ECE 2713
Test 1

Tuesday, March 12, 2019
12:00 PM - 1:15 PM

Spring 2019 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. Calculators are allowed. All work must be your own. You have 75 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

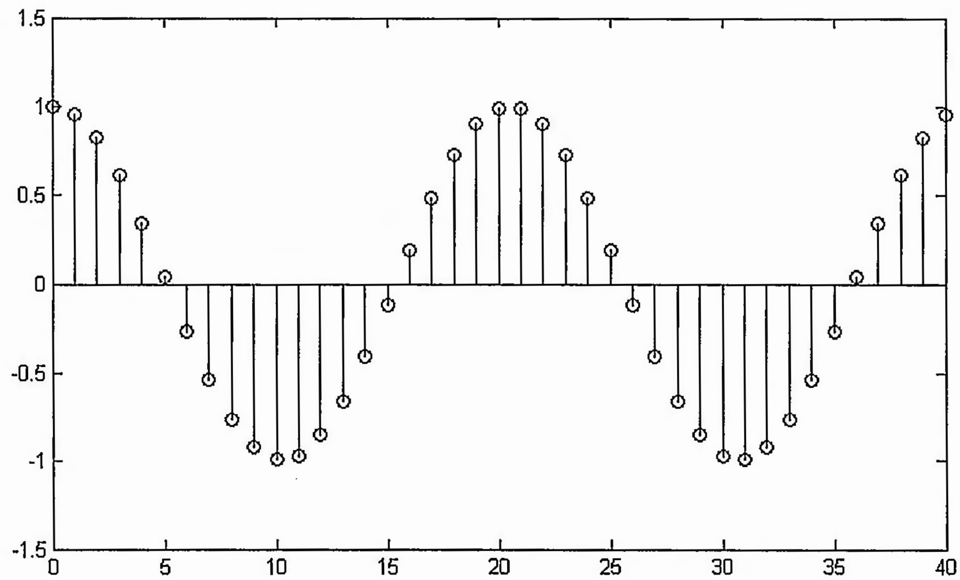
TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A **periodic** discrete-time sinusoidal signal $x[n]$ is given by $x[n] = \cos(\omega_0 n)$.

   The figure below shows a graph of ***exactly one period:***



Notice that the graph starts at $n = 0$ and ends at $n = 40$. This is exactly one period. Find the frequency ω_0.

Hint: according to the formula sheet, if a discrete-time sinusoidal signal $x[n]$ is periodic, then $\frac{\omega_0}{2\pi} = \frac{m}{N}$ where m and N are integers.
Find m and N from the graph and then solve for ω_0.

N = fundamental period = length of graph = 41

m = "number of times the graph "goes around" " = 2

$$\frac{\omega_0}{2\pi} = \frac{m}{N} = \frac{2}{41}$$

Cross multiply:

$$\boxed{\omega_0 = \frac{4\pi}{41}}$$

2. **25 pts.** A continuous-time signal $x(t)$ is given by

$$x(t) = 4\cos\left(\frac{\pi}{7}t + \frac{2\pi}{3}\right) + 2\cos\left(\frac{\pi}{7}t + \frac{3\pi}{4}\right).$$

Use phasor addition to express $x(t)$ in the form

$$x(t) = A\cos\left(\frac{\pi}{7}t + \phi\right).$$

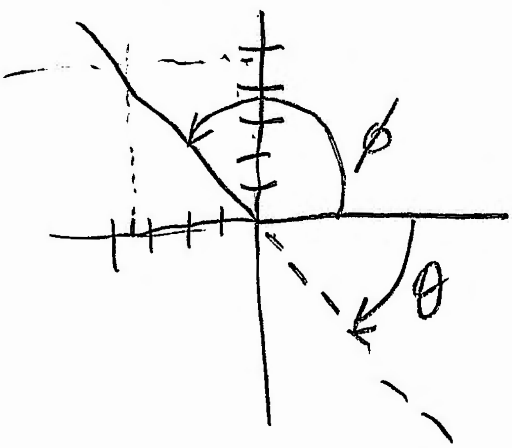
phasor for $4\cos(\frac{\pi}{7}t + \frac{2\pi}{3})$: $X_1 = 4e^{j2\pi/3}$

phasor for $2\cos(\frac{\pi}{7}t + \frac{3\pi}{4})$: $X_2 = 2e^{j3\pi/4}$

phasor for $x(t)$:

$$\begin{aligned} X &= X_1 + X_2 \\ &= 4e^{j2\pi/3} + 2e^{j3\pi/4} \\ &= (4\cos\tfrac{2\pi}{3} + j4\sin 2\pi/3) + (2\cos 3\pi/4 + j2\sin 3\pi/4) \\ &= (-2 + j2\sqrt{3}) + (-\sqrt{2} + j\sqrt{2}) \\ &= (-2-\sqrt{2}) + j(2\sqrt{3} + \sqrt{2}) \\ &= -3.41421 + j4.87832 \end{aligned}$$

$$A = |X| = \sqrt{(-3.41241)^2 + (4.87832)^2} = \sqrt{35.4548} = 5.95439$$



Since ϕ is in the 2^{nd} quadrant, "atan" will give the wrong angle. atan will give the angle θ. Then $\phi = \theta \pm \pi$.

