

ECE 2713

Test 1

Thursday, March 5, 2020

12:00 PM - 1:15 PM

Spring 2020

Dr. Havlicek

Name: SOLUTION

Student Num: _____

Directions: This test is closed book and closed notes. Calculators are allowed. All work must be your own. You have 75 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) _____

2. (25) _____

3. (25) _____

4. (25) _____

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: _____

Date: _____

1. 25 pts. Consider the discrete-time signal

$$x[n] = 5 \sin\left(\frac{4\pi}{27}n\right). \quad \omega_0 = \frac{4\pi}{27}$$

(a) 13 pts. Is $x[n]$ periodic? If you say *no*, then explain why not. If you say *yes*, then find the fundamental period.

$$\frac{\omega_0}{2\pi} = \frac{4\pi/27}{2\pi} = \frac{4\pi}{27} \cdot \frac{1}{2\pi} = \frac{2}{27} = \frac{m}{N} : \quad \begin{matrix} m=2 \\ N=27 \end{matrix}$$

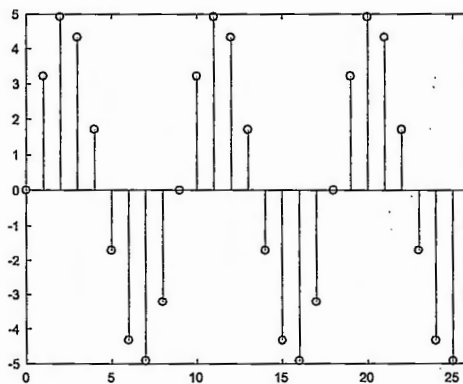
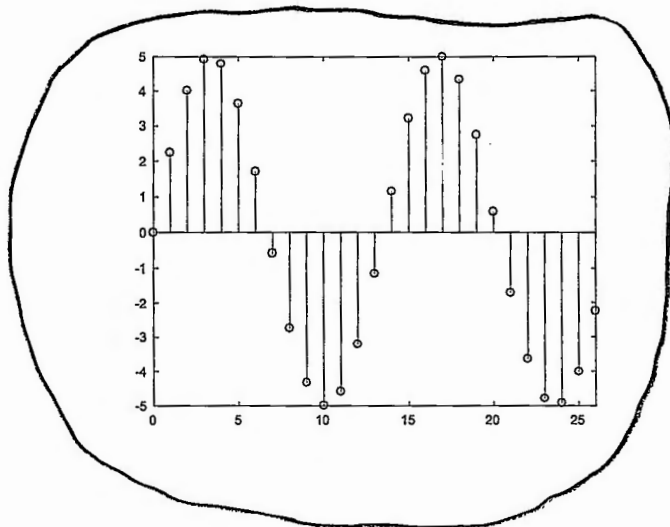
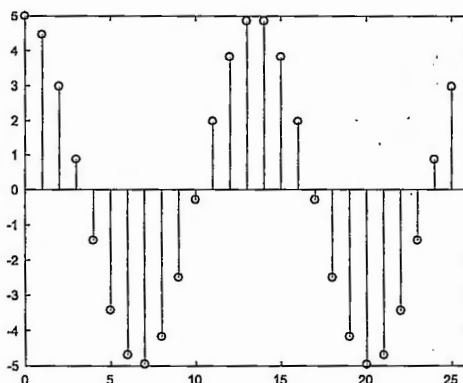
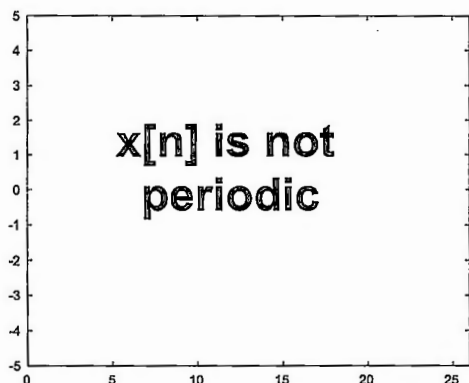
→ This is a ratio of two integers.

→ So: $x[n]$ is periodic

Fundamental period = $N = 27$

$m=2$: graph goes around 2 times for one period.

