ECE 2713
Test 1

Thursday, March 9, 2023
12:00 PM - 1:15 PM

Spring 2023 | Name: SOLUTION

Dr. Havlicek | Student Num:__________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. All work must be your own. You have 75 minutes to complete the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ______
2. (25) ______
3. (25) ______
4. (25) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** Consider the discrete-time signal

$$x[n] = 5\sin\left(\frac{4\pi}{27}n\right).$$

(a) **13 pts.** Is $x[n]$ periodic? If you say *no*, then explain why not. If you say *yes*, then find the fundamental period.

$$\frac{\omega_0}{2\pi} = \frac{4\pi/27}{2\pi} = \frac{4\pi}{27}\cdot\frac{1}{2\pi} = \frac{2}{27} = \frac{m}{N} \; : \quad m = 2, \; N = 27$$

→ This is a ratio of two integers, i.e. $\frac{\omega_0}{2\pi} \in \mathbb{Q}$.

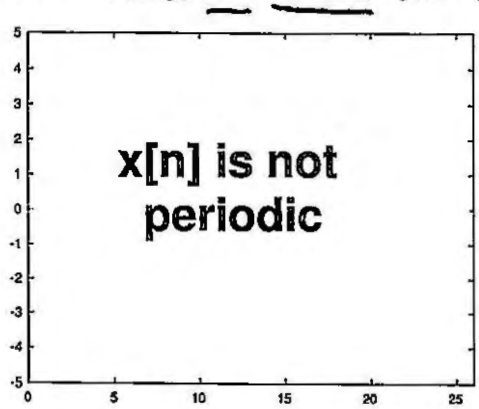
→ Therefore, $\boxed{x[n] \text{ is periodic.}}$

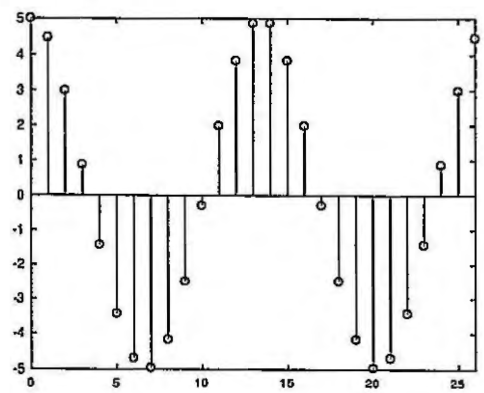
$\boxed{\text{Fundamental Period} = N = 27}$

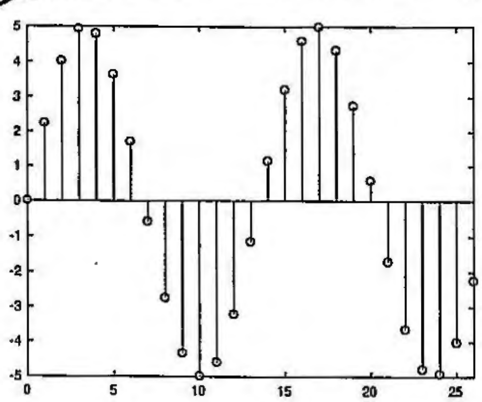
(b) **12 pts.** Circle the graph that shows one period of $x[n]$:

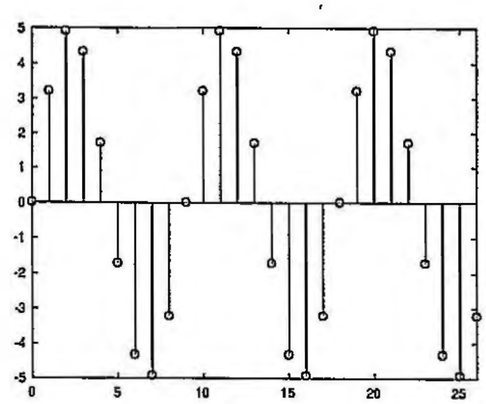
$m=2$ → graph goes around 2 times for one period

