ECE 2713
Test 2

Thursday, April 19, 2018
12:00 PM - 1:15 PM

Spring 2018

Dr. Havlicek

Name: SOLUTION

Student Num:

Directions: This test is closed book and closed notes. Calculators are allowed. All work must be your own. You have 75 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ________________ Date: ________________

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(\frac{1}{9}\right)^n u[n]$$

and input

$$x[n] = \left(-\frac{1}{3}\right)^n u[n].$$

Note: for $x[n]$, this means that $a = -\frac{1}{3}$.

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

Table: $H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{9}e^{-j\omega}}$

Table: $X(e^{j\omega}) = \dfrac{1}{1+\frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \frac{1}{(1-\frac{1}{9}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})} = \frac{A}{1-\frac{1}{9}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

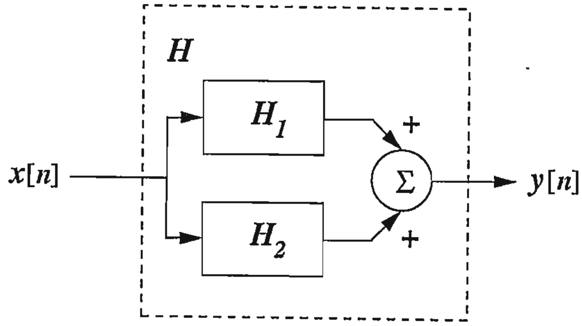
$$A = \left.\frac{1}{1+\frac{1}{3}\theta}\right|_{\theta=9} = \frac{1}{1+3} = \frac{1}{4}$$

$$B = \left.\frac{1}{1-\frac{1}{9}\theta}\right|_{\theta=-3} = \frac{1}{1+\frac{1}{3}} = \frac{1}{4/3} = \frac{3}{4}$$

$$Y(e^{j\omega}) = \frac{1/4}{1-\frac{1}{9}e^{-j\omega}} + \frac{3/4}{1+\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{y[n] = \frac{1}{4}\left(\frac{1}{9}\right)^n u[n] + \frac{3}{4}\left(-\frac{1}{3}\right)^n u[n]}$

2. **25 pts.** The discrete-time LTI system H is a parallel connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below.



The frequency response of system H_1 is given by

$$H_1(e^{j\omega}) = \frac{9}{\left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{2}{3}e^{-j\omega}\right)}.$$

The impulse response of system H_2 is given by

$$h_2[n] = 3\left(\frac{1}{9}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the impulse response $h[n]$ of the overall system H.

Table: $H_2(e^{j\omega}) = \dfrac{3}{1 - \frac{1}{9}e^{-j\omega}}$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$= \frac{9}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})} + \frac{3}{(1-\frac{1}{9}e^{-j\omega})}$$

$$= \frac{9(1-\frac{1}{9}e^{-j\omega}) + 3(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{9}e^{-j\omega})}$$

→ This is a proper fraction, so I could do the PFE right now without bothering to multiply out the numerator...

But let's go ahead and multiply out upstairs anyway...

$$H(e^{j\omega}) = \frac{9 - e^{-j\omega} + 3\left(1 - \frac{2}{3}e^{-j\omega} - \frac{1}{3}e^{-j\omega} + \frac{2}{9}e^{-j2\omega}\right)}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{9}e^{-j\omega})}$$

$$= \frac{9 - e^{-j\omega} + 3\left(1 - e^{-j\omega} + \frac{2}{9}e^{-j2\omega}\right)}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{9}e^{-j\omega})}$$

$$= \frac{9 - e^{-j\omega} + 3 - 3e^{-j\omega} + \frac{2}{3}e^{-j2\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{9}e^{-j\omega})}$$

$$= \frac{12 - 4e^{-j\omega} + \frac{2}{3}e^{-j2\omega}}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{-j\omega})(1-\frac{1}{9}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{2}{3}e^{-j\omega}} + \frac{C}{1-\frac{1}{9}e^{-j\omega}}$$

$$A = \left.\frac{12 - 4\theta + \frac{2}{3}\theta^2}{(1-\frac{2}{3}\theta)(1-\frac{1}{9}\theta)}\right|_{\theta=3} = \frac{12-12+\frac{2}{3}9}{(1-2)(1-\frac{1}{3})} = \frac{6}{-2/3}$$

$$= -\frac{3}{2}\cdot 6 = -9$$

$\longrightarrow$

More Workspace for Problem 2...