(b) 12 pts. Circle the graph that shows one period of $x[n]$: → $\sin = 0$ at $n=0$



This is the same as P2 on T1 from SP 2018

2. 25 pts. A continuous-time signal $x(t)$ is given by

$$x(t) = 5 \cos\left(\omega_0 t + \frac{3\pi}{2}\right) + 4 \cos\left(\omega_0 t + \frac{2\pi}{3}\right).$$

Use phasor addition to express $x(t)$ in the form

$$x(t) = A \cos(\omega_0 t + \phi).$$

Phasor for $5 \cos\left(\omega_0 t + \frac{3\pi}{2}\right)$: $X_1 = 5e^{j3\pi/2}$

Phasor for $4 \cos\left(\omega_0 t + \frac{2\pi}{3}\right)$: $X_2 = 4e^{j2\pi/3}$

Phasor for $x(t)$: $X = X_1 + X_2$

$$= 5e^{j3\pi/2} + 4e^{j2\pi/3}$$

$$= \left(5 \cos \frac{3\pi}{2} + j 5 \sin \frac{3\pi}{2}\right) + \left(4 \cos \frac{2\pi}{3} + j 4 \sin \frac{2\pi}{3}\right)$$

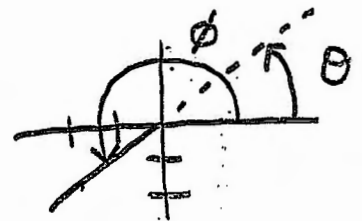
$$= (0 - j5) + (-2 + j2\sqrt{3})$$

$$= -2 + j(2\sqrt{3} - 5) = -2 - j1.53590$$

$$A = |X| = \sqrt{(-2)^2 + (1.53590)^2} = \sqrt{6.35898} = 2.52170$$

$$\phi = \arg X = \arctan\left(\frac{-1.53590}{-2}\right)$$

since ϕ is in the third quadrant, "atan" will give the wrong angle. atan will give θ .



\Rightarrow Then $\phi = \theta \pm \pi$.

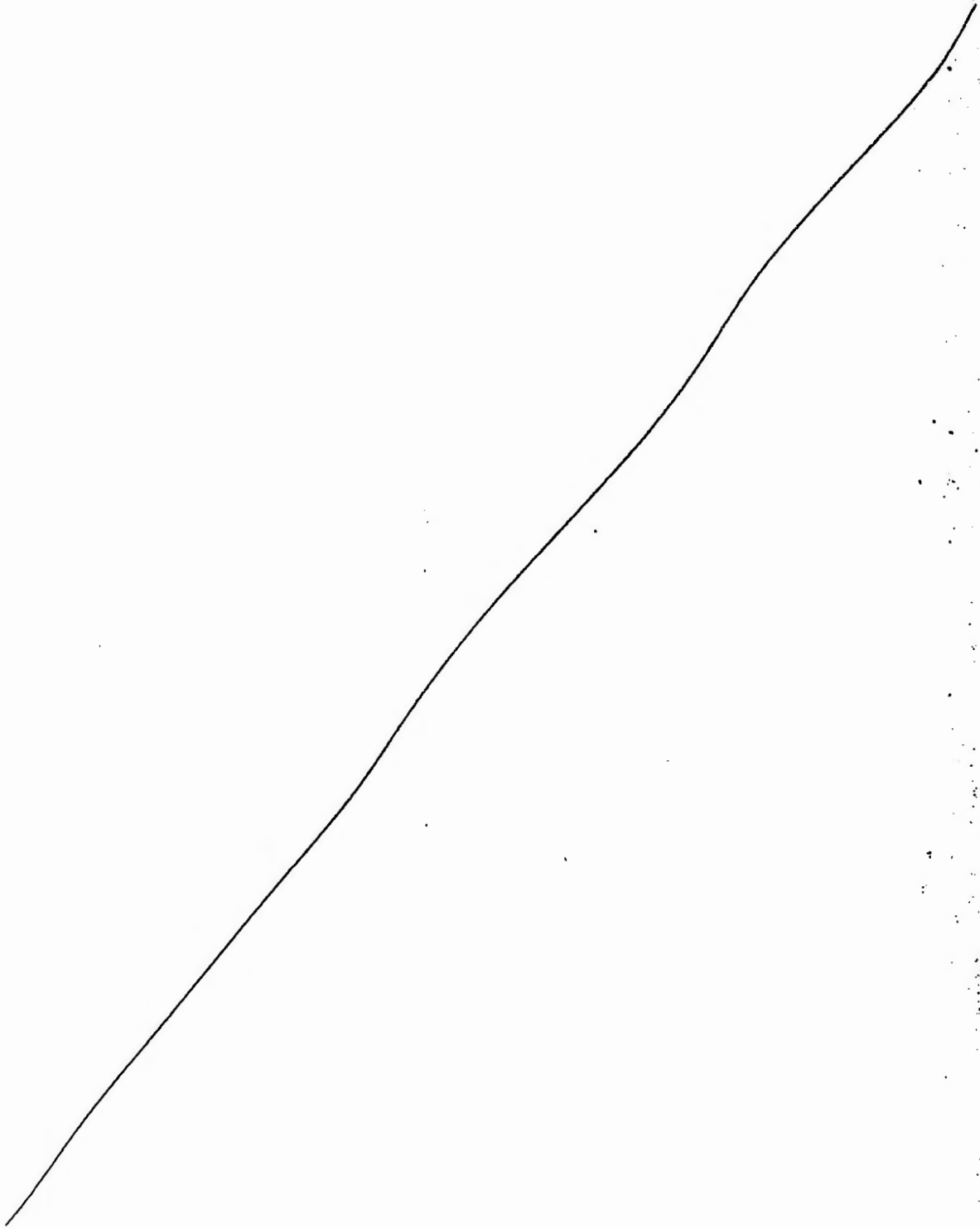
$$\theta = \text{atan}\left(\frac{-1.53590}{-2}\right) = 0.654890 \text{ rad}$$