[$\sin(0) = 0$ at $n=0$]









2. **25 pts.** A continuous-time signal $x(t)$ is given by

$$x(t) = 4\cos\left(\frac{\pi}{7}t + \frac{2\pi}{3}\right) + 2\cos\left(\frac{\pi}{7}t + \frac{3\pi}{4}\right).$$

Use phasor addition to express $x(t)$ in the form

phasor $= Ae^{j\phi}$

$$x(t) = A\cos\left(\frac{\pi}{7}t + \phi\right).$$

phasor for $4\cos\left(\frac{\pi}{7}t + \frac{2\pi}{3}\right)$: $A = 4$, $\phi = \frac{2\pi}{3}$: $X_1 = 4e^{j2\pi/3}$

phasor for $2\cos\left(\frac{\pi}{7}t + \frac{3\pi}{4}\right)$: $A = 2$, $\phi = \frac{3\pi}{4}$: $X_2 = 2e^{j3\pi/4}$

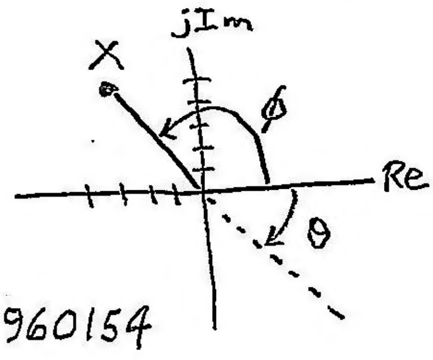
phasor for $x(t)$: $X = X_1 + X_2 = 4e^{j2\pi/3} + 2e^{j3\pi/4}$

$$X = \left[4\cos(2\pi/3) + j4\sin(2\pi/3)\right] + \left[2\cos(3\pi/4) + j2\sin(3\pi/4)\right]$$
$$= \left[4(-\tfrac{1}{2}) + j4(\tfrac{1}{2}\sqrt{3})\right] + \left[2(-\tfrac{1}{2}\sqrt{2}) + j2(\tfrac{1}{2}\sqrt{2})\right]$$
$$= \left[-2 + j2\sqrt{3}\right] + \left[-\sqrt{2} + j\sqrt{2}\right] = (-2-\sqrt{2}) + j(2\sqrt{3}+\sqrt{2})$$

$X = -3.41421 + j4.87832$

$A = |X| = \sqrt{(-3.41421)^2 + (4.87832)^2} = \sqrt{35.4548} = 5.95439$

→ Since ϕ is in the 2nd quadrant, "atan" will give the wrong angle... atan will give the angle θ shown at right. Then we must add (or subtract) π to get $\phi = \theta \pm \pi$.



$\Rightarrow \theta = \operatorname{atan}\left(\frac{4.87832}{-3.41421}\right) = \operatorname{atan}(-1.42883) = -0.960154$

$\Rightarrow \phi = \theta + \pi = -0.960154 + \pi = 2.18144$

phasor for $x(t) = X = Ae^{j\phi} = 5.95439e^{j2.18144}$

$$\boxed{x(t) = A\cos\left(\frac{\pi}{7}t + \phi\right) = 5.95439\cos\left(\frac{\pi}{7}t + 2.18144\right)}$$

More Workspace for Problem 2...

3. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2].$$

The system input is given by

$$x[n] = 2\delta[n+1] - 4\delta[n] + 2\delta[n-1].$$

Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = x[n] * (\delta[n] + 2\delta[n-1] + 3\delta[n-2])$$

$$= x[n] * \delta[n] + 2x[n] * \delta[n-1] + 3x[n] * \delta[n-2]$$

$$= x[n] + 2x[n-1] + 3x[n-2]$$

$$= 2\delta[n+1] - 4\delta[n] + 2\delta[n-1] \quad \leftarrow x[n]$$

$$+ 4\delta[n] - 8\delta[n-1] + 4\delta[n-2] \quad \leftarrow 2x[n-1]$$

$$+ 6\delta[n-1] - 12\delta[n-2] + 6\delta[n-3] \quad \leftarrow 3x[n-2]$$

$$= 2\delta[n+1] + 0\delta[n] + 0\delta[n-1] - 8\delta[n-2] + 6\delta[n-3]$$

$$= 2\delta[n+1] - 8\delta[n-2] + 6\delta[n-3]$$

$$\boxed{y[n] = 2\delta[n+1] - 8\delta[n-2] + 6\delta[n-3]}$$

③ "OTHER WAY"

$$y[n] = h[n] * x[n]$$

$$= h[n] * 2\delta[n+1] - 4\delta[n] + 2\delta[n-1]$$

$$= 2h[n] * \delta[n+1] - 4h[n] * \delta[n] + 2h[n] * \delta[n-1]$$

$$= 2h[n+1] - 4h[n] + 2h[n+1]$$

$$= 2(\delta[n+1] + 2\delta[n] + 3\delta[n-1])$$
$$-4(\delta[n] + 2\delta[n-1] + 3\delta[n-2])$$
$$+2(\delta[n-1] + 2\delta[n-2] + 3\delta[n-3])$$

$$= 2\delta[n+1] + 4\delta[n] + 6\delta[n-1]$$
$$- 4\delta[n] - 8\delta[n-1] - 12\delta[n-2]$$
$$+ 2\delta[n-1] + 4\delta[n-2] + 6\delta[n-3]$$

$$y[n] = 2\delta[n+1] + 0\delta[n] + 0\delta[n-1] - 8\delta[n-2] + 6\delta[n-3]$$

$$\boxed{y[n] = 2\delta[n+1] - 8\delta[n-2] + 6\delta[n-3]}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n u[n].$$

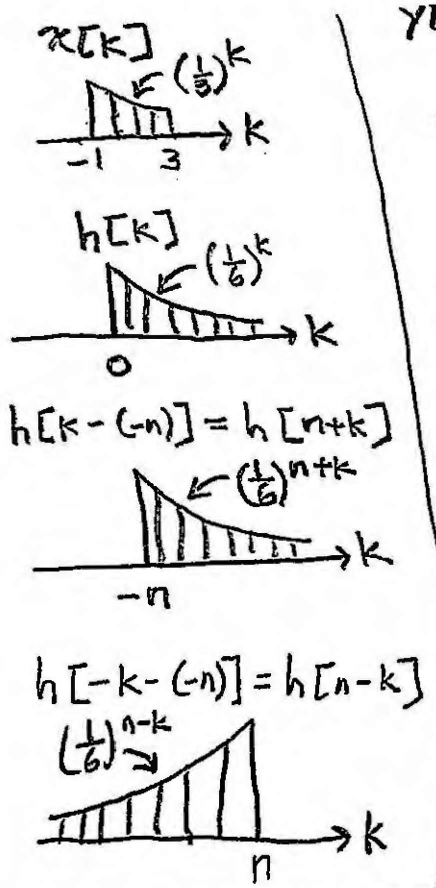
The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^n (u[n+1] - u[n-4]) = \begin{cases} \left(\frac{1}{3}\right)^n, & -1 \le n \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the system output $y[n]$.

Hint: you should follow these steps:

1. write $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.
2. Use the definition of $x[n]$ given above to draw the graph of $x[k]$.
3. Use the definition of $h[n]$ given above to draw the graph of $h[k]$.
4. Slide the graph of $h[k]$ to the right by $-n$ to get the graph of $h[k-(-n)] = h[n+k]$.
5. Flip the graph of $h[n+k]$ with respect to k to get the graph of $h[-k-(-n)] = h[n-k]$.
6. For the n's in each region, multiply the graph of $x[k]$ with the graph of $h[n-k]$ and add up the product graph to get $y[n]$.