$$\Rightarrow \theta = \operatorname{atan}\left(\frac{4.87832}{-3.41421}\right) = -0.960154$$

$$\Rightarrow \phi = \theta + \pi = 2.18144$$

$$X = 5.95439\, e^{j2.18144}$$

$$\boxed{x(t) = A\cos\left(\tfrac{\pi}{7}t + \phi\right) = 5.95439\cos\left(\tfrac{\pi}{7}t + 2.18144\right)}$$

More Workspace for Problem 2...

3. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1].$$

The system input is given by

$$x[n] = 2\delta[n] - 4\delta[n-1] + 6\delta[n-2].$$

Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = x[n] * \left(\tfrac{1}{2}\delta[n+1] + \tfrac{1}{2}\delta[n-1]\right)$$
$$= \tfrac{1}{2}x[n+1] + \tfrac{1}{2}x[n-1]$$
$$= \dots$$

$\frac{1}{2}x[n+1]: \longrightarrow \delta[n+1] - 2\delta[n] + 3\delta[n-1]$

$\frac{1}{2}x[n-1]: \longrightarrow + \delta[n-1] - 2\delta[n-2] + 3\delta[n-3]$

$$\boxed{y[n] = \delta[n+1] - 2\delta[n] + 4\delta[n-1] - 2\delta[n-2] + 3\delta[n-3]}$$

Other Way:

$$y[n] = h[n] * x[n] = h[n] * (2\delta[n] - 4\delta[n-1] + 6\delta[n-2])$$
$$= 2h[n] - 4h[n-1] + 6h[n-2]$$
$$= \dots$$

$2h[n]: \longrightarrow \delta[n+1] + \delta[n-1]$

$-4h[n-1]: \longrightarrow -2\delta[n] - 2\delta[n-2]$

$6h[n-2]: \longrightarrow +3\delta[n-1] + 3\delta[n-3]$

$$\boxed{y[n] = \delta[n+1] - 2\delta[n] + 4\delta[n-1] - 2\delta[n-2] + 3\delta[n-3]}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{4}\right)^n u[n].$$

The system input is given by

$$x[n] = \left(\frac{1}{2}\right)^n u[n-1].$$

Find the system output $y[n]$.

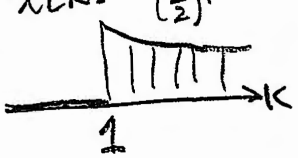
NOTE: $u[n]$ "turns on" at $n = 0$. So $u[n-1]$ "turns on" at $n = 1$. This means that the graph of $x[n]$ starts at $n = 1$, *not* at $n = 0$!

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

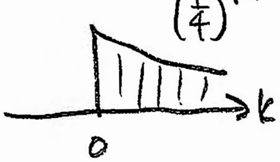
Hint: you should follow these steps:

1. write $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.
2. Use the definition of $x[n]$ given above to draw the graph of $x[k]$.
3. Use the definition of $h[n]$ given above to draw the graph of $h[k]$.
4. Slide the graph of $h[k]$ to the right by $-n$ to get the graph of $h[k-(-n)] = h[n+k]$.
5. Flip the graph of $h[n+k]$ with respect to k to get the graph of $h[-k-(-n)] = h[n-k]$.
6. For the n's in each region, multiply the graph of $x[k]$ with the graph of $h[n-k]$ and add up the product graph to get $y[n]$.

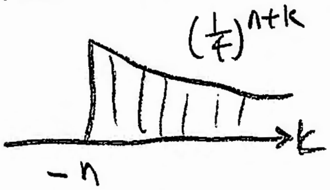
$x[k]$ $\left(\frac{1}{2}\right)^k$



$h[k]$ $\left(\frac{1}{4}\right)^k$



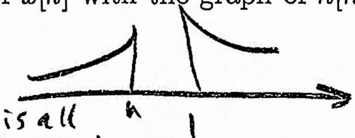
$h[k-(-n)] = h[n+k]$ $\left(\frac{1}{4}\right)^{n+k}$



$h[-k-(-n)] = h[n-k]$ $\left(\frac{1}{4}\right)^{n-k}$

Case I) $n < 1$:

→ product graph is all zero



$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$

Case II) $n \geq 1$:

→ The product graph goes from $k=1$ to $k=n$.

$$y[n] = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-k} = \left(\frac{1}{4}\right)^n \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k 4^k = \left(\frac{1}{4}\right)^n \sum_{k=1}^{n} \left(\frac{4}{2}\right)^k = \left(\frac{1}{4}\right)^n \sum_{k=1}^{n} 2^k$$

$$= \left(\frac{1}{4}\right)^n \frac{2^1 - 2^{n+1}}{1-2} = \left(\frac{1}{4}\right)^n \frac{2 - 2\cdot 2^n}{-1} = \left(\frac{1}{4}\right)^n \left[2\cdot 2^n - 2\right]$$

$$= 2\cdot 2^n \left(\frac{1}{4}\right)^n - 2\left(\frac{1}{4}\right)^n = 2\left(\frac{2}{4}\right)^n - 2\left(\frac{1}{4}\right)^n = 2\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n$$

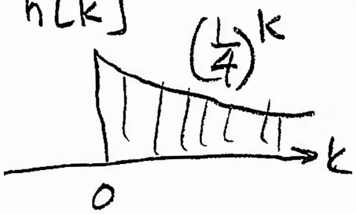
all Together:

$$\boxed{y[n] = \begin{cases} 0, & n < 1 \\ 2\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n, & n \geq 1 \end{cases} \qquad y[n] = 2\left(\frac{1}{2}\right)^n u[n-1] - 2\left(\frac{1}{4}\right)^n u[n-1]}$$

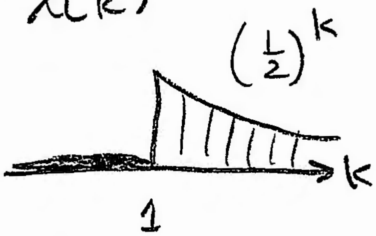
"OTHER WAY"

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

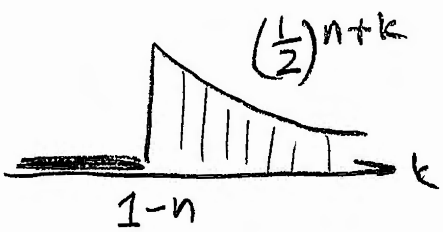
$h[k]$



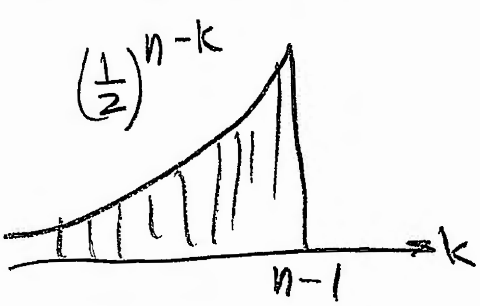
$x[k]$



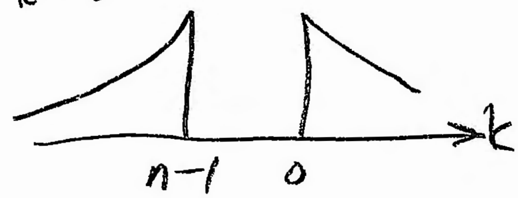
$x[k-(-n)] = x[n+k]$



$x[-k-(-n)] = x[n-k]$



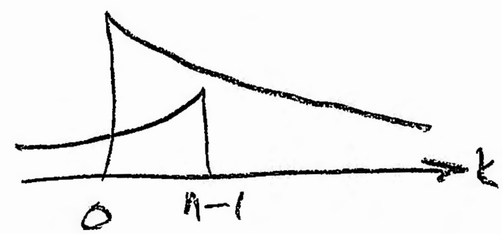
Case I) $n-1<0$
$n<1$



product graph is everywhere equal to zero.

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq 1$



→ The product graph is nonzero from $k=0$ to $k=n-1$

$$y[n] = \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{n-k} = \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{4}\right)^k 2^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{2}{4}\right)^k = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{2}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \frac{\left(\tfrac{1}{2}\right)^0 - \left(\tfrac{1}{2}\right)^n}{1-\tfrac{1}{2}} = \left(\tfrac{1}{2}\right)^n \frac{1-\left(\tfrac{1}{2}\right)^n}{1/2}$$

$$= 2\left(\tfrac{1}{2}\right)^n \left[1-\left(\tfrac{1}{2}\right)^n\right]$$

$$= 2\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^n = 2\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n$$

all Together:

$$y[n] = \begin{cases} 0, & n<1 \\ 2\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n, & n \geq 1 \end{cases}$$

$$y[n] = 2\left(\tfrac{1}{2}\right)^n u[n-1] - 2\left(\tfrac{1}{4}\right)^n u[n-1]$$