$$\phi = \theta + \pi = 3.79648 \text{ rad}$$

$$x(t) = 2.52170 \cos(\omega_0 t + 3.79648)$$

$$X = Ae^{j\phi} = 2.52170 e^{j3.79648}$$

More Workspace for Problem 2...



3. 25 pts. A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \frac{1}{3}\delta[n] - \frac{1}{3}\delta[n-1].$$

The system input is given by

$$x[n] = 3\delta[n] + 6\delta[n-1] - 9\delta[n-2].$$

Find the system output $y[n]$.

$$\begin{aligned}y[n] &= x[n] * h[n] \\&= (3\delta[n] + 6\delta[n-1] - 9\delta[n-2]) * \left(\frac{1}{3}\delta[n] - \frac{1}{3}\delta[n-1]\right) \\&= \frac{1}{3}\delta[n] * (3\delta[n] + 6\delta[n-1] - 9\delta[n-2]) \\&\quad - \frac{1}{3}\delta[n-1] * (3\delta[n] + 6\delta[n-1] - 9\delta[n-2]) \\&= \delta[n] + 2\delta[n-1] - 3\delta[n-2] \\&\quad - \delta[n-1] - 2\delta[n-2] + 3\delta[n-3]\end{aligned}$$

$$y[n] = \delta[n] + \delta[n-1] - 5\delta[n-2] + 3\delta[n-3]$$

This problem is the same as the example on p. 2.27 of the notes file ECE2713NotesConvolution.pdf

4. 25 pts. A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n (u[n] - u[n-6]) = \begin{cases} \left(\frac{1}{6}\right)^n, & 0 \leq n \leq 5, \\ 0, & \text{otherwise.} \end{cases}$$

The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n+2].$$

Find the system output $y[n]$.

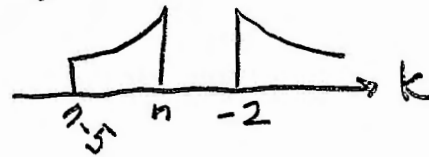
NOTE: $u[n]$ "turns on" at $n = 0$. So $u[n+2]$ "turns on" at $n = -2$. This means that the graph of $x[n]$ starts at $n = -2$, not at $n = 0$!

Hint: you should follow these steps:

1. write $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.
2. Use the definition of $x[n]$ given above to draw the graph of $x[k]$.
3. Use the definition of $h[n]$ given above to draw the graph of $h[k]$.
4. Slide the graph of $h[k]$ to the right by $-n$ to get the graph of $h[k - (-n)] = h[n+k]$.
5. Flip the graph of $h[n+k]$ with respect to k to get the graph of $h[-k - (-n)] = h[n-k]$.
6. For the n 's in each region, multiply the graph of $x[k]$ with the graph of $h[n-k]$ and add up the product graph to get $y[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Case I) $n < -2$:

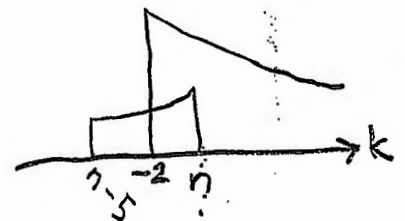


$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq -2$ and $n-5 < -2$

$$n \geq -2 \text{ and } n < 3$$

$$-2 \leq n < 3$$



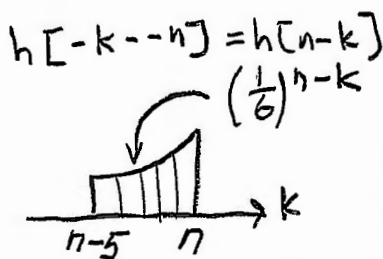
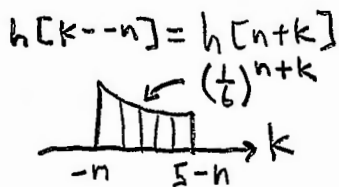
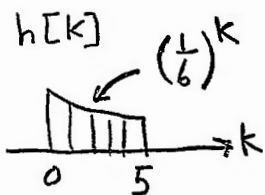
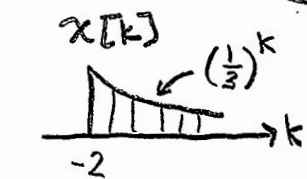
$$y[n] = \sum_{k=-2}^n \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k} = \left(\frac{1}{6}\right)^n \sum_{k=-2}^n \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{-k}$$

$$= \left(\frac{1}{6}\right)^n \sum_{k=-2}^n \left(\frac{1}{3}\right)^k 6^k = \left(\frac{1}{6}\right)^n \sum_{k=-2}^n \left(\frac{6}{3}\right)^k = \left(\frac{1}{6}\right)^n \sum_{k=-2}^n 2^k$$

$$= \left(\frac{1}{6}\right)^n \frac{2^{-2} - 2^{n+1}}{1-2} = \left(\frac{1}{6}\right)^n \frac{-2 \cdot 2^n}{-1}$$

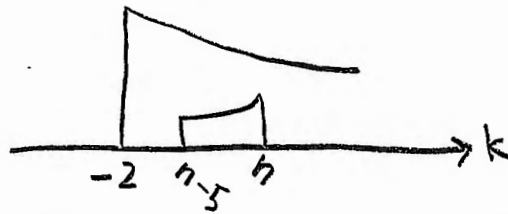
$$= \left(\frac{1}{6}\right)^n (2 \cdot 2^n - \frac{1}{4}) = 2 \cdot 2^n \left(\frac{1}{6}\right)^n - \frac{1}{4} \left(\frac{1}{6}\right)^n \rightarrow$$

$$= 2 \left(\frac{2}{6}\right)^n - \frac{1}{4} \left(\frac{1}{6}\right)^n = 2 \left(\frac{1}{3}\right)^n - \frac{1}{4} \left(\frac{1}{6}\right)^n$$



More Workspace for Problem 4...

Case III) $n-5 \geq -2$
 $n \geq 3$



$$\begin{aligned}y[n] &= \sum_{k=n-5}^n \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k} = \left(\frac{1}{6}\right)^n \sum_{k=n-5}^n \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{-k} = \left(\frac{1}{6}\right)^n \sum_{k=n-5}^n \left(\frac{1}{3}\right)^k 6^k \\&= \left(\frac{1}{6}\right)^n \sum_{k=n-5}^n \left(\frac{6}{3}\right)^k = \left(\frac{1}{6}\right)^n \sum_{k=n-5}^n 2^k = \left(\frac{1}{6}\right)^n \frac{2^{n-5} - 2^{n+1}}{1-2} \\&= \left(\frac{1}{6}\right)^n \frac{2^{-5} \cdot 2^n - 2 \cdot 2^n}{-1} = \left(\frac{1}{6}\right)^n \left[2 \cdot 2^n - \frac{1}{32} 2^n \right] \\&= \left(\frac{1}{6}\right)^n 2^n \left(2 - \frac{1}{32} \right) = \left(\frac{2}{6}\right)^n \left(\frac{64}{32} - \frac{1}{32} \right) = \left(\frac{1}{3}\right)^n \frac{63}{32} \\&= \frac{63}{32} \left(\frac{1}{3}\right)^n\end{aligned}$$

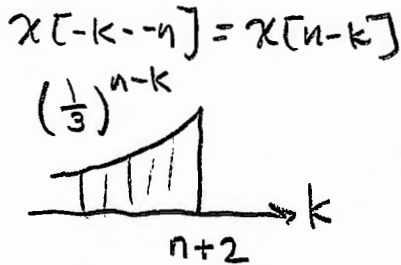
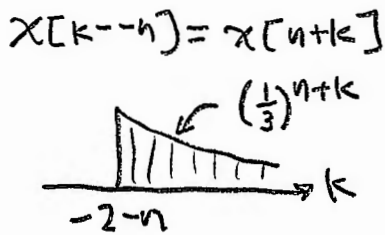
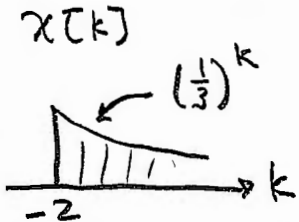
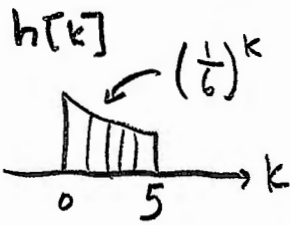
All Together:

$$y[n] = \begin{cases} 0 & , \quad n < -2 \\ 2\left(\frac{1}{3}\right)^n - \frac{1}{4}\left(\frac{1}{6}\right)^n & , \quad -2 \leq n < 3 \\ \frac{63}{32} \left(\frac{1}{3}\right)^n & , \quad n \geq 3 \end{cases}$$

"OTHER WAY"

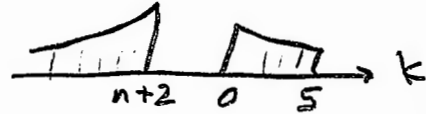
More Workspace for Problem 4...

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$



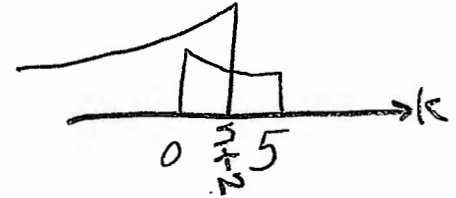
Case I) $n+2 < 0$
 $n < -2$

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$



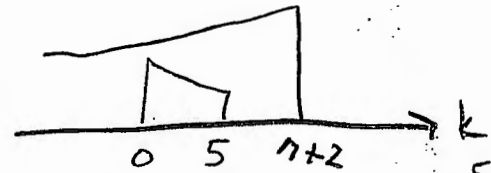
Case II) $n+2 > 0$ and $n+2 < 5$
 $n > -2$ and $n < 3$
 $-2 \leq n < 3$

$$\begin{aligned} y[n] &= \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{n-k} = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{-k} \\ &= \left(\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(\frac{1}{6}\right)^k 3^k = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(\frac{3}{6}\right)^k = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n+2} \left(\frac{1}{2}\right)^k \\ &= \left(\frac{1}{3}\right)^n \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{n+3}}{1 - \frac{1}{2}} = \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{1}{2}\right)^{n+3}}{\frac{1}{2}} \\ &= 2 \left(\frac{1}{3}\right)^n \left[1 - \frac{1}{8} \left(\frac{1}{2}\right)^n\right] = 2 \left(\frac{1}{3}\right)^n - \frac{2}{8} \left(\frac{1}{3}\right)^n \left(\frac{1}{2}\right)^n \\ &= 2 \left(\frac{1}{3}\right)^n - \frac{1}{4} \left(\frac{1}{6}\right)^n \end{aligned}$$



Case III) $n+2 > 5$
 $n > 3$

$$\begin{aligned} y[n] &= \sum_{k=0}^5 \left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{n-k} = \left(\frac{1}{3}\right)^n \sum_{k=0}^5 \left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{-k} = \left(\frac{1}{3}\right)^n \sum_{k=0}^5 \left(\frac{1}{6}\right)^k 3^k \\ &= \left(\frac{1}{3}\right)^n \sum_{k=0}^5 \left(\frac{3}{6}\right)^k = \left(\frac{1}{3}\right)^n \sum_{k=0}^5 \left(\frac{1}{2}\right)^k \\ &= \left(\frac{1}{3}\right)^n \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}} = \left(\frac{1}{3}\right)^n \frac{1 - \frac{1}{64}}{\frac{1}{2}} \end{aligned}$$



$$\begin{aligned} &= \left(\frac{1}{3}\right)^n \frac{63}{64} = \frac{63}{32} \left(\frac{1}{3}\right)^n \\ &= 2 \left(\frac{1}{3}\right)^n \left[1 - \frac{1}{64}\right] = 2 \left(\frac{1}{3}\right)^n \frac{63}{64} = \frac{63}{32} \left(\frac{1}{3}\right)^n \end{aligned}$$

All Together:

$$y[n] = \begin{cases} 0 & , n < -2 \\ 2 \left(\frac{1}{3}\right)^n - \frac{1}{4} \left(\frac{1}{6}\right)^n & , -2 \leq n < 3 \\ \frac{63}{32} \left(\frac{1}{3}\right)^n & , n \geq 3 \end{cases}$$