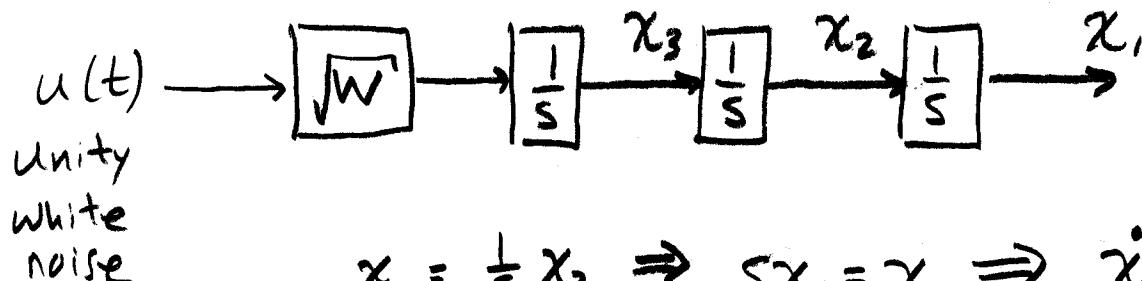


TEST I SOLUTION

① 5.2) Block Diagram:



$$\dot{x}_1 = \frac{1}{s} x_2 \Rightarrow s\dot{x}_1 = x_2 \Rightarrow \dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{s} x_3 \Rightarrow s\dot{x}_2 = x_3 \Rightarrow \dot{x}_2 = x_3$$

$$\dot{x}_3 = \frac{\sqrt{W}}{s} u \Rightarrow s\dot{x}_3 = \sqrt{W} u \Rightarrow \dot{x}_3 = \sqrt{W} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_F \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sqrt{W} \end{bmatrix} u(t)$$

$$\Phi_k = \mathcal{L}^{-1} \left\{ (sI - F)^{-1} \right\} \Big|_{t=\Delta t}$$

$$sI - F = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}$$

$$\text{Q.} \dots \text{ Let } (sI - F)^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$(1) sa_{11} - a_{21} = 1 \quad (4) sa_{12} - a_{22} = 0 \quad (7) sa_{13} - a_{23} = 0$$

$$(2) sa_{21} - a_{31} = 0 \quad (5) sa_{22} - a_{32} = 1 \quad (8) sa_{23} - a_{33} = 0$$

$$(3) sa_{31} = 0 \quad (6) sa_{32} = 0 \quad (9) sa_{33} = 1$$

$$(9) : a_{33} = s^{-1}$$

$$(8) : sa_{23} = s^{-1} \rightarrow a_{23} = s^{-2}$$

$$(7) : sa_{13} = s^{-2} \rightarrow a_{13} = s^{-3}$$

$$(6) : a_{32} = 0$$

$$(5) : sa_{22} = 1 \rightarrow a_{22} = s^{-1}$$

$$(4) : sa_{12} = s^{-1} \rightarrow a_{12} = s^{-2}$$

$$(3) : a_{31} = 0$$

$$(2) : sa_{21} = 0 \rightarrow a_{21} = 0$$

$$(1) : sa_{11} = 1 \rightarrow a_{11} = s^{-1}$$

$$(sI - F)^{-1} = \begin{bmatrix} s^{-1} & s^{-2} & s^{-3} \\ 0 & s^{-1} & s^{-2} \\ 0 & 0 & s^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

①

$$\Phi_k = \mathcal{Z}^{-1} \left\{ \begin{bmatrix} \frac{1}{s^3} & \frac{1}{s^2} & \frac{1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \right\} \Big|_{t=\Delta t}$$

$$= \begin{bmatrix} u(t) & tu(t) & \frac{1}{2}t^2u(t) \\ 0 & u(t) & tu(t) \\ 0 & 0 & u(t) \end{bmatrix} \Big|_{t=\Delta t}$$

" $u(t)$ " here is
unit step function

$$\underline{\underline{\Phi_k = \begin{bmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}}}$$

