ECE 2713
Test 2

Thursday, April 23, 2020
12:00 PM - 1.15 PM

Spring 2020 Name: SOLUTION

Dr. Havlicek Student Num: ____

Directions: This test is closed book and closed notes. Calculators are allowed. All work must be your own. You have 75 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ____
2. (25) ____
3. (25) ____
4. (25) ____

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____ Date: ____

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(-\frac{1}{2}\right)^n u[n]$$

and input

$$x[n] = 2\left(-\frac{1}{3}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

Table: $H(e^{j\omega}) = \dfrac{1}{1+\frac{1}{2}e^{-j\omega}}$

Table $X(e^{j\omega}) = \dfrac{2}{1+\frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{2}{(1+\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}$$

$$= \frac{A}{1+\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

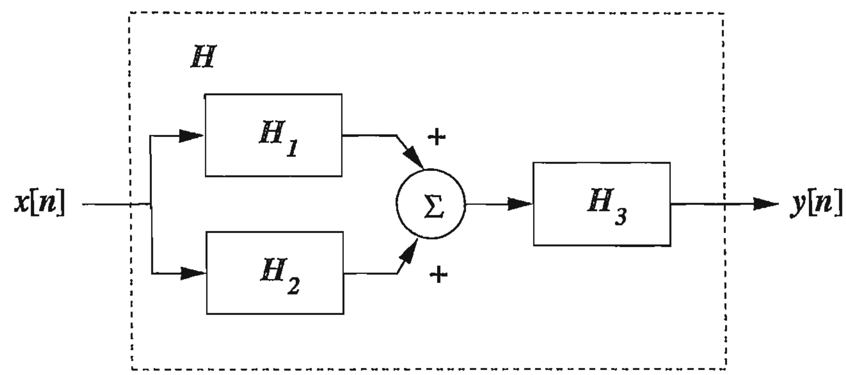
$$A = \left.\frac{2}{1+\frac{1}{3}\theta}\right|_{\theta=-2} = \frac{2}{1-2/3} = \frac{2}{1/3} = 6$$

$$B = \left.\frac{2}{1+\frac{1}{2}\theta}\right|_{\theta=-3} = \frac{2}{1-\frac{3}{2}} = \frac{2}{-1/2} = -4$$

$$H(e^{j\omega}) = \frac{6}{1+\frac{1}{2}e^{j\omega}} - \frac{4}{1+\frac{1}{3}e^{j\omega}}$$

Table: $\boxed{h[n] = 6\left(-\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{3}\right)^n u[n]}$

2. **25 pts.** A discrete-time LTI system H is formed by connecting three discrete-time LTI systems H_1, H_2, and H_3 as shown in the figure below:



The impulse responses of the individual systems H_1, H_2, and H_3 are given by

$$h_1[n] = \frac{3}{2}\left(\frac{1}{2}\right)^n u[n],$$
$$h_2[n] = -\frac{1}{2}\left(\frac{1}{3}\right)^n u[n],$$
$$h_3[n] = \left(\frac{1}{4}\right)^n u[n].$$

Table:

$$H_1(e^{j\omega}) = \frac{3/2}{1-\frac{1}{2}e^{-j\omega}}$$

$$H_2(e^{j\omega}) = \frac{-1/2}{1-\frac{1}{3}e^{-j\omega}}$$

$$H_3(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

(a) **12 pts.** Find the overall system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \left[H_1(e^{j\omega}) + H_2(e^{j\omega})\right]H_3(e^{j\omega})$$

$$= \left[\frac{3/2}{1-\frac{1}{2}e^{-j\omega}} - \frac{1/2}{1-\frac{1}{3}e^{-j\omega}}\right]\frac{1}{1-\frac{1}{4}e^{-j\omega}} = \left[\frac{3/2\left(1-\frac{1}{3}e^{-j\omega}\right)-\frac{1}{2}\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}\right] \times \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{\frac{3}{2} - \frac{1}{2}e^{-j\omega} - \frac{1}{2} + \frac{1}{4}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{\cancel{1-\frac{1}{4}e^{-j\omega}}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)\cancel{\left(1-\frac{1}{4}e^{-j\omega}\right)}} = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}}$$

Problem 2, cont ..

