ECE 2713
Test 2

Thursday, April 25, 2019
12:00 PM - 1:15 PM

Spring 2019 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. Calculators are allowed. All work must be your own. You have 75 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

and input

$$x[n] = 2\left(\frac{1}{4}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

Table: $H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

Table: $X(e^{j\omega}) = \dfrac{2}{1-\frac{1}{4}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

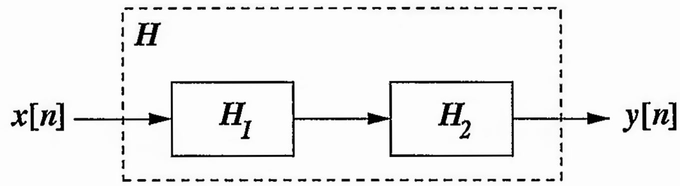
$$A = \left.\frac{2}{1-\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2}{1-\frac{1}{2}} = \frac{2}{1/2} = 4$$

$$B = \left.\frac{2}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2}{1-2} = \frac{2}{-1} = -2$$

$$Y(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{j\omega}}$$

Table: $\boxed{y[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$

2. **25 pts.** The discrete-time LTI system H is a series connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below.



The impulse response of system H_1 is given by

$$h_1[n] = 15\left(\frac{1}{2}\right)^n u[n] - 14\left(\frac{1}{3}\right)^n u[n].$$

The impulse response of system H_2 is given by

$$h_2[n] = \left(-\frac{1}{4}\right)^n u[n].$$

(a) **12 pts.** Find the overall system frequency response $H(e^{j\omega})$.

Table: $$H_1(e^{j\omega}) = \frac{15}{1-\frac{1}{2}e^{-j\omega}} - \frac{14}{1-\frac{1}{3}e^{-j\omega}} = \frac{15\left(1-\frac{1}{3}e^{-j\omega}\right) - 14\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{15-5e^{-j\omega}-14+7e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)} = \frac{1+2e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

Table: $$H_2(e^{j\omega}) = \frac{1}{1+\frac{1}{4}e^{-j\omega}}$$

$$\boxed{H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{1+2e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1+\frac{1}{4}e^{-j\omega}\right)}}$$

Problem 2, cont...

(b) **13 pts.** Find the overall system impulse response $h[n]$.

$$H(e^{j\omega}) = \frac{1+2e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{4}e^{-j\omega})} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{3}e^{-j\omega}} + \frac{C}{1+\frac{1}{4}e^{-j\omega}}$$

$$A = \left.\frac{1+2\theta}{(1-\frac{1}{3}\theta)(1+\frac{1}{4}\theta)}\right|_{\theta=2} = \frac{1+4}{(1-\frac{2}{3})(1+\frac{1}{2})} = \frac{5}{(\frac{1}{3})(\frac{3}{2})} = \frac{5}{1/2} = 10$$

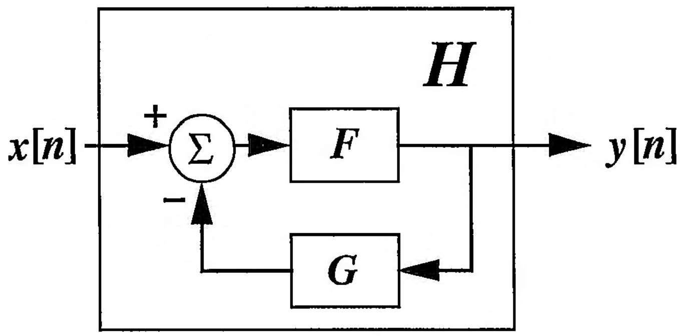
$$B = \left.\frac{1+2\theta}{(1-\frac{1}{2}\theta)(1+\frac{1}{4}\theta)}\right|_{\theta=3} = \frac{1+6}{(1-\frac{3}{2})(1+\frac{3}{4})} = \frac{7}{(-\frac{1}{2})(\frac{7}{4})} = \frac{7}{-7/8} = -8$$

$$C = \left.\frac{1+2\theta}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=-4} = \frac{1-8}{(1+2)(1+\frac{4}{3})} = \frac{-7}{3(7/3)} = \frac{-7}{7} = -1$$

$$H(e^{j\omega}) = \frac{10}{1-\frac{1}{2}e^{-j\omega}} - \frac{8}{1-\frac{1}{3}e^{-j\omega}} - \frac{1}{1+\frac{1}{4}e^{-j\omega}}$$

Table: $\boxed{h[n] = 10\left(\frac{1}{2}\right)^n u[n] - 8\left(\frac{1}{3}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]}$

3. **25 pts.** Consider the causal discrete-time LTI system H shown below.



F and G are both causal discrete-time LTI systems.

The impulse response of F is given by $f[n] = \left(\frac{1}{3}\right)^n u[n]$.

The impulse response of G is given by $g[n] = \left(\frac{1}{2}\right)^n u[n]$.

(a) **13 pts.** Find the overall system transfer function $H(z)$. You do **not** have to specify the ROC. You do **not** have to factor the denominator (it has complex roots).