Q1: use the "Transfer Function Method"

$$G(u \text{ to } x_1) = G_1 = \frac{\sqrt{w}}{s^3}$$

$$g_1(t) = \frac{\sqrt{w}}{2} t^2 u(t) \leftarrow$$

$$G(u \text{ to } x_2) = G_2 = \frac{\sqrt{w}}{s^2}$$

$$g_2(t) = \sqrt{w} t u(t) \leftarrow \text{unit step function}$$

$$G(u \text{ to } x_3) = G_3 = \frac{\sqrt{w}}{s}$$

$$g_3(t) = \sqrt{w} u(t) \leftarrow$$

(D.) "u(t)" here is the unit PSD input noise.

$$\begin{aligned} E[x_1 x_1] &= \int_0^{\Delta t} \int_0^{\Delta t} g_1(\xi) g_1(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\ &= \int_0^{\Delta t} \int_0^{\Delta t} \frac{\sqrt{W}}{2} \xi^2 \frac{\sqrt{W}}{2} \eta^2 \delta(\xi - \eta) d\xi d\eta \\ &= \frac{W}{4} \int_0^{\Delta t} \xi^4 d\xi = \frac{W}{20} (\Delta t)^5 \end{aligned}$$

$$\begin{aligned} E[x_1 x_2] &= \int_0^{\Delta t} \int_0^{\Delta t} g_1(\xi) g_2(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\ &= \int_0^{\Delta t} \int_0^{\Delta t} \frac{\sqrt{W}}{2} \xi^2 \sqrt{W} \eta \delta(\xi - \eta) d\xi d\eta \\ &= \frac{W}{2} \int_0^{\Delta t} \xi^3 d\xi = \frac{W}{8} (\Delta t)^4 \end{aligned}$$

$$\begin{aligned} E[x_1 x_3] &= \int_0^{\Delta t} \int_0^{\Delta t} g_1(\xi) g_3(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\ &= \int_0^{\Delta t} \int_0^{\Delta t} \frac{\sqrt{W}}{2} \xi^2 \sqrt{W} \delta(\xi - \eta) d\xi d\eta \\ &= \frac{W}{2} \int_0^{\Delta t} \xi^2 d\xi = \frac{W}{6} (\Delta t)^3 \end{aligned}$$

$$\begin{aligned}
 \textcircled{1} \dots \\
 E[x_2 x_2] &= \int_0^{\Delta t} \int_0^{\Delta t} g_2(\xi) g_2(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\
 &= \int_0^{\Delta t} \int_0^{\Delta t} \sqrt{W} \xi \sqrt{W} \eta \delta(\xi - \eta) d\xi d\eta \\
 &= W \int_0^{\Delta t} \xi^2 d\xi = \frac{W}{3} (\Delta t)^3 \\
 E[x_2 x_3] &= \int_0^{\Delta t} \int_0^{\Delta t} g_2(\xi) g_3(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\
 &= \int_0^{\Delta t} \int_0^{\Delta t} \sqrt{W} \xi \sqrt{W} \eta \delta(\xi - \eta) d\xi d\eta \\
 &= W \int_0^{\Delta t} \xi d\xi = \frac{W}{2} (\Delta t)^2 \\
 E[x_3 x_3] &= \int_0^{\Delta t} \int_0^{\Delta t} g_3(\xi) g_3(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\
 &= \int_0^{\Delta t} \int_0^{\Delta t} \sqrt{W} \sqrt{W} \delta(\xi - \eta) d\xi d\eta = W \int_0^{\Delta t} d\xi = W \Delta t
 \end{aligned}$$

Collecting results,

$$Q_K = \begin{bmatrix} \frac{W}{20} (\Delta t)^5 & \frac{W}{8} (\Delta t)^4 & \frac{W}{6} (\Delta t)^3 \\ \frac{W}{8} (\Delta t)^4 & \frac{W}{3} (\Delta t)^3 & \frac{W}{2} (\Delta t)^2 \\ \frac{W}{6} (\Delta t)^3 & \frac{W}{2} (\Delta t)^2 & W \Delta t \end{bmatrix}$$

(2) 5.11)

- The random bias is described on p. 196 of the text. The state model is

$$\dot{x} = Ox + ou(t)$$

$$y = x$$

subject to the initial condition $y(0) = a_0$, where a_0 has a known distribution.

- The Wiener process is described on p. 196 and analyzed in Ex. 5.7 (p. 220). The state model is

$$\dot{x} = Ox + ku(t)$$

$$y = x$$

subject to the initial condition $x(0) = 0$, where $u(t)$ is a white Gaussian noise.

