

MODULE 9

LINEARIZATION & ADDITIONAL TOPICS

9: ADDITIONAL TOPICS

TOPIC 1: LINEARIZATION

- Formulation of the Kalman filter depended on linearity of both the system dynamical equation and the measurement equation:

$$X_{k+1} = \Phi_k X_k + w_k \quad (5.5.1)$$

$$Z_k = H_k X_k + v_k \quad (5.5.2)$$

- In many practical situations, one or both of these equations may be nonlinear.
- Modified formulations of the Kalman filter can often still be obtained by linearizing the nonlinear equations.
- For a state vector with n elements, the values of the state vector X_k plotted as k increases trace out a curve in \mathbb{R}^n . This curve is called a "trajectory".
- Linearizing the system dynamical equation and/or measurement equation about a nominal trajectory X_k^* that does not depend on the observations leads to a modified formulation called the "linearized Kalman filter".

- Alternatively, the dynamical and measurement equations may be linearized about a trajectory x_k^* that is updated by the actual observations z_k as the filter runs.

→ This leads to a modified formulation called the "extended Kalman filter".

Linearized Kalman Filter

- The dynamical and measurement equations are assumed to be known and of the form

$$\dot{x} = f(x, u_d, t) + u(t) \quad (9.1.1)$$

$$z = h(x, t) + v(t) \quad (9.1.2)$$

where u_d is a deterministic forcing function.

- We also assume that a nominal or "expected"¹⁾ state vector trajectory $x^*(t)$ has been obtained.

- The true trajectory of the state vector is

$$x(t) = x^*(t) + \Delta x(t). \quad (9.1.3)$$

- Plugging the true trajectory into the nonlinear system equations gives

$$\dot{x}^* + \Delta \dot{x} = f(x^* + \Delta x, u_d, t) + u(t) \quad (9.1.4)$$

$$z = h(x^* + \Delta x, t) + v(t) \quad (9.1.5)$$

- Assuming Δx is small compared to x^* , we expand $f(\cdot)$ and $h(\cdot)$ in Taylor series about the nominal trajectory x^* and discard all terms of order 2 and greater:

$$f(x^* + \Delta x, u_d, t) \approx f(x^*, u_d, t) + \left. \frac{\partial f}{\partial x} \right|_{x^*} \Delta x$$

$$h(x^* + \Delta x, t) \approx h(x^*, t) + \left. \frac{\partial h}{\partial x} \right|_{x^*} \Delta x,$$

where, e.g.,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \ddots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (9.1.8)$$

- Plugging the Taylor series into the system equations, we get

$$\dot{x}^* + \Delta \dot{x} \approx f(x^*, u_d, t) + \Delta x \left(\frac{\partial f}{\partial x} \Big|_{x^*} \right) + u(t) \quad (9.1.6)$$

$$z \approx h(x^*, t) + \Delta x \left(\frac{\partial h}{\partial x} \Big|_{x^*} \right) + v(t) \quad (9.1.7)$$

\Rightarrow The nominal trajectory x^* must be chosen so that it is a solution of the (generally) nonlinear differential equation

$$\dot{x}^* = f(x^*, u_d, t) \quad (9.1.9)$$

- Subtracting (9.1.9) from both sides of the dynamical equation (9.1.6), we have

$$\Delta \dot{x} = \left(\frac{\partial f}{\partial x} \Big|_{x^*} \right) \Delta x + u(t) \quad (9.1.10)$$

- Subtracting the known quantity $h(x^*, t)$ from both sides of the measurement equation (9.1.7), we get

$$[z - h(x^*, t)] = \left(\frac{\partial h}{\partial x} \Big|_{x^*} \right) \Delta x + v(t) \quad (9.1.11)$$

- Thus, we have a linearized model that fits into the "usual" Kalman filtering framework.
 - The linearized state vector is Δx ,
 - The observation input to the "modified" kalman filter is $z - h(x^*, t)$.
- The system equations are

$$\Delta \dot{x} = F \Delta x + u(t)$$

$$z = H \Delta x + v(t)$$

where

$$F = \left. \frac{\partial f}{\partial x} \right|_{x^*}$$

$$H = \left. \frac{\partial h}{\partial x} \right|_{x^*}$$

- A "linearized" kalman filter can now be developed for the modified system using the techniques of Chapter 5.