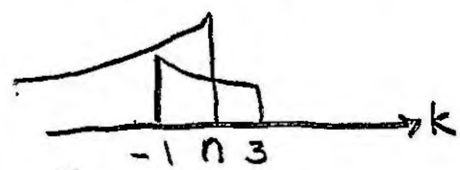
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

case I) $n < -1$

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n \geq -1$ and $n < 3$

$-1 \le n < 3$



$$y[n] = \sum_{k=-1}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k}$$

$$= \sum_{k=-1}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^n \left(\tfrac{1}{6}\right)^{-k} = \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{n} \left(\tfrac{1}{3}\right)^k 6^k$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{n} \left(\tfrac{6}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{n} 2^k = \left(\tfrac{1}{6}\right)^n \frac{2^{-1} - 2^{n+1}}{1-2}$$

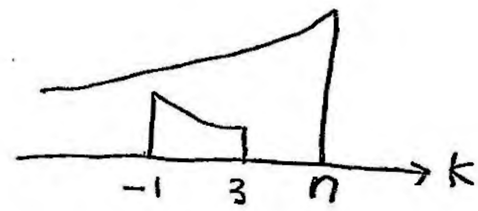
$$= \left(\tfrac{1}{6}\right)^n \frac{\frac{1}{2} - 2\cdot 2^n}{-1} = \left(\tfrac{1}{6}\right)^n \left[2\cdot 2^n - \tfrac{1}{2}\right]$$

$$= 2\left(\tfrac{1}{6}\right)^n 2^n - \tfrac{1}{2}\left(\tfrac{1}{6}\right)^n = 2\left(\tfrac{2}{6}\right)^n - \tfrac{1}{2}\left(\tfrac{1}{6}\right)^n$$

$$= 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{2}\left(\tfrac{1}{6}\right)^n$$

More Workspace for Problem 4...

Case III) $n \geq 3$



$$y[n] = \sum_{k=-1}^{3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k}$$

$$= \sum_{k=-1}^{3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^n \left(\tfrac{1}{6}\right)^{-k} = \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{3} \left(\tfrac{1}{3}\right)^k 6^k$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{3} \left(\tfrac{6}{3}\right)^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-1}^{3} 2^k = \left(\tfrac{1}{6}\right)^n \frac{2^{-1} - 2^4}{1-2}$$

$$= \left(\tfrac{1}{6}\right)^n \frac{\tfrac{1}{2} - 16}{-1} = \left(\tfrac{1}{6}\right)^n \left[16 - \tfrac{1}{2}\right] = \left(\tfrac{1}{6}\right)^n \left[\tfrac{32}{2} - \tfrac{1}{2}\right]$$

$$= \tfrac{31}{2} \left(\tfrac{1}{6}\right)^n$$

All Together:

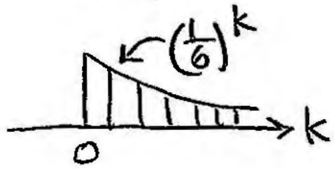
$$\boxed{y[n] = \begin{cases} 0, & n < -1 \\ 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{2}\left(\tfrac{1}{6}\right)^n, & -1 \leq n < 3 \\ \tfrac{31}{2}\left(\tfrac{1}{6}\right)^n, & n \geq 3 \end{cases}}$$

"OTHER WAY"

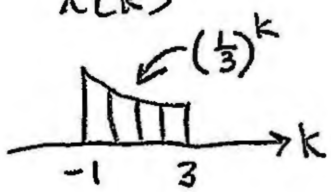
More Workspace for Problem 4...