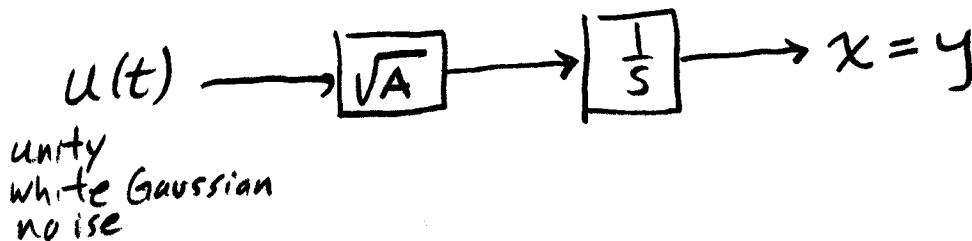
- In this problem, we have a scalar process $y(t)$ that is the sum of a Wiener process and a random bias.

→ Observations z_k are taken at unit time intervals beginning at $t=0$. The measurement noise variance is R .

②...

- The random bias is distributed $N(0, \sigma^2)$.
- The Wiener process has variance $A\Delta t$.

a) Single-state model:



The state equation is

$$\dot{x} = 0x + \sqrt{A} u(t)$$

$$y = x$$

where $x(0)$ is distributed $N(0, \sigma^2)$.

- The "F matrix" is zero.

$$\text{So } S\mathbf{I} - F = S\mathbf{I} = S.$$

$$\text{So } (S\mathbf{I} - F)^{-1} = \frac{1}{S}$$

$$\Phi_k = \mathcal{J}^{-1}\{(S\mathbf{I} - F)^{-1}\}_{t=1} = \mathcal{J}^{-1}\left\{\frac{1}{S}\right\}_{t=1}$$

$$= u(t) \Big|_{t=1} = 1.$$

To find Q_k , we can use the same technique as in problem 5.2 or use (5.3.6) directly (as is done in (5.6.3)). Either way,

②... we obtain

$$Q_k = \int_0^1 \int_0^1 \sqrt{A} \delta(\xi - \eta) \sqrt{A} d\xi d\eta$$

$$= A \int_0^1 d\xi = A.$$

From the state model, it is clear that $H_k = I$, and we are given $R_k = R$.

Since $E[x(0)] = E[y(0)] = 0$, we pick $\hat{x}_0^- = 0$.

Since $x(0)$ is distributed $N(0, \sigma^2)$, this gives us $P_0^- = \sigma^2$.

- We now enter the loop of Fig. 5.8 (p. 219):

$$\begin{aligned} \underline{k=0} \quad K_0 &= P_0^- H_0^T (H_0 P_0^- H_0^T + R_0)^{-1} \\ &= \sigma^2 (\sigma^2 + R)^{-1} = \frac{\sigma^2}{\sigma^2 + R} \end{aligned}$$

$$\begin{aligned} \hat{y}_0 &= \hat{x}_0 = \hat{x}_0^- + K_0 (z_0 - H_0 \hat{x}_0^-) \\ &= 0 + \frac{\sigma^2}{\sigma^2 + R} (z_0 - 0) = \frac{\sigma^2}{\sigma^2 + R} z_0 \end{aligned}$$

$$P_0 = (I - K_0 H_0) P_0^- = \left(1 - \frac{\sigma^2}{\sigma^2 + R}\right) \sigma^2$$

$$= \left(\frac{\sigma^2 + R}{\sigma^2 + R} - \frac{\sigma^2}{\sigma^2 + R}\right) \sigma^2 = \frac{R \sigma^2}{\sigma^2 + R}$$

②...

$$P_i^- = \phi_0 P_0 \phi_0^T + Q_0$$

$$= P_0 + Q_0 = \frac{R\sigma^2}{R+\sigma^2} + A = \frac{R\sigma^2 + A\sigma^2 + AR}{R+\sigma^2}$$

$$\hat{x}_i^- = \phi_0 \hat{x}_0 = \hat{x}_0 = \frac{z_0 \sigma^2}{\sigma^2 + R}$$