$$B = \left.\frac{12-4\theta+\frac{2}{3}\theta^2}{(1-\frac{1}{3}\theta)(1-\frac{1}{9}\theta)}\right|_{\theta=\frac{3}{2}} = \frac{12-6+\frac{2}{3}\cdot\frac{9}{4}}{(1-\frac{1}{2})(1-\frac{1}{6})} = \frac{6+\frac{3}{2}}{(\frac{1}{2})(\frac{5}{6})}$$

$$= \frac{\frac{12}{2}+\frac{3}{2}}{5/12} = \frac{\frac{15}{2}}{5/12} = \frac{12}{5}\cdot\frac{15}{2}$$

$$= 3\cdot 6 = 18$$

$$C = \left.\frac{12-4\theta+\frac{2}{3}\theta^2}{(1-\frac{1}{3}\theta)(1-\frac{2}{3}\theta)}\right|_{\theta=9} = \frac{12-36+\frac{2}{3}\cdot 81}{(1-3)(1-6)} = \frac{-24+54}{(-2)(-5)}$$

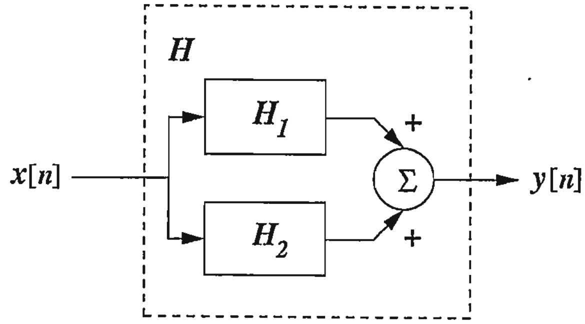
$$= \frac{30}{10} = 3$$

$$H(e^{j\omega}) = \frac{-9}{1-\frac{1}{3}e^{-j\omega}} + \frac{18}{1-\frac{2}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{9}e^{-j\omega}}$$

Table:

$$\boxed{h[n] = -9\left(\tfrac{1}{3}\right)^n u[n] + 18\left(\tfrac{2}{3}\right)^n u[n] + 3\left(\tfrac{1}{9}\right)^n u[n]}$$

2. **25 pts.** The discrete-time LTI system H is a parallel connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below.



HERE IS A SHORTER WAY TO DO PROBLEM 2.

The frequency response of system H_1 is given by

$$H_1(e^{j\omega}) = \frac{9}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{2}{3}e^{-j\omega}\right)}.$$

The impulse response of system H_2 is given by

$$h_2[n] = 3\left(\frac{1}{9}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the impulse response $h[n]$ of the overall system H.

$$H_1(e^{j\omega}) = \frac{9}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{2}{3}e^{j\omega})} = \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{2}{3}e^{-j\omega}}$$

$$A = \left.\frac{9}{1-\frac{2}{3}\theta}\right|_{\theta=3} = \frac{9}{1-2} = -9$$

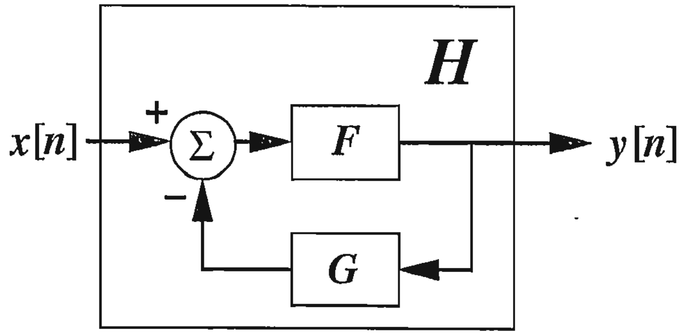
$$B = \left.\frac{9}{1-\frac{1}{3}\theta}\right|_{\theta=\frac{3}{2}} = \frac{9}{1-\frac{1}{2}} = \frac{9}{1/2} = 18$$

$$H_1(e^{j\omega}) = \frac{18}{1-\frac{2}{3}e^{-j\omega}} - \frac{9}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $h_1[n] = 18\left(\frac{2}{3}\right)^n u[n] - 9\left(\frac{1}{3}\right)^n u[n]$

$$h[n] = h_1[n] + h_2[n] = 18\left(\frac{2}{3}\right)^n u[n] - 9\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{9}\right)^n u[n]$$

3. **25 pts.** Consider the **causal** discrete-time LTI system H shown below.