Table: $F(z) = \dfrac{1}{1-\frac{1}{3}z^{-1}},\ |z| > \frac{1}{3}$ Table: $G(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}},\ |z| > \frac{1}{2}$

$$H(z) = \frac{F(z)}{1+F(z)G(z)} = \frac{\frac{1}{1-\frac{1}{3}z^{-1}}}{1+\frac{1}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}} \cdot \underbrace{\frac{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}}_{\text{ONE}}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})+1}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}-\frac{1}{2}z^{-1}+\frac{1}{6}z^{-2}+1} = \frac{1-\frac{1}{2}z^{-1}}{2-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$\boxed{H(z) = \frac{1-\frac{1}{2}z^{-1}}{2-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}}$$

Problem 3, cont...

(b) **12 pts.** Find the difference equation (I/O equation) that relates the overall system input $x[n]$ to the output $y[n]$.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2}z^{-1}}{2 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

Cross Multiply: $Y(z)\left[2 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right] = X(z)\left[1 - \frac{1}{2}z^{-1}\right]$

$$2Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$$

$\mathcal{Z}^{-1}$: $\boxed{2y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]}$

4. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by the difference equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n] - 4x[n-1].$$

(a) **7 pts.** Find the transfer function $H(z)$. **Note:** in this part you do not have enough information to specify the ROC.

$$Z: \quad Y(z) + \frac{5}{2}z^{-1}Y(z) - \frac{3}{2}z^{-2}Y(z) = X(z) - 4z^{-1}X(z)$$

$$Y(z)\left[1 + \frac{5}{2}z^{-1} - \frac{3}{2}z^{-2}\right] = X(z)\left[1 - 4z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-4z^{-1}}{1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}} = \frac{1-4z^{-1}}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

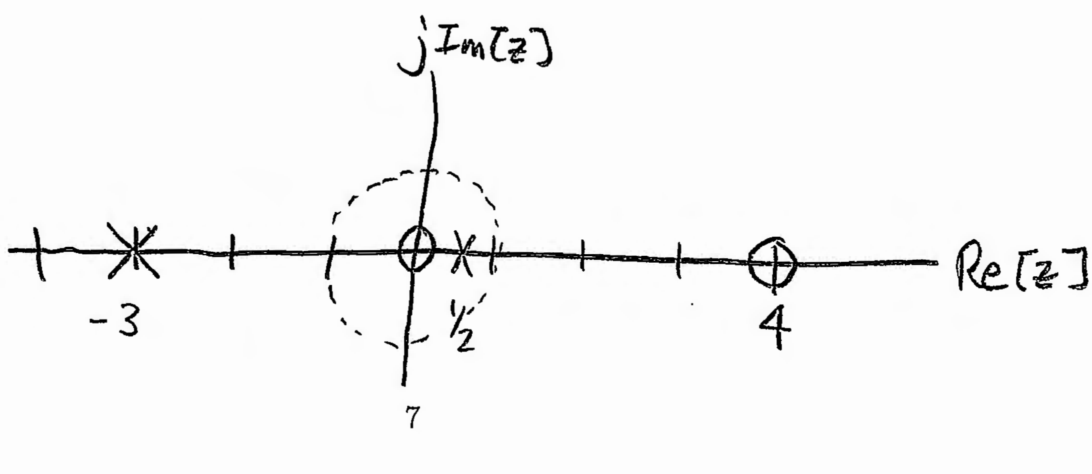
$$\boxed{H(z) = \frac{1-4z^{-1}}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}}$$

(b) **6 pts.** Give a pole-zero plot for $H(z)$. **Note:** in this part you still do not have enough information to specify the ROC.

$$H(z)\cdot\frac{z^2}{z^2} = \frac{z(z-4)}{(z+3)(z-\frac{1}{2})}$$

zeros: $z = 0, 4$

poles: $z = -3, \frac{1}{2}$



Problem 4, cont...

(c) **6 pts.** Now assume that the system H is **stable.** For this assumption, give the ROC of $H(z)$ and find the impulse response $h[n]$.

If H is stable, then the ROC of H(z) must include the unit circle. The ROC also cannot contain any poles. So the ROC of H(z) must be $\underline{\underline{\frac{1}{2} < |z| < 3}}$

– To find h[n], we must do a PFE on H(z):

$$H(z) = \frac{1-4z^{-1}}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{A}{1+3z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{1-4\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}} = \frac{1+\frac{4}{3}}{1+\frac{1}{6}} = \frac{7/3}{7/6} = \frac{6}{7}\cdot\frac{7}{3} = \frac{6}{3} = 2$$

$$B = \left.\frac{1-4\theta}{1+3\theta}\right|_{\theta=2} = \frac{1-8}{1+6} = \frac{-7}{7} = -1$$

$$H(z) = \underbrace{\frac{2}{1+3z^{-1}}}_{|z|<3} - \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table: $\boxed{h[n] = -2(-3)^n u[-n-1] - (\tfrac{1}{2})^n u[n]}$

Problem 4, cont...

(d) **6 pts.** Now assume instead that the system H is **causal**. For this assumption, give the ROC of $H(z)$ and find the impulse response $h[n]$.

For the system to be causal, the ROC of $H(z)$ must be exterior to the largest pole. So it must be: $\underline{\underline{|z|>3}}$.

- The PFE is the <u>same</u> as in part (c), but the ROCs/ for the individual terms are not the same. For this part, they must both be <u>exterior</u>.

$$H(z) = \underbrace{\frac{2}{1+3z^{-1}}}_{|z|>3} - \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}}$$

Table: $\boxed{h[n] = 2(-3)^n u[n] - \left(\tfrac{1}{2}\right)^n u[n]}$