K=1

$$K_i = P_i^- H_i^T (H_i P_i^- H_i^T + R_i)^{-1}$$

$$= \frac{R\sigma^2 + A\sigma^2 + AR}{R+\sigma^2} \left(\frac{R\sigma^2 + A\sigma^2 + AR}{R+\sigma^2} + R \right)^{-1}$$

$$= \frac{R\sigma^2 + A\sigma^2 + AR}{R+\sigma^2} \left(\frac{R\sigma^2 + A\sigma^2 + AR + R(R+\sigma^2)}{R+\sigma^2} \right)^{-1}$$

$$= \frac{R\sigma^2 + A\sigma^2 + AR}{R\sigma^2 + A(R+\sigma^2) + R(R+\sigma^2)}$$

$$= \frac{R\sigma^2 + A(\sigma^2 + R)}{R\sigma^2 + (A+R)(R+\sigma^2)}$$

$$\textcircled{2} \quad \hat{y}_1 = \hat{x}_1 = \hat{x}_1^- + K_1 (z_1 - H_1 \hat{x}_1^-)$$

$$= \frac{z_0 \sigma^2}{\sigma^2 + R} + \frac{R \sigma^2 + A(\sigma^2 + R)}{R \sigma^2 + (A+R)(\sigma^2 + R)} \left(z_1 - \frac{z_0 \sigma^2}{\sigma^2 + R} \right)$$

Two-State model:

- State x_1 is random bias
- State x_2 is the Wiener process.
- Combining the state models from p. 6,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{A} \end{bmatrix} u(t)$$

$$y = [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Subject to initial conditions $x_1(0)$ is distributed $\mathcal{N}(0, \sigma^2)$ and $x_2(0) = 0$.

- The "F matrix" is again zero.

$$sI - F = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\textcircled{2} \dots [SI - F]^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

$$\phi_k = \mathcal{F}^{-1} \left\{ (SI - F)^{-1} \right\} \Big|_{t=\Delta t} = \begin{bmatrix} u(t) & 0 \\ 0 & u(t) \end{bmatrix}_{t=1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G(u \text{ to } x_1) = G_1 = 0 \rightarrow g_1(t) = 0$$

$$G(u \text{ to } x_2) = G_2 = \frac{\sqrt{A}}{S} \rightarrow g_2(t) = \sqrt{A} u_q(t)$$

$$\begin{aligned} E[x_1 x_1] &= \int_0^1 \int_0^1 g_1(\xi) g_1(\eta) E[u(\xi) u(\eta)] d\xi d\eta \quad \text{unit step fcn} \\ &= \int_0^1 \int_0^1 0 \cdot 0 \cdot \delta(\xi - \eta) d\xi d\eta = \int_0^1 0 d\xi = 0 \end{aligned}$$

$$\begin{aligned} E[x_1 x_2] &= \int_0^1 \int_0^1 g_1(\xi) g_2(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\ &= \int_0^1 \int_0^1 0 \cdot \sqrt{A} \delta(\xi - \eta) d\xi d\eta = \int_0^1 0 d\xi = 0 \end{aligned}$$

$$\begin{aligned} E[x_2 x_2] &= \int_0^1 \int_0^1 g_2(\xi) g_2(\eta) E[u(\xi) u(\eta)] d\xi d\eta \\ &= \int_0^1 \int_0^1 \sqrt{A} \sqrt{A} \delta(\xi - \eta) d\xi d\eta = A \int_0^1 d\xi = A \end{aligned}$$

$$Q_k = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$$

②.. We have still that $R_k = R$.

From the two-state model on p. 10, it is clear that $H_k = [1 \ 1]$.

Since $E\hat{x}_0 = E\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we take $\hat{x}_0^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

→ The variance of the error in the first component is σ^2 .

→ There is no error in the second component.

$$P_0^- = E[(x_0 - \hat{x}_0^-)(x_0 - \hat{x}_0^-)^T] = \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \begin{bmatrix} \sigma & 0 \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\underline{K=0}}$$

$$K_0 = P_0^- H_0^T (H_0 P_0^- H_0^T + R_0)^{-1}$$

$$P_0^- H_0^T = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} [1] = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} H_0 P_0^- H_0^T + R_0 &= [1 \ 1] \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} [1] + R \\ &= [\sigma^2 \ 0] [1] + R = \sigma^2 + R \end{aligned}$$

$$K_0 = \begin{bmatrix} \frac{\sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix}$$

②...