(b) **13 pts.** Find the overall system impulse response $h[n]$

$$H(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{3}e^{-j\omega}}$$

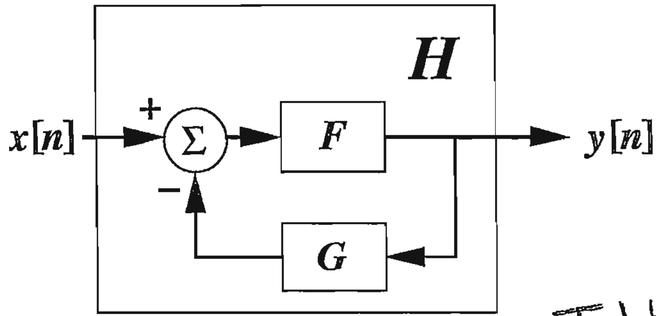
$$A = \left.\frac{1}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

$$B = \left.\frac{1}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{1}{1-\frac{3}{2}} = \frac{1}{-1/2} = -2$$

$$H(e^{j\omega}) = \frac{3}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

Table; $\boxed{h[n] = 3(\tfrac{1}{2})^n u[n] - 2(\tfrac{1}{3})^n u[n]}$

3. **25 pts**. Consider the causal discrete-time LTI system H shown below.



F and G are both causal discrete-time LTI systems.
The impulse response of F is given by $f[n] = \left(\frac{1}{4}\right)^n u[n]$.
The impulse response of G is given by $g[n] = -\frac{5}{8}\delta[n]$.

Table

$$F(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

$$G(z) = -\frac{5}{8}, \quad \text{all } z$$

(a) **9 pts**. Find the overall system transfer function $H(z)$.

$$H(z) = \frac{F(z)}{1+F(z)G(z)} = \frac{\frac{1}{1-\frac{1}{4}z^{-1}}}{1 - \frac{5/8}{1-\frac{1}{4}z^{-1}}} \cdot \underbrace{\frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}}_{\text{one}}$$

$$= \frac{1}{1-\frac{1}{4}z^{-1} - 5/8} = \frac{1}{\frac{3}{8} - \frac{1}{4}z^{-1}}$$

$$= \frac{1}{\frac{3}{8} - \frac{1}{4}z^{-1}} \cdot \frac{8/3}{8/3} = \frac{8/3}{1-\frac{2}{3}z^{-1}}, \quad |z| > \frac{2}{3}$$

$$\boxed{H(z) = \frac{8/3}{1-\frac{2}{3}z^{-1}}, \quad |z| > \frac{2}{3}}$$

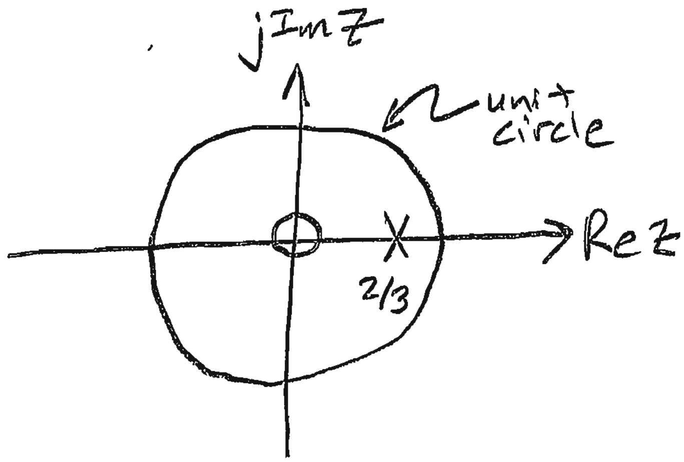
(The ROC must be exterior because H is given to be causal)

Problem 3, cont. .

(b) **8 pts.** Give a pole-zero plot and specify the ROC of $H(z)$.

