ECE 2713
Test 2

Thursday, April 20, 2023
12:00 PM - 1:15 PM

Spring 2023 — Name: SOLUTION

Dr. Havlicek — Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. All work must be your own. You have 75 minutes to complete the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________________ Date: ____________________

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(-\frac{1}{2}\right)^n u[n]$$

and input

$$x[n] = \left(\frac{1}{5}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

Table (with $a = -\frac{1}{2}$): $H(e^{j\omega}) = \dfrac{1}{1+\frac{1}{2}e^{-j\omega}}$

Table: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{5}e^{-j\omega}}$

DTFT convolution property:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\frac{1}{5}e^{-j\omega})(1+\frac{1}{2}e^{-j\omega})}$$

PFE:

$$\frac{1}{(1-\frac{1}{5}\theta)(1+\frac{1}{2}\theta)} = \frac{A}{1-\frac{1}{5}\theta} + \frac{B}{1+\frac{1}{2}\theta}$$

$$A = \left.\frac{1}{1+\frac{1}{2}\theta}\right|_{\theta=5} = \frac{1}{1+5/2} = \frac{1}{7/2} = \frac{2}{7}$$

$$B = \left.\frac{1}{1-\frac{1}{5}\theta}\right|_{\theta=-2} = \frac{1}{1+\frac{2}{5}} = \frac{1}{7/5} = \frac{5}{7}$$

$$Y(e^{j\omega}) = \frac{2/7}{1-\frac{1}{5}e^{-j\omega}} + \frac{5/7}{1+\frac{1}{2}e^{-j\omega}}$$

Table: $\boxed{y[n] = \frac{2}{7}\left(\frac{1}{5}\right)^n u[n] + \frac{5}{7}\left(-\frac{1}{2}\right)^n u[n]}$

2. **25 pts.** Consider the discrete-time LTI system H shown in the figure below:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

When the input signal $x[n]$ is given by

$$x[n] = \left(\frac{1}{4}\right)^n u[n],$$

the output signal $y[n]$ is observed to be

$$y[n] = (n+1)\left(\frac{3}{4}\right)^n u[n].$$

(a) **8 pts.** Use the DTFT to find the system frequency response $H(e^{j\omega})$.

Hint: $X(e^{j\omega})$ and $Y(e^{j\omega})$ can both be found in your DTFT table.

Table: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$ $\quad Y(e^{j\omega}) = \dfrac{1}{(1-\frac{3}{4}e^{-j\omega})^2}$

DTFT convolution property: $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

$$\rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left[\frac{1}{(1-\frac{3}{4}e^{-j\omega})^2}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{3}{4}e^{-j\omega})^2}}$$

Problem 2, cont...

(b) **6 pts.** Find the difference equation (I/O equation) for the system H.

Hint: the difference equation is the equation which says that a linear combination of the shifts of $y[n]$ is equal to a linear combination of the shifts of $x[n]$.

$$H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{3}{4}e^{-j\omega})(1-\frac{3}{4}e^{-j\omega})} = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}-\frac{3}{4}e^{-j\omega}+\frac{9}{16}e^{-j2\omega}}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{3}{2}e^{-j\omega}+\frac{9}{16}e^{-j2\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

cross multiply:

$$Y(e^{j\omega})\left[1-\frac{3}{2}e^{-j\omega}+\frac{9}{16}e^{-j2\omega}\right] = X(e^{j\omega})\left[1-\frac{1}{4}e^{-j\omega}\right]$$

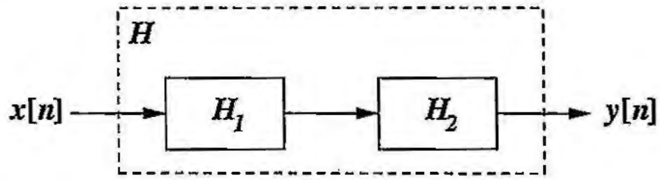
$$Y(e^{j\omega})-\frac{3}{2}e^{-j\omega}Y(e^{j\omega})+\frac{9}{16}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})-\frac{1}{4}e^{-j\omega}X(e^{j\omega})$$

IDTFT:

$$\boxed{y[n]-\frac{3}{2}y[n-1]+\frac{9}{16}y[n-2] = x[n]-\frac{1}{4}x[n-1]}$$

Problem 2, cont...

(c) **6 pts.** After further investigation, it is discovered that the system H from parts (a) and (b) is actually a series connection of two discrete-time LTI system H_1 and H_2 as shown in the figure below:



The impulse response of system H_1 is given by

$$h_1[n] = \left(\frac{3}{4}\right)^n u[n] - \frac{1}{2}\left(\frac{3}{4}\right)^{n-1} u[n-1].$$

Use the DTFT to find the frequency response $H_2(e^{j\omega})$ of the system H_2.

Hint: the overall system frequency response $H(e^{j\omega})$ is still exactly the same as it was in part (a). Use your DTFT table to find the transforms of the two terms in $h_1[n]$ (you will need to also use the time shift property for the second term) and combine them to find $H_1(e^{j\omega})$. Then use the series connection formula to combine this with $H(e^{j\omega})$ from part (a) and solve for $H_2(e^{j\omega})$.

Table: $\left(\frac{3}{4}\right)^n u[n] \xleftrightarrow{\text{DTFT}} \dfrac{1}{1-\frac{3}{4}e^{-j\omega}}$

Time Shift Property: $\frac{1}{2}\left(\frac{3}{4}\right)^{n-1} u[n-1] \xleftrightarrow{\text{DTFT}} \dfrac{\frac{1}{2}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}}$

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}} - \frac{\frac{1}{2}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}} = \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}}$$

SERIES CONNECTION: $H(e^{j\omega}) = H_1(e^{j\omega})\,H_2(e^{j\omega})$

$$\Rightarrow H_2(e^{j\omega}) = \frac{H(e^{j\omega})}{H_1(e^{j\omega})} = H(e^{j\omega}) \times \frac{1}{H_1(e^{j\omega})}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{\left(1-\frac{3}{4}e^{-j\omega}\right)^2} \times \frac{1-\frac{3}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$\boxed{H_2(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{\left(1-\frac{3}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}}$$

Problem 2, cont...

(d) **5 pts.** Find