$$\hat{X}_0 = \hat{X}_0^- + K_0 (z_0 - H_0 \hat{X}_0^-)$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix} (z_0 - [1 \ 1] \begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

$$= \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix}; \boxed{\hat{Y}_0 = H_0 \hat{X}_0 = [1 \ 1] \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} = \frac{z_0 \sigma^2}{\sigma^2 + R}}$$

$$P_0 = (I - K_0 H_0) P_0^- = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix} [1 \ 1] \right) \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sigma^2 & \sigma^2 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{\sigma^2}{\sigma^2 + R} & -\frac{\sigma^2}{\sigma^2 + R} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma^2 + R - \sigma^2}{\sigma^2 + R} & -\frac{\sigma^2}{\sigma^2 + R} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{R}{\sigma^2 + R} & -\frac{\sigma^2}{\sigma^2 + R} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{R \sigma^2}{\sigma^2 + R} & 0 \\ 0 & 0 \end{bmatrix}$$

(2) ...

$$\hat{\vec{x}}_t^- = \phi_o \hat{\vec{x}}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix}$$

$$P_t^- = \phi_o P_0 \phi_o^T + Q_K = P_0 + Q_K$$

$$= \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} & 0 \\ 0 & A \end{bmatrix}$$

K=1

$$K_1 = P_t^- H_1^T (H_1 P_t^- H_1^T + R_1)^{-1}$$

$$= \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + R \right)^{-1}$$

$$= \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} \\ A \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} \\ A \end{bmatrix} + R \right)^{-1}$$

$$= \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} \\ A \end{bmatrix} \left(\frac{R\sigma^2}{\sigma^2 + R} + A + R \right)^{-1}$$

$$= \begin{bmatrix} \frac{R\sigma^2}{\sigma^2 + R} \\ A \end{bmatrix} \left(\frac{R\sigma^2 + (A+R)(R+\sigma^2)}{\sigma^2 + R} \right)^{-1} = \begin{bmatrix} \frac{R\sigma^2}{R\sigma^2 + (A+R)(R+\sigma^2)} \\ \frac{A(R+\sigma^2)}{R\sigma^2 + (A+R)(R+\sigma^2)} \end{bmatrix}$$

(2) ...

$$\hat{x}_1 = \hat{x}_1^- + k_1 (z_1 - h_1 \hat{x}_1^-)$$

$$= \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} + \frac{1}{R \sigma^2 + (A+R)(R+\sigma^2)} \begin{bmatrix} R \sigma^2 \\ A(R+\sigma^2) \end{bmatrix} \left(z_1 - [1 \ 1] \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} + \frac{1}{R \sigma^2 + (A+R)(R+\sigma^2)} \begin{bmatrix} R \sigma^2 \\ A(R+\sigma^2) \end{bmatrix} \left(z_1 - \frac{z_0 \sigma^2}{\sigma^2 + R} \right)$$

$$\hat{y}_1 = H_1 \hat{x}_1 = [1 \ 1] \begin{bmatrix} \hat{x}_1^- \\ \uparrow \end{bmatrix}$$

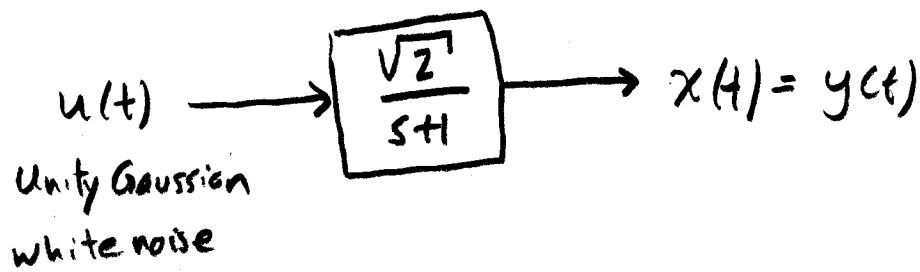
$$= [1 \ 1] \begin{bmatrix} \frac{z_0 \sigma^2}{\sigma^2 + R} \\ 0 \end{bmatrix} + \frac{1}{R \sigma^2 + (A+R)(\sigma^2 + R)} [1 \ 1] \begin{bmatrix} R \sigma^2 \\ A(R+\sigma^2) \end{bmatrix} \left(z_1 - \frac{z_0 \sigma^2}{\sigma^2 + R} \right)$$

$$= \frac{z_0 \sigma^2}{\sigma^2 + R} + \frac{R \sigma^2 + A(R+\sigma^2)}{R \sigma^2 + (A+R)(\sigma^2 + R)} \left(z_1 - \frac{z_0 \sigma^2}{\sigma^2 + R} \right)$$

②...