NOTE: To form the observations for the linearized Kalman filter, we must subtract from the true observations the "expected" observations that would be obtained from the nominal state vector trajectory if there was no measurement noise.

- An example of this with linear "constant velocity" state dynamics and a nonlinear measurement equation is given in Ex. 9.1 on pages 337-340 of the book.
- Example 9.2 on pages 340-343 of the book works through a situation where both the system dynamical equation and the measurement equation are nonlinear.

EXTENDED KALMAN FILTER

- For the linearized Kalman filter, we "linearized" the dynamical and measurement equations using Taylor series about a nominal state trajectory x^* that did not depend on the observations.
- For the extended Kalman filter, the nominal trajectory x^* is taken equal to the Kalman filter's estimated state trajectory, and therefore depends on the observations.
⇒ Clearly, the Kalman gains cannot be precomputed in this case.

- The expected value of the performance of the extended Kalman filter (averaged over an ensemble) is always equal to or better than the expected performance of the linearized Kalman Filter.
- In any single run, however, the linearized Kalman filter may perform better.
- Also, the extended Kalman filter is far more susceptible to divergence problems.
- The dynamical and measurement equations for the extended Kalman filter are the same as those for the linearized filter (drop "t" dependency to save notation and sample to convert to discrete time):

$$\Delta x_{k+1} = \left(\frac{\partial f}{\partial x} |_{x^*} \right) \Delta x_k + u_k$$

$$z_k - h(x_k^*) = \left(\frac{\partial h}{\partial x} |_{x^*} \right) \Delta x_k + v_k$$

where the nominal trajectory x^* is the total trajectory \hat{x}_k estimated by the Kalman filter.

- For the "usual" Kalman filter, as formulated in Chapter 5, the state variables are usually "total" quantities such as position, velocity, acceleration, etc.
- In the linearized Kalman filter, the state variables were "incremental" quantities ΔX , where $X = X^* + \Delta X$.
- In contrast to the linearized Kalman filter, the extended Kalman filter (EKF) is usually implemented in terms of total quantities.
- Discretizing the linearized measurement equation (9.1.11) from PAGE 9.4, we have

$$z_k - h(x_k^*) = H_k \Delta X_k + v_k \quad (9.1.31)$$

- The estimated (incremental) state vector update equation is

$$\Delta \hat{X}_k = \Delta \hat{X}_k^- + K_k [z_k - h(x_k^*) - H_k \Delta \hat{X}_k^-] \quad (9.1.32)$$



- But, from PAGE 9.3. (Taylor series about x^*), we have

$$h(x_k^* + \Delta x_k) \approx h(x_k^*) + \left(\frac{\partial h}{\partial x}\Big|_{x_k^*}\right) \Delta x_k \\ = h(x_k^*) + H_k \Delta x_k \quad (*)$$

\Rightarrow Since $E[v_k] = 0$, (from 9.1.5 on p. 9.3)

$$E[z_k] = \underbrace{h(x_k^* + \Delta x_k)}_{}$$

expectation of "total" observation.

\Rightarrow Comparing this to (*) above, we see that the Kalman filter's predicted "total" observation is

$$\hat{z}_k^- = h(x_k^*) + H_k \Delta \hat{x}_k^- \quad (**)$$

- Plugging (**) into (9.1.32) on PAGE 9.8, we get

$$\Delta \hat{x}_k = \Delta \hat{x}_k^- + K_k [z_k - \hat{z}_k^-] \quad (9.1.34)$$

- The incremental quantities $\Delta \hat{x}_k$ and $\Delta \hat{x}_k^-$ can be removed by adding x_k^* to each side of (9.1.34) on PAGE 9.9 and substituting the total quantities:

$$\hat{x}_k = x_k^* + \Delta \hat{x}_k \quad (*)$$

$$\hat{x}_k^- = x_k^* + \Delta \hat{x}_k^-$$

- We get:

$$\hat{x}_k = \hat{x}_k^- + k_k (z_k - \hat{z}_k^-) \quad (9.1.35)$$

NOTE : Since the EKF takes $x_k^* = \hat{x}_k$ by definition, $\Delta \hat{x}_k$ in (*) above is always equal to zero.