F and G are both **causal** discrete-time LTI systems.

The impulse response of G is given by $g[n] = \frac{3}{2}\delta[n-1]$.

When the overall system input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n-1] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1],$$

the overall system output is observed to be

$$y[n] = n\left(\frac{1}{2}\right)^n u[n].$$

(a) **7 pts.** Find the overall system transfer function $H(z)$.

Table + Time Shift : $X(z) = \dfrac{\frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}$

Table : $Y(z) = \dfrac{\frac{1}{2} z^{-1}}{(1 - \frac{1}{2} z^{-1})^2}, \quad |z| > \frac{1}{2}$

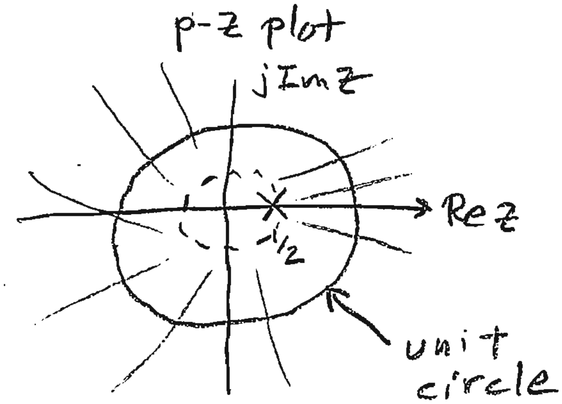
$$H(z) = \frac{Y(z)}{X(z)} = Y(z)\frac{1}{X(z)} = \frac{\frac{1}{2} z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \cdot \frac{1-\frac{1}{2}z^{-1}}{\frac{1}{2} z^{-1}}$$

$$\boxed{H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}}$$

Problem 3, cont...

(b) **4 pts.** Is the system H stable? *Justify your answer.*

YES. Because the ROC of $H(z)$ includes the unit circle.



(c) **10 pts.** Find the impulse response $f[n]$ of system F.

Table: $G(z) = \frac{3}{2} z^{-1}$, $|z| > 0$
(+ Time Shift)

$$H(z) = \frac{F(z)}{1 + F(z)G(z)} \Rightarrow \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{F(z)}{1+\frac{3}{2}z^{-1}F(z)}$$

$$1 + \frac{3}{2}z^{-1}F(z) = \left(1 - \frac{1}{2}z^{-1}\right)F(z) = F(z) - \frac{1}{2}z^{-1}F(z)$$

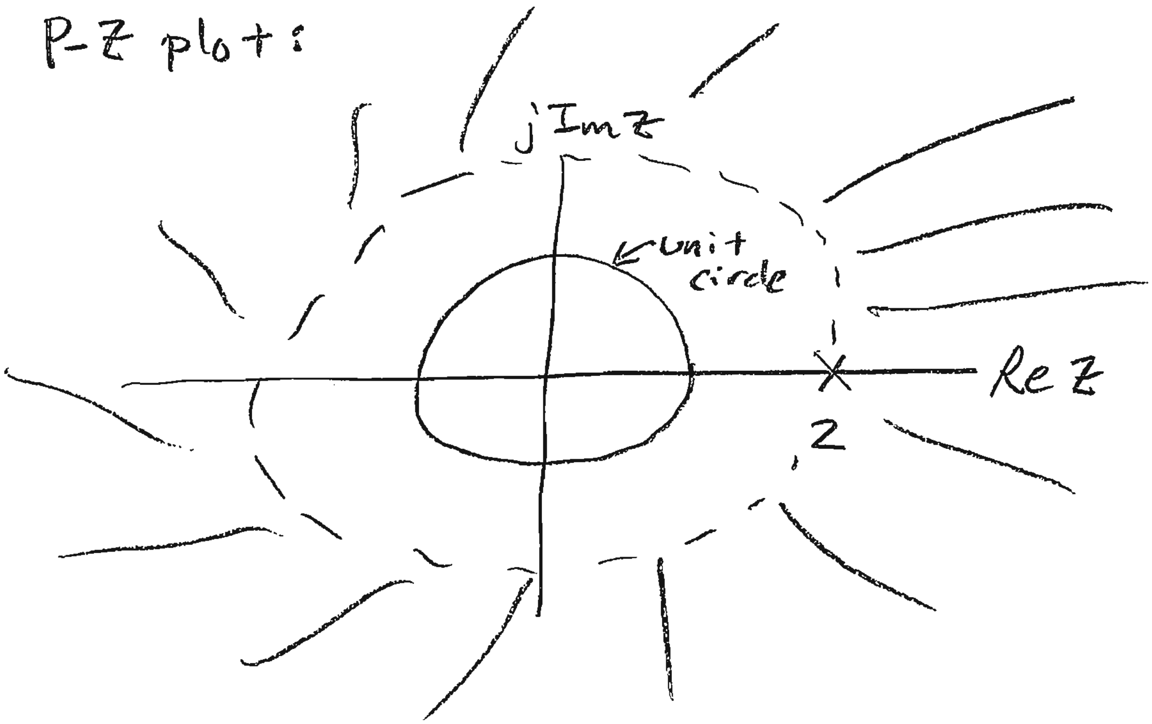
$$1 = F(z) - \frac{1}{2}z^{-1}F(z) - \frac{3}{2}z^{-1}F(z)$$