- The boxed expression for \hat{y}_0 on p-8 from the one-state model is identical to the one on p-13 for the two-state model.
 - The boxed expression for \hat{y}_1 on p. 10 from the one-state model is also identical to the one on p-15 for the two-state model.
⇒ Both models produce identical estimates \hat{y}_{ik} .
- b) If only the sum is of interest, then the one-state model is preferred because it is computationally simpler.

③ 5.12)



a) $x = \frac{\sqrt{2}}{s+1} u$

$$sx + x = \sqrt{2} u$$

$$sx = -x + \sqrt{2} u$$

$$\begin{cases} \dot{x} = -x + \sqrt{2} u(t) \\ y = x \end{cases}$$

State
model

Given: $\Delta t = 0.02$ sec., $R_k = 1$,

From state model: $F = -1 \rightarrow sI - F = s + 1 \rightarrow (sI - F)^{-1} = \frac{1}{s+1}$

$$\mathcal{L}^{-1}\{(sI - F)^{-1}\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} u(t)$$

$$\phi_k = e^{-t} u(t) \Big|_{t=\Delta t} = e^{-t} u(t) \Big|_{t=0.02} = e^{-0.02} = 0.9802$$

From the state model, $H_k = 1$.

$$G(u \text{ to } x) = G_1 = \frac{\sqrt{2}}{s+1}$$

$$g_1(t) = \sqrt{2} e^{-t} u(t)$$

$$\begin{aligned}
 ③ \dots Q_K &= E[W_K^2] = E[X^2] \\
 &= \int_0^{0.02} \int_0^{0.02} g_1(\xi) g_1(\eta) E[u(\xi)u(\eta)] d\xi d\eta \\
 &= \int_0^{0.02} \int_0^{0.02} \sqrt{2} e^{-\xi} \sqrt{2} e^{-\eta} \delta(5-\eta) d\xi d\eta \\
 &= 2 \int_0^{0.02} e^{-2\xi} d\xi = -[e^{-2\xi}]_0^{0.02} \\
 &= -[e^{-2(0.02)} - 1] = 1 - e^{-2(0.02)} = 0.03921
 \end{aligned}$$

Given: the autocorrelation is $R_X(\tau) = e^{-|\tau|}$

Since $\lim_{T \rightarrow \infty} R_X(\tau) = 0$, we have $E[X(t)] = 0$.

$$\begin{aligned}
 \text{VAR}[X(t)] &= E[X^2(t)] - (\underbrace{E[X(t)]}_0)^2 \\
 &= E[X^2(t)] = R_X(0) = 1
 \end{aligned}$$

Then $E[X(0)] = 0$ and $\text{Var}[X(0)] = 1$,

so we take $\hat{X}_0^- = 0$ with $P_0^- = 1$.

Summary of Model:

$$(3) \dots \phi_k = e^{-0.02} = 0.9802 \quad \Delta t = 0.02$$

$$H_k = 1$$

$$Q_k = 1 - e^{-2(0.02)} = 0.03921$$

$$R_k = 1$$

$$\hat{x}_0^- = 0$$

$$P_0^- = 1$$

Enter the loop:

$$k=0, \quad t=0$$

$$K_0 = P_0^- H_0^T (H_0 P_0^- H_0^T + R_0)^{-1}$$

$$= 1 \cdot 1 \cdot (1 \cdot 1 \cdot 1 + 1)^{-1} = 1 \cdot (2)^{-1} = \frac{1}{2}$$

$$\hat{x}_0 = \hat{x}_0^- + K_0 (z_0 - H_0 \hat{x}_0^-)$$

$$= 0 + \frac{1}{2} (z_0 - 1 \cdot 0) = \frac{1}{2} z_0$$

$$P_0 = (I - K_0 H_0) P_0^- = (1 - \frac{1}{2} \cdot 1) 1 = \frac{1}{2}$$

$$\hat{x}_1^- = \phi_0 \hat{x}_0 = e^{-0.02} \cdot \frac{1}{2} z_0 = \frac{e^{-0.02}}{2} z_0 = 0.4901 z_0$$

$$P_1^- = \phi_0 P_0 \phi_0^T + Q_0 = e^{-0.02} \cdot \frac{1}{2} \cdot e^{-0.02} + 1 - e^{-0.04}$$

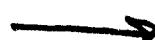
$$= \frac{1}{2} e^{-0.04} + 1 - e^{-0.04}$$

$$= 1 - \frac{1}{2} e^{-0.04} = 0.5196$$

③... $k=1, t=0.02$

$$\begin{aligned}K_1 &= P_1^{-1} H_1^T (H_1 P_1^{-1} H_1^T + R_1)^{-1} \\&= (1 - \frac{1}{2} e^{-0.04}) \cdot 1 \cdot [1 \cdot (1 - \frac{1}{2} e^{-0.04}) \cdot 1 + 1]^{-1} \\&= (1 - \frac{1}{2} e^{-0.04}) [2 - \frac{1}{2} e^{-0.04}]^{-1} \\&= \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} = 0.3419\end{aligned}$$

$$\begin{aligned}\hat{x}_1 &= \hat{x}_1^- + K_1 (z_1 - H_1 \hat{x}_1^-) \\&= \frac{1}{2} e^{-0.02} z_0 + \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} (z_1 - 1 \cdot \frac{1}{2} e^{-0.02} z_0) \\&= \frac{1}{2} e^{-0.02} z_0 + \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} z_1 - \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} \cdot \frac{1}{2} \cdot e^{-0.02} z_0 \\&= \frac{1}{2} \left[1 - \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} \right] e^{-0.02} z_0 + \frac{1}{2} \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} z_1 \\&= \frac{1}{2} \left[\frac{2 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} - \frac{1 - \frac{1}{2} e^{-0.04}}{2 - \frac{1}{2} e^{-0.04}} \right] e^{-0.02} z_0 + \frac{2 - e^{-0.04}}{4 - e^{-0.04}} z_1 \\&= \frac{1}{2} \left[\frac{1}{2 - \frac{1}{2} e^{-0.04}} \right] e^{-0.02} z_0 + \frac{2 - e^{-0.04}}{4 - e^{-0.04}} z_1\end{aligned}$$



$$③ \dots = \frac{e^{-0.02}}{4 - e^{-0.04}} z_0 + \frac{2 - e^{-0.04}}{4 - e^{-0.04}} z_1$$

$$= 0.32251 z_0 + 0.34193 z_1$$

\hat{x}_1 , obtained
by Kalman
filter in
part (a).

b) Weight function (Wiener Filter) approach:

$$\begin{bmatrix} E[z_0 z_0] & E[z_0 z_1] \\ E[z_0 z_1] & E[z_1 z_1] \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} E[z_0 x_1] \\ E[z_1 x_1] \end{bmatrix} \quad (*)$$

$$\begin{aligned} E[z_0 z_0] &= E[(x_0 + v_0)(x_0 + v_0)] \\ &= E[x_0^2 + 2x_0 v_0 + v_0^2] \\ &= E[x_0^2] + \underbrace{2E[x_0 v_0]}_{\text{zero}} + E[v_0^2] \\ &= P_0^- + 0 + R_0 = 1 + 0 + 1 = 2 \end{aligned}$$

$$\begin{aligned}
 ③ \dots E[z_0 z_1] &= E[(x_0 + v_0)(x_1 + v_1)] \\
 &= E[x_0 x_1 + x_0 v_1 + v_0 x_1 + v_0 v_1] \\
 &= E[x_0 x_1] + \underbrace{E[x_0 v_1]}_{\text{all zero}} + \underbrace{E[v_0 x_1]}_{\text{all zero}} + E[v_0 v_1] \\
 &= E[x_0 \phi_1 x_0] = E[x_0^2] \phi_1 = \phi_1 = e^{-0.02}
 \end{aligned}$$

$$\begin{aligned}
 E[z_1 z_1] &= E[(x_1 + v_1)(x_1 + v_1)] = E[x_1^2 + 2x_1 v_1 + v_1^2] \\
 &= E[x_1^2] + \underbrace{2E[x_1 v_1]}_{\text{zero}} + E[v_1^2] \\
 &= 1 + 0 + 1 = 2.
 \end{aligned}$$

$$\begin{aligned}
 E[z_0 x_1] &= E[(x_0 + v_0) x_1] = E[x_0 x_1] \\
 &= E[x_0 \phi_1 x_0] = E[x_0^2] \phi_1 = \phi_1 = e^{-0.02}
 \end{aligned}$$

$$\begin{aligned}
 E[z_1 x_1] &= E[(x_1 + v_1) x_1] = E[x_1^2 + v_1 x_1] \\
 &= E[x_1^2] + \underbrace{E[v_1 x_1]}_{\text{zero}} = E[x_1^2] = 1
 \end{aligned}$$

