ECE 2713
Test 2

Monday, April 26 – Thursday, April 29, 2021

Spring 2021

Dr. Havlicek

Name: SOLUTION

Student Num.__________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. All work must be your own You may work the test on this test paper or you may use your own blank paper. Upload a scan or photograph of your test paper to the course Canvas page no later than 11:59 PM on Thursday, April 29, 2021.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(\frac{1}{3}\right)^n u[n]$$

and input

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$

Table: $H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{3}e^{-j\omega}}$; $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

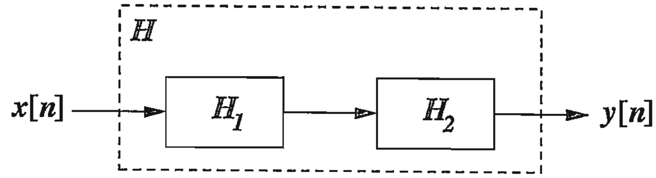
$$A = \left.\frac{1}{1-\frac{1}{4}\theta}\right|_{\theta=3} = \frac{1}{1-3/4} = \frac{1}{1/4} = 4$$

$$B = \left.\frac{1}{1-\frac{1}{3}\theta}\right|_{\theta=4} = \frac{1}{1-4/3} = \frac{1}{-1/3} = -3$$

$$Y(e^{j\omega}) = \frac{4}{1-\frac{1}{3}e^{-j\omega}} - \frac{3}{1-\frac{1}{4}e^{-j\omega}}$$

Table: $\boxed{y[n] = 4\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n]}$

2. **25 pts.** The discrete-time LTI system H is a series connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below.



The impulse response of system H_1 is given by

$$h_1[n] = \left(-\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1].$$

The impulse response of system H_2 is given by

$$h_2[n] = \left(\frac{1}{3}\right)^n u[n].$$

Table:

$$H_2(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

(a) **12 pts.** Find the overall system frequency response $H(e^{j\omega})$.

Table: $\left(-\frac{1}{2}\right)^n u[n] \leftrightarrow \dfrac{1}{1+\frac{1}{2}e^{-j\omega}}$; $\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

Time shift: $\left(\frac{1}{2}\right)^{n-1} u[n-1] \leftrightarrow \dfrac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

$$H_1(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}} - \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega} - e^{-j\omega}\left(1+\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega} - e^{-j\omega} - \frac{1}{2}e^{-j2\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)} = \frac{1-\frac{3}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{1-\frac{3}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$\boxed{H(e^{j\omega}) = \frac{1-\frac{3}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}}$$

Problem 2, cont...

(b) **13 pts.** Find the overall system impulse response $h[n]$.

$$H(\theta) = \frac{1-\frac{3}{2}\theta-\frac{1}{2}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1+\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{2}\theta} + \frac{C}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1-\frac{3}{2}\theta-\frac{1}{2}\theta^2}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=-2} = \frac{1+3-\frac{1}{2}(4)}{(1+1)(1+2/3)} = \frac{4-2}{(2)(5/3)} = \frac{2}{10/3}$$

$$= \frac{1}{5/3} = 3/5 \,//$$

$$B = \left.\frac{1-\frac{3}{2}\theta-\frac{1}{2}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=2} = \frac{1-3-\frac{1}{2}(4)}{(1+1)(1-\frac{2}{3})} = \frac{-2-2}{2(1/3)} = \frac{-4}{2(1/3)} = \frac{-2}{1/3}$$

$$= 3(-2) = -6 \,//$$

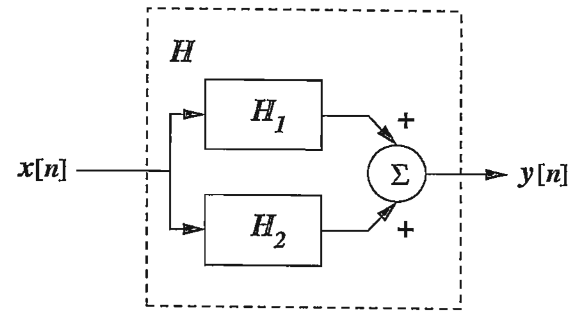
$$C = \left.\frac{1-\frac{3}{2}\theta-\frac{1}{2}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=3} = \frac{1-9/2-9/2}{(1+\frac{3}{2})(1-\frac{3}{2})} = \frac{1-9}{(\frac{5}{2})(-\frac{1}{2})} = \frac{-8}{-5/4}$$

$$= \left(-\frac{4}{5}\right)(-8) = 32/5 \,//$$

$$H(e^{j\omega}) = \frac{3/5}{1+\frac{1}{2}e^{-j\omega}} - \frac{6}{1-\frac{1}{2}e^{-j\omega}} + \frac{32/5}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{h[n] = \frac{3}{5}\left(-\frac{1}{2}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] + \frac{32}{5}\left(\frac{1}{3}\right)^n u[n]}$

3. **25 pts.** The discrete-time LTI system H is a parallel connection of two discrete-time LTI systems H_1 and H_2 as shown in the figure below



The impulse response of system H_1 is given by

$$h_1[n] = (n+1)\left(\frac{1}{2}\right)^n u[n].$$

The impulse response of system H_2 is given by

$$h_2[n] = \left(\frac{1}{4}\right)^n u[n].$$

Find the difference equation (I/O equation) that relates the input $x[n]$ and output $y[n]$ of the overall system H.