- To find \hat{x}_{k+1}^- , we must solve the (generally) nonlinear differential equation

$$\dot{x} = f(x, u_k, t)$$

subject to the initial condition $x = \hat{x}_k$ at $t = t_k$ and evaluate the solution at $t = t_{k+1}$.

(This is usually the hardest part of the EKF).

- Once \hat{x}_{k+1}^- is found, the predicted measurement is given by

$$\hat{z}_{k+1}^- = h(\hat{x}_{k+1}^-)$$
- When observation z_{k+1} arrives, the measurement residual $z_{k+1} - \hat{z}_{k+1}^-$ needed in the state vector update equation (9.1.35) can be formed.
- The filtered and predicted state vector error covariance update equations are unchanged from the "usual" Kalman filter as developed in Chapter 5:

$$P_k^- = (I - K_k H_k) P_k^- \quad (9.1.37)$$

$$P_{k+1}^- = \Phi_k P_k^- \Phi_k^T + Q_k \quad (9.1.38)$$

$\Rightarrow \Phi_k$, H_k , and Q_k come from the linearized model.

NOTE: if \hat{x}_0^- is poor, then there may be large errors in the first-order Taylor series approximation used to linearize the system. This generally causes the EKF to diverge.

TOPIC 2: Correlated Process and Measurement Noises

- Suppose that the process noise w and measurement noise v are correlated.
- This requires modifying the Kalman filtering equations as follows.
- As in Chapter 5, the system model is

$$x_{k+1} = \phi_k x_k + w_k \quad (9.2.1)$$

$$z_k = h_k x_k + v_k \quad (9.2.2)$$

$$E[w_k w_i^T] = Q_k \delta_{k-i} \quad (9.2.3)$$

$$E[v_k v_i^T] = R_k \delta_{k-i} \quad (9.2.4)$$

- We now assume nonzero correlation between the process and measurement noises.

NOTE: w_k does not contribute to z_k in (9.2.2).

Rather, it is w_{k-1} that contributes to z_k through

$$x_k = \phi_{k-1} x_{k-1} + w_{k-1}, \quad (9.2.6)$$

so that

$$z_k = h_k [\phi_{k-1} x_{k-1} + w_{k-1}] + v_k$$

- Therefore, we assume that

$$E[w_{k-1} v_k^T] = C_k \quad (9.2.5)$$

- As in Chapter 5, the general state vector update equation is

$$\hat{x}_k = \hat{x}_{k-} + K_k (z_k - H_k \hat{x}_{k-}) \quad (9.2.7)$$

- The error in the filtered state vector estimate is

$$\begin{aligned} e_k &= x_k - \hat{x}_k \\ &= x_k - [\hat{x}_{k-} + K_k (z_k - H_k \hat{x}_{k-})] \\ &= x_k - \hat{x}_{k-} - K_k z_k + K_k H_k \hat{x}_{k-} \\ &\quad \underbrace{\qquad\qquad\qquad z_k = H_k x_k + v_k} \\ &= x_k - \hat{x}_{k-} + K_k H_k \hat{x}_{k-} - K_k (H_k x_k + v_k) \\ &= x_k - \hat{x}_{k-} + K_k H_k \hat{x}_{k-} - K_k H_k x_k - K_k v_k \\ &= (x_k - \hat{x}_{k-}) - K_k H_k (x_k - \hat{x}_{k-}) - K_k v_k \\ &= (I - K_k H_k) (x_k - \hat{x}_{k-}) - K_k v_k \\ &= (I - K_k H_k) e_{k-} - K_k v_k \quad (9.2.8) \end{aligned}$$

- Since e_k^- depends on \hat{x}_{k-1}^- , which depends on w_{k-1} , we expect nontrivial correlation between e_k^- and v_k :

$$\begin{aligned}
 E[e_k^- v_k^T] &= E[(x_k - \hat{x}_k^-) v_k^T] \\
 &= E[(\phi_{k-1} x_{k-1} + w_{k-1} - \phi_{k-1} \hat{x}_{k-1}^-) v_k^T] \quad (9.2.9) \\
 &= E[\phi_{k-1} x_{k-1} v_k^T + w_{k-1} v_k^T - \phi_{k-1} \hat{x}_{k-1}^- v_k^T] \\
 &= \underbrace{\phi_{k-1} E[x_{k-1} v_k^T]}_{\text{zero because } v_k \text{ is white}} + E[w_{k-1} v_k^T] - \underbrace{\phi_{k-1} E[\hat{x}_{k-1}^- v_k^T]}_{\text{zero because } v_k \text{ is white}} \\
 &= E[w_{k-1} v_k^T] \\
 &= C_k. \quad (9.2.10)
 \end{aligned}$$