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

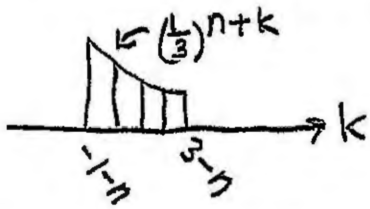
$h[k]$



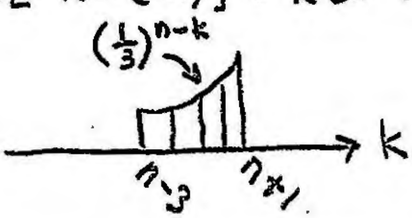
$x[k]$



$x[k-(-n)] = x[n+k]$



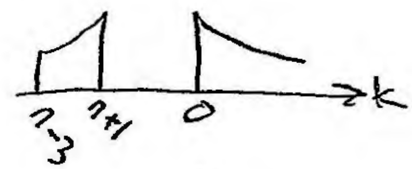
$x[-k-(-n)] = x[n-k]$



case I) $n+1 < 0$

$n < -1$

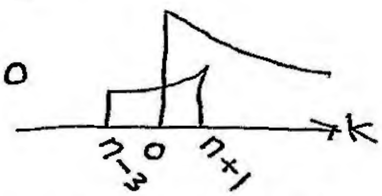
$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$



case II) $n+1 \geq 0$ and $n-3 < 0$

$n \geq -1$ and $n < 3$

$-1 \leq n < 3$



$$y[n] = \sum_{k=0}^{n+1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{n-k} = \sum_{k=0}^{n+1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n+1} \left(\tfrac{1}{6}\right)^k 3^k = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n+1} \left(\tfrac{3}{6}\right)^k$$

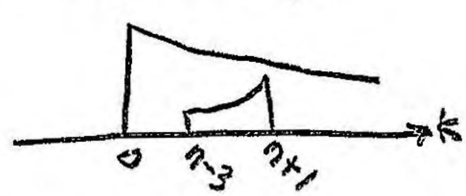
$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n+1} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{3}\right)^n \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{n+2}}{1 - 1/2}$$

$$= \left(\tfrac{1}{3}\right)^n \frac{1 - \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^n}{1/2} = 2\left(\tfrac{1}{3}\right)^n \left[1 - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^n\right]$$

$$= 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{2}\left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{2}\right)^n = 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{2}\left(\tfrac{1}{6}\right)^n$$

Case III) $n-3 \geq 0$

$n \geq 3$



$$y[n] = \sum_{k=n-3}^{n+1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{n-k} = \sum_{k=n-3}^{n+1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=n-3}^{n+1} \left(\tfrac{1}{6}\right)^k 3^k = \left(\tfrac{1}{3}\right)^n \sum_{k=n-3}^{n+1} \left(\tfrac{3}{6}\right)^k = \left(\tfrac{1}{3}\right)^n \sum_{k=n-3}^{n+1} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{3}\right)^n \frac{\left(\frac{1}{2}\right)^{n-3} - \left(\frac{1}{2}\right)^{n+2}}{1 - 1/2}$$

$$= \left(\tfrac{1}{3}\right)^n \frac{\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{-3} - \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^2}{1/2} = 2\left(\tfrac{1}{3}\right)^n \left[2^3 \left(\tfrac{1}{2}\right)^n - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^n\right]$$

$$= 2\left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{2}\right)^n \left[8 - \tfrac{1}{4}\right] = 2\left(\tfrac{1}{6}\right)^n \left[\tfrac{32}{4} - \tfrac{1}{4}\right] = 2\left(\tfrac{31}{4}\right)\left(\tfrac{1}{6}\right)^n = \tfrac{31}{2}\left(\tfrac{1}{6}\right)^n$$

All Together:

$$\boxed{y[n] = \begin{cases} 0 & ; \ n < -1 \\ 2\left(\frac{1}{3}\right)^n - \frac{1}{2}\left(\frac{1}{6}\right)^n & ; \ -1 \leq n < 3 \\ \frac{31}{2}\left(\frac{1}{6}\right)^n & , \ n \geq 3 \end{cases}}$$