}(i+1|i)$$

* for $j = k+1$,

$$\hat{x}(k|k+1) = \hat{x}(k|k) + B(k+1) [\hat{x}(k+1|k+1) - \hat{x}(k+1|k)]$$

$$B(k+1) = A(k) = P(k|k) \Phi^T(k+1|k) P^{-1}(k+1|k)$$

- $\hat{x}(k|k)$, $\hat{x}(k+1|k+1)$, $\hat{x}(k+1|k)$, $P(k|k)$, $P(k+1|k)$, are computed by

* for $j = k+2$

$$\hat{x}(k|k+2) = \hat{x}(k|k+1) + B(k+2) [\hat{x}(k+2|k+2) - \hat{x}(k+2|k+1)]$$

$$B(k+2) = B(k+1) \cdot P(k+1|k+1) \Phi^T(k+2|k+1) P^{-1}(k+2|k+1)$$

& so on.....

Fixed-Lag Smoothing:

- we seek to estimate the state vector at a fixed-time interval lagging the time of the current measurement.

- this type of smoothing trades off estimate latency for more -----

- Fixed-interval smoothing algorithm could be used to perform fixed-lag smoothing, when the number of backward steps equal to the time lag.
- this is fine as long as the number of backward steps is small, what about large number of backward steps!
- Fixed-lag smoothing algorithm exhibits startup problem.