Table: $H_1(e^{j\omega}) = \dfrac{1}{(1-\frac{1}{2}e^{-j\omega})^2}$; $H_2(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$= \frac{(1-\frac{1}{4}e^{-j\omega}) + (1-\frac{1}{2}e^{-j\omega})^2}{(1-\frac{1}{2}e^{-j\omega})^2(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega} + (1 - \frac{1}{2}e^{-j\omega} - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega})}{(1-\frac{1}{2}e^{-j\omega} - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega} + 1 - e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}{(1-e^{-j\omega}+\frac{1}{4}e^{-j2\omega})(1-\frac{1}{4}e^{-j\omega})} \longrightarrow$$

More Workspace for Problem 3. .

$$\ldots H(e^{j\omega}) = \frac{2 - \frac{5}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}{(1 - e^{-j\omega} + \frac{1}{4}e^{-j2\omega}) - \frac{1}{4}e^{-j\omega}(1 - e^{-j\omega} + \frac{1}{4}e^{-j2\omega})}$$

$$= \frac{2 - \frac{5}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}{1 - e^{-j\omega} + \frac{1}{4}e^{-j2\omega} - \frac{1}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} - \frac{1}{16}e^{-j3\omega}}$$

$$H(e^{j\omega}) = \frac{2 - \frac{5}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}{1 - \frac{5}{4}e^{-j\omega} + \frac{1}{2}e^{-j2\omega} - \frac{1}{16}e^{-j3\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$X(e^{j\omega})\left[2 - \frac{5}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}\right] = Y(e^{j\omega})\left[1 - \frac{5}{4}e^{-j\omega} + \frac{1}{2}e^{-j2\omega} - \frac{1}{16}e^{-j3\omega}\right]$$

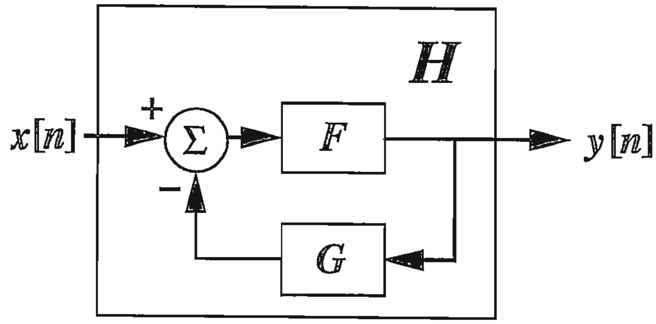
$$2X(e^{j\omega}) - \frac{5}{4}e^{-j\omega}X(e^{j\omega}) + \frac{1}{4}e^{-j2\omega}X(e^{j\omega})$$

$$= Y(e^{j\omega}) - \frac{5}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{2}e^{-j2\omega}Y(e^{j\omega}) - \frac{1}{16}e^{-j3\omega}Y(e^{j\omega})$$

Time Shift Property:

$$\boxed{y[n] - \frac{5}{4}y[n-1] + \frac{1}{2}y[n-2] - \frac{1}{16}y[n-3] = 2x[n] - \frac{5}{4}x[n-1] + \frac{1}{4}x[n-2]}$$

4. **25 pts.** Consider the discrete-time LTI system H shown below.



F and G are both discrete-time LTI systems.
The impulse response of F is given by $f[n] = \left(\frac{1}{3}\right)^n u[n]$.
The impulse response of G is given by $g[n] = \left(-\frac{1}{3}\right)^n u[n]$.

Table:

$$F(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$G(e^{j\omega}) = \frac{1}{1+\frac{1}{3}e^{-j\omega}}$$

(a) **13 pts.** Find the overall system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{F(e^{j\omega})}{1+F(e^{j\omega})G(e^{j\omega})} = \frac{\frac{1}{1-\frac{1}{3}e^{-j\omega}}}{1+\frac{1}{(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}}$$

$$H(e^{j\omega}) \cdot \underbrace{\left[\frac{(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}\right]}_{\text{one}} = \frac{1+\frac{1}{3}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})+1}$$

$$= \frac{1+\frac{1}{3}e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega}+\frac{1}{3}e^{-j\omega}-\frac{1}{9}e^{-j2\omega})+1} = \frac{1+\frac{1}{3}e^{-j\omega}}{2-\frac{1}{9}e^{-j2\omega}}$$

$$\boxed{H(e^{j\omega}) = \frac{1+\frac{1}{3}e^{-j\omega}}{2-\frac{1}{9}e^{-j2\omega}}}$$

Problem 4, cont..

(b) **12 pts.** Find the difference equation (I/O equation) that relates the input $x[n]$ and output $y[n]$ of the overall system H.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\frac{1}{3}e^{-j\omega}}{2-\frac{1}{9}e^{-j2\omega}}$$

$$Y(e^{j\omega})\left[2-\frac{1}{9}e^{-j2\omega}\right] = X(e^{j\omega})\left[1+\frac{1}{3}e^{-j\omega}\right]$$

$$2Y(e^{j\omega}) - \frac{1}{9}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{3}e^{-j\omega}X(e^{j\omega})$$

Time Shift Property:

$$\boxed{2y[n] - \frac{1}{9}y[n-2] = x[n] + \frac{1}{3}x[n-1]}$$