- Using (9.2.8) on page 9.13, we have for the filtered state vector error covariance matrix

$$\begin{aligned}
 P_k &= E[e_k^- e_k^{T-}] \\
 &= E\left\{ [(I - K_k H_k) e_k^- - K_k v_k] [(I - K_k H_k) e_k^- - K_k v_k]^T \right\} \quad (9.2.11) \\
 &= E[(I - K_k H_k) e_k^- (e_k^-)^T (I - K_k H_k)^T - K_k v_k v_k^T K_k^T] \\
 &= (I - K_k H_k) P_k (I - K_k H_k)^T + K_k R_k K_k^T \\
 &\quad - E[(I - K_k H_k) e_k^- v_k^T K_k^T] - E[K_k v_k (e_k^-)^T (I - K_k H_k)^T]
 \end{aligned}$$

- OR,

$$\begin{aligned}
 P_k &= (I - K_k H_k) P_k (I - K_k H_k)^T + K_k R_k K_k^T \\
 &\quad - (I - K_k H_k) C_k K_k^T - K_k C_k^T (I - K_k H_k)^T \quad (9.2.12)
 \end{aligned}$$

- To find the optimal Kalman gain sequence K_k , we differentiate (9.2.12) with respect to K_k and set the result equal to zero.
- This gives the following expression for the optimal gain sequence:

$$K_k = (P_k^- - H_k^T + C_k) \left[H_k P_k^- H_k^T + R_k + H_k C_k + C_k^T H_k^T \right]^{-1} \quad (9.2.13)$$

NOTE : In the limit as $C_k \rightarrow 0$, (9.2.13) goes to the usual Kalman gain formulation developed in chapter 5.

- Plugging (9.2.13) back into (9.2.12) gives us the modified P update equation

$$P_k = P_k^- - K_k \left[H_k P_k^- H_k^T + R_k + H_k C_k + C_k^T H_k^T \right] K_k^T \quad (9.2.14)$$

which has natural symmetry.

- Sacrificing the natural symmetry, this can be further simplified to

$$P_k = (I - K_k H_k) P_k^- - K_k C_k^T \quad (9.2.15)$$

- Because V_k is a white noise, the transformation equations from filter to predictor are not affected by the correlation between w_{k-1} and V_k :

$$\hat{X}_{k+1}^- = \phi_k \hat{X}_k \quad (9.2.16)$$

$$P_{k+1}^- = \phi_k P_k \phi_k^T + Q_k \quad (9.2.17)$$

- The complete set of equations for implementing the modified Kalman filter to account for the correlation between w_{k-1} and u_k are given by;

$$\hat{x}_k = \hat{x}_{k-} + K_k (z_k - H_k \hat{x}_{k-}) \quad (9.2.7)$$

$$K_k = (P_{k-} H_k^T + C_k) [H_k P_{k-} H_k^T + R_k + H_k C_k + C_k^T H_k^T]^{-1} \quad (9.2.13)$$

$k=k+1$

$$P_k = (I - K_k H_k) P_{k-} - K_k C_k^T \quad (9.2.15)$$

$$\hat{x}_{k+1}^- = \Phi_k \hat{x}_k \quad (9.2.16)$$

$$P_{k+1}^- = \Phi_k P_k \Phi_k^T + Q_k \quad (9.2.17)$$

Topic ZB: Delayed-State Kalman filter

- Often, observations are obtained from sensors that integrate the physical quantities of interest.
- In other cases, the physical quantities are intentionally integrated to reduce noise.
- In such situations, we have, e.g.,

Measurement at $t=t_k = \int_{t_{k-1}}^{t_k} (\text{velocity}) dt + \text{discrete noise}$

$$= (\text{position at } t_k) - (\text{position at } t_{k-1}) \\ + (\text{discrete noise})$$