From H(z) in part (a), there is a pole @ $z = 2/3$.

$H(z)\frac{z}{z} = \frac{z}{z - \frac{2}{3}} \longrightarrow$ there is a zero @ $z=0$



The ROC is $|z| > \frac{2}{3}$

$\rightarrow$ must be <u>exterior</u> because H is given to be causal.

(c) **8 pts.** Find the impulse response $h[n]$.

Table: $\boxed{h[n] = \frac{8}{3}\left(\frac{2}{3}\right)^n u[n]}$

4. **25 pts** The input $x[n]$ and output $y[n]$ of a **causal**, **stable** discrete-time LTI system H are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] - \frac{3}{16}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

(a) **10 pts.** Find the transfer function $H(z)$. Be sure to specify the ROC.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) - \frac{3}{16}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}\right] = X(z)\left[1 - \frac{1}{2}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{3}{16}z^{-2}} = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-1})(1-\frac{3}{4}z^{-1})}$$

H is given to be causal $\rightarrow$ ROC must be exterior to the largest pole... must be $|z| > \frac{3}{4}$

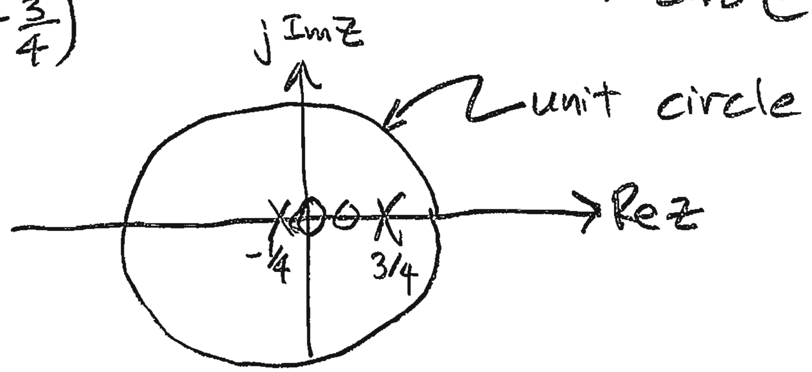
$$\boxed{H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-1})(1-\frac{3}{4}z^{-1})}, \quad |z| > \frac{3}{4}}$$

(b) **5 pts.** Give a pole-zero plot for $H(z)$

From $H(z)$ in part (a) there are poles @ $z = -\frac{1}{4}, \frac{3}{4}$

zeros @ $z = 1/2$

$$H(z)\frac{z^2}{z^2} = \frac{z(z-\frac{1}{2})}{(z+\frac{1}{4})(z-\frac{3}{4})}$$ $\rightarrow$ There is another zero @ $z=0$



Problem 4, cont...

(c) **10 pts**. Find the impulse response $h[n]$.

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1+\frac{1}{4}z^{-1})(1-\frac{3}{4}z^{-1})} = \frac{A}{1+\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{3}{4}z^{-1}}$$

$$A = \left.\frac{1-\frac{1}{2}\theta}{1-\frac{3}{4}\theta}\right|_{\theta=-4} = \frac{1+2}{1+3} = \frac{3}{4}$$

$$B = \left.\frac{1-\frac{1}{2}\theta}{1+\frac{1}{4}\theta}\right|_{\theta=\frac{4}{3}} = \frac{1-\frac{1}{2}\cdot\frac{4}{3}}{1+\frac{1}{4}\cdot\frac{4}{3}} = \frac{1-\frac{2}{3}}{1+\frac{1}{3}} = \frac{1/3}{4/3} = \frac{1}{4}$$

$$H(z) = \underbrace{\frac{3/4}{1+\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} + \underbrace{\frac{1/4}{1-\frac{3}{4}z^{-1}}}_{|z|>\frac{3}{4}}, \quad |z| > \frac{3}{4}$$

Table: $\boxed{h[n] = \frac{3}{4}\left(-\frac{1}{4}\right)^n u[n] + \frac{1}{4}\left(\frac{3}{4}\right)^n u[n]}$