$$1 = F(z) - 2z^{-1}F(z) = F(z)\left[1 - 2z^{-1}\right]$$

$$F(z) = \frac{1}{1-2z^{-1}} , \quad |z| > 2$$

Table: $\boxed{f[n] = 2^n u[n]}$

P-Z plot:



(d) **4 pts.** Is the system F stable? *Justify your answer.*

NO. The ROC of F(z) does not contain the unit circle.

4. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by the difference equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n] - 4x[n-1].$$

(a) **10 pts.** For this part, assume that the system is **stable**. Find the impulse response $h[n]$.

$\mathcal{Z}$: $Y(z) + \frac{5}{2}z^{-1}Y(z) - \frac{3}{2}z^{-2}Y(z) = X(z) - 4z^{-1}X(z)$

$Y(z)\left[1+\frac{5}{2}z^{-1} - \frac{3}{2}z^{-2}\right] = X(z)\left[1-4z^{-1}\right]$

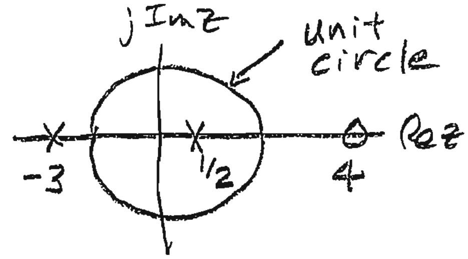
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-4z^{-1}}{1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}} = \frac{1-4z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+3z^{-1}}$$

$A = \left.\frac{1-4\theta}{1+3\theta}\right|_{\theta=2} = \frac{1-8}{1+6} = \frac{-7}{7} = -1$

$B = \left.\frac{1-4\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}} = \frac{1+\frac{4}{3}}{1+\frac{1}{6}} = \frac{7/3}{7/6} = \frac{7}{3}\cdot\frac{6}{7} = 2$

p-z plot



$$H(z) = \underbrace{\frac{-1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{2}{1+3z^{-1}}}_{|z|<3}, \quad \frac{1}{2}<|z|<3$$

From the poles, there are three possibilities for the ROC:

$|z|<\frac{1}{2}$; $\frac{1}{2}<|z|<3$; $|z|>3$

But we are given that H is stable. So the ROC must include the unit circle.

$\Rightarrow$ The ROC must be $\frac{1}{2}<|z|<3$.

Table:

$$\boxed{h[n] = -\left(\tfrac{1}{2}\right)^n u[n] - 2(-3)^n u[-n-1]}$$

Problem 4, cont...

(b) **10 pts.** Now assume that the system is **causal**, but not necessarily stable. Find the impulse response $h[n]$.

The PFE for $H(z)$ is still the same, but the ROC is changed. We are given that H is causal, so the ROC of $H(z)$ must be exterior. It must be $|z|>3$.

$$H(z) = \underbrace{\frac{-1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{2}{1+3z^{-1}}}_{|z|>3}, \quad |z|>3$$

Table: $\boxed{h[n] = -\left(\frac{1}{2}\right)^n u[n] + 2(-3)^n u[n]}$

(c) **5 pts.** Does a system exist that is causal **and** stable **and** has its input and output related by this difference equation? *Justify your answer.*

NO. For H to be causal and stable, the ROC of $H(z)$ must be exterior and it must contain the unit circle. But this $H(z)$ has a pole at $z=3$, which is outside the unit circle. So the only possible exterior ROC is $|z|>3$, which does not contain the unit circle. So it's impossible for the ROC to be exterior and include the unit circle.