- or, more generally,

$$z_k = H_k x_k + J_k x_{k-1} + v_k \quad (9.2.19)$$

- The Kalman filtering equations must be reformulated to account for the presence of the delayed state vector x_{k-1} in the measurement equation.
- The system dynamical equation for x_k is

$$x_k = \phi_{k-1} x_{k-1} + w_{k-1} \quad (9.2.20)$$

- Solving this for x_{k-1} gives

$$x_{k-1} = \phi_{k-1}^{-1} x_k - \phi_{k-1}^{-1} w_{k-1} \quad (9.2.21)$$

- Plugging this into the measurement equation (9.2.19) above, we get

$$\begin{aligned} z_k &= H_k x_k + J_k (\phi_{k-1}^{-1} x_k - \phi_{k-1}^{-1} w_{k-1}) + v_k \\ &= \underbrace{(H_k + J_k \phi_{k-1}^{-1})}_{\tilde{H}_k} x_k + \underbrace{(-J_k \phi_{k-1}^{-1} w_{k-1} + v_k)}_{\tilde{v}_k} \\ &= \tilde{H}_k x_k + \tilde{v}_k \quad (9.2.22) \end{aligned}$$

- The modified system model

$$X_{k+1} = \Phi_k X_k + w_k$$

$$z_k = \tilde{H}_k X_k + \tilde{v}_k$$

Obviously has correlation between w_{k-1} and \tilde{v}_k , as we treated in the last section.

- Assuming that the original (unmodified) w_{k-1} and v_k were uncorrelated, we have for the covariance of \tilde{v}_k the following:

$$\begin{aligned}\tilde{R}_k &= E[\tilde{v}_k \tilde{v}_k^T] \\ &= E[(-J_k \Phi_{k-1}^{-1} w_{k-1} + v_k)(-J_k \Phi_{k-1}^{-1} w_{k-1} + v_k)^T] \\ &= E[J_k \Phi_{k-1}^{-1} w_{k-1} w_{k-1}^T \Phi_{k-1}^{-1 T} J_k^T + v_k v_k^T + 0 + 0] \\ &= J_k \Phi_{k-1}^{-1} Q_{k-1} \Phi_{k-1}^{-1 T} J_k^T + R_k \quad (9.2.24)\end{aligned}$$

- The crosscorrelation between w_{k-1} and \tilde{v}_k is

$$\begin{aligned}C_k &= E[w_{k-1} \tilde{v}_k^T] \\ &= E[w_{k-1} (-J_k \Phi_{k-1}^{-1} w_{k-1} + v_k)^T] \\ &= E[w_{k-1} w_{k-1}^T] \Phi_{k-1}^{-1 T} (-J_k)^T + \underbrace{E[w_{k-1} v_k^T]}_{\text{zero}} \\ &= -Q_{k-1} \Phi_{k-1}^{-1 T} J_k^T \quad (9.2.25)\end{aligned}$$

- The inverse of the state transition matrix, ϕ_k^{-1} , can be eliminated from the equations used to implement the modified Kalman filter using

$$Q_{k-1} = P_k^- - \phi_{k-1} P_{k-1} \phi_{k-1}^T \quad (\text{from P update eq.})$$

$$\phi_k^{-1 T} = (\phi_k^T)^{-1}$$

- The resulting equations to implement the modified Kalman filter are

$$\hat{z}_k^- = H_k \hat{x}_k^- + J_k \hat{x}_{k-1} \quad (9.2.31)$$

$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - \hat{z}_k^-) \quad (9.2.30)$$

$$K_k = [P_k^- H_k^T + \phi_{k-1} P_{k-1} J_k^T] [H_k P_k^- H_k^T + R_k + J_k P_{k-1} \phi_{k-1}^T H_k^T + H_k \phi_{k-1} P_{k-1} J_k^T + J_k P_{k-1} J_k^T]^{-1} \quad (9.2.32)$$

$$P_k = P_k^- - K_k L_k K_k^T \quad (9.2.33)$$

where

$$L_k = H_k P_k^- H_k^T + R_k + J_k P_{k-1} \phi_{k-1}^T H_k^T + H_k \phi_{k-1} P_{k-1} J_k^T + J_k P_{k-1} J_k^T \quad (9.2.34)$$

- The projection equations are unaffected:

$$\hat{x}_{k+1}^- = \phi_k \hat{x}_k \quad (9.2.35)$$

$$\bar{P}_{k+1} = \phi_k P_k \phi_k^T + Q_k \quad (9.2.36)$$