③ - (*) on p-21 then becomes

$$\begin{bmatrix} 2 & e^{-0.02} \\ e^{-0.02} & 2 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} e^{-0.02} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} 2 & e^{-0.02} \\ e^{-0.02} & 2 \end{bmatrix}^{-1} \begin{bmatrix} e^{-0.02} \\ 1 \end{bmatrix} \quad (*)$$

$$\begin{bmatrix} 2 & e^{-0.02} \\ e^{-0.02} & 2 \end{bmatrix}^{-1} = \frac{1}{4 - e^{-0.04}} \begin{bmatrix} 2 & -e^{-0.02} \\ -e^{-0.02} & 2 \end{bmatrix} \quad (**)$$

- Plug (**) into (*):

$$\begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \frac{1}{4 - e^{-0.04}} \begin{bmatrix} 2 & -e^{-0.02} \\ -e^{-0.02} & 2 \end{bmatrix} \begin{bmatrix} e^{-0.02} \\ 1 \end{bmatrix}$$

$$= \frac{1}{4 - e^{-0.04}} \begin{bmatrix} 2e^{-0.02} & -e^{-0.02} \\ -e^{-0.04} & +2 \end{bmatrix}$$

$$k_0 = \frac{e^{-0.02}}{4 - e^{-0.04}} = 0.32251$$

$$k_1 = \frac{2 - e^{-0.04}}{4 - e^{-0.04}} = 0.34193$$

Now, $\hat{x}_1 = k_0 z_0 + k_1 z$, which agrees precisely with part (a). ✓

5.17

$$f_{x^*|z} = \frac{f_{3|x^*} f_{x^*}}{f_3}, \quad \begin{array}{l} \text{In the notation on p. 242-243,} \\ x^* \rightarrow x_k \\ z \rightarrow z_k \\ P^{*-1} \rightarrow P_k^{-1} \end{array}$$

All densities are normal, so we can write

$$f_{3|x^*} = \frac{1}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}}} e^{-\frac{1}{2}(z - Hx^*)^T R^{-1} (z - Hx^*)}$$

$$f_{x^*} = \frac{1}{(2\pi)^{\frac{n}{2}} |P^*|^{\frac{1}{2}}} e^{-\frac{1}{2}(x^* - m^*)^T P^{*-1} (x^* - m^*)}$$

$$f_3 = \frac{1}{(2\pi)^{\frac{n}{2}} \underbrace{|HP^*H^T + R|^{\frac{1}{2}}}_{C_3}} e^{-\frac{1}{2}(z - m_3)^T C_3^{-1} (z - m_3)}$$

$$\therefore f_{x^*|z} = \frac{|C_3|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}} |R|^{\frac{1}{2}} |P^*|^{\frac{1}{2}}} e^{-\frac{1}{2} [\text{sum of above exponents}]}$$

Writing out the indicate sum yields

$$[\] = (z - Hx^*)^T R^{-1} (z - Hx^*) + (x^* - m^*)^T P^{*-1} (x^* - m^*) \\ - (z - m_3)^T C_3^{-1} (z - m_3)$$

Now, expand out the sum and collect quadratic, linear, and zero-order terms in x^* . This leads to:

$$[\] = x^{*T} (H^T R^{-1} H + P^{*-1}) x^* \\ - x^{*T} (H^T R^{-1} z + P^{*-1} m^*) - (z^T R^{-1} H + m^{*T} P^{*-1}) x^* \\ + [\text{zero-order terms}]$$

The desired form is:

$$(x^* - m)^T C^{-1} (x^* - m)$$

It is now obvious that

$$C^{-1} = H^T R^{-1} H + P^{*-1}$$

or

$$C = [P^{*-1} + H^T R^{-1} H]^{-1}$$