2. **25 pts.** A discrete-time LTI system H has impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n]$ and input $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Use the discrete-time Fourier transform (DTFT) to find the system output $y[n]$.

Table: $H(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{4}e^{-j\omega}}$

Table: $X(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}}$$

$$A = \left.\frac{1}{1 - \frac{1}{4}\theta}\right|_{\theta = 2} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{1/2} = 2$$

$$B = \left.\frac{1}{1 - \frac{1}{2}\theta}\right|_{\theta = 4} = \frac{1}{1 - 2} = \frac{1}{-1} = -1$$

$$Y(e^{j\omega}) = \frac{2}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

Table: $\boxed{y(t) = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]}$

3. **20 pts.** A discrete-time LTI system H has input

$$x[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]$$

and output

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 8\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n].$$

(a) **5 pts.** Find $X(e^{j\omega})$.

Table: $$X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

(b) **5 pts.** Find $Y(e^{j\omega})$.

Table: $$Y(e^{j\omega}) = \frac{6}{1-\frac{1}{2}e^{-j\omega}} - \frac{8}{1-\frac{1}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{6\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right) - 8\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right) + 3\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{6\left(1-\frac{1}{3}e^{-j\omega}-\frac{1}{4}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}\right) - 8\left(1-\frac{1}{4}e^{-j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right) + 3\left(1-\frac{1}{3}e^{-j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{6}e^{-j2\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{6\left(1-\frac{7}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}\right) - 8\left(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right) + 3\left(1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-j2\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{6-\frac{7}{2}e^{-j\omega}+\frac{1}{2}e^{-j2\omega} - 8 + 6e^{-j\omega} - e^{-j2\omega} + 3 - \frac{5}{2}e^{-j\omega} + \frac{1}{2}e^{-j2\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{6-8+3 + e^{-j\omega}\left[\overbrace{-\frac{7}{2}+6-\frac{5}{2}}^{0}\right] + e^{-j2\omega}\left[\overbrace{\frac{1}{2}-1+\frac{1}{2}}^{0}\right]}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)} \quad ///$$

Problem 3, cont...

(c) **5 pts**. Find the system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1-\frac{1}{2}e^{-j\omega})^2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} \quad ///$$

(d) **5 pts**. Find the impulse response $h[n]$.

$$\frac{1-\frac{1}{2}\theta}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)} = \frac{A}{1-\frac{1}{3}\theta} + \frac{B}{1-\frac{1}{4}\theta}$$

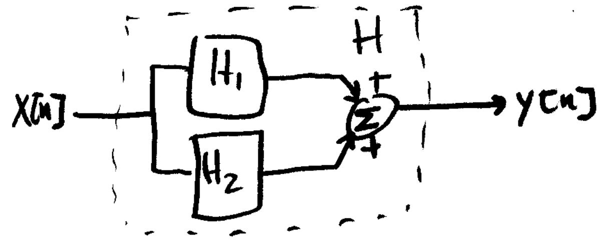
$$A = \left.\frac{1-\frac{1}{2}\theta}{1-\frac{1}{4}\theta}\right|_{\theta=3} = \frac{1-\frac{3}{2}}{1-\frac{3}{4}} = \frac{-\frac{1}{2}}{1/4} = 4(-\tfrac{1}{2}) = -2$$

$$B = \left.\frac{1-\frac{1}{2}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=4} = \frac{1-2}{1-\frac{4}{3}} = \frac{-1}{-1/3} = (-3)(-1) = 3$$

$$H(e^{j\omega}) = \frac{3}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{h[n] = 3\left(\tfrac{1}{4}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[n]}$

3. **25 pts**. An LTI system H_1 with impulse response



$$h_1[n] = \left(\frac{1}{3}\right)^n u[n]$$

is connected in parallel with another LTI system H_2 with impulse response $h_2[n]$ to form the overall system H.

The input $x[n]$ and output $y[n]$ of H are related by the difference equation

$$12y[n] - 7y[n-1] + y[n-2] = -12x[n] + 5x[n-1].$$

(a) **10 pts**. Find the overall system frequency response $H(e^{j\omega})$.

$$Y(e^{j\omega})\left[12 - 7e^{-j\omega} + e^{-j2\omega}\right] = X(e^{j\omega})\left[-12 + 5e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{\frac{5e^{-j\omega} - 12}{e^{-j2\omega} - 7e^{-j\omega} + 12}}}$$

$$= \frac{5e^{-j\omega} - 12}{(e^{-j\omega} - 3)(e^{-j\omega} - 4)} = \frac{\frac{5}{12}e^{-j\omega} - 1}{(1 - \frac{1}{3}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

Problem 3, cont...

(b) **15 pts**. Find the impulse response $h_2[n]$ of system H_2.

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$\rightarrow H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$$

$$= \frac{\frac{5}{12}e^{-j\omega} - 1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} - \frac{1\,(1-\frac{1}{4}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{\frac{5}{12}e^{-j\omega} - 1 - 1 + \frac{1}{4}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{\frac{2}{3}e^{-j\omega} - 2}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{-2(1-\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{h_2[n] = -2\left(\tfrac{1}{4}\right)^n u[n]}$$

4. **25 pts.** Consider a causal LTI system described by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

(a) **10 pts.** Find the system frequency response $H(e^{j\omega})$ **and** impulse response $h[n]$.

$$Y(e^{j\omega})\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right] = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j\omega}} = \frac{2}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \left.\frac{2}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2}{-1} = -2 \; ; \; B = \left.\frac{2}{1-\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2}{1-\frac{1}{2}} = 4$$

$$\boxed{H(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}}$$

$$\boxed{h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$$

(b) **5 pts.** Is the system stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{4}\left|4\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n\right| \leq 4\sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^n + 2\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= 4\frac{1}{1-\frac{1}{2}} + 2\frac{1}{1-\frac{1}{4}} = 8 + 8/3 = \frac{32}{3} < \infty.$$

The system is stable

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Ignore part (b)

Problem 4, cont...

(c) **10 pts.** Find the system response $y[n]$ when the input is

$$x[n] = \left(\frac{1}{3}\right)^n u[n].$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \frac{2}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}} + \frac{C}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \left.\frac{2}{(1-\frac{1}{4}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=3} = \frac{2}{(1-\frac{3}{4})(1-\frac{3}{2})} = \frac{2}{(\frac{1}{4})(-\frac{1}{2})}$$

$$= \frac{2}{-1/8} = -16$$

$$B = \left.\frac{2}{(1-\frac{1}{3}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=4} = \frac{2}{(1-\frac{4}{3})(1-2)} = \frac{2}{(-\frac{1}{3})(-1)} = \frac{2}{1/3} = 6$$

$$C = \left.\frac{2}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)}\right|_{\theta=2} = \frac{2}{(1-\frac{2}{3})(1-\frac{1}{2})} = \frac{2}{(\frac{1}{3})(\frac{1}{2})} = \frac{2}{1/6} = 12$$

$$Y(e^{j\omega}) = \frac{6}{1-\frac{1}{4}e^{-j\omega}} + \frac{12}{1-\frac{1}{2}e^{-j\omega}} - \frac{16}{1-\frac{1}{3}e^{-j\omega}}$$

$$\boxed{y[n] = 6\left(\frac{1}{4}\right)^n u[n] + 12\left(\frac{1}{2}\right)^n u[n] - 16\left(\frac{1}{3}\right)^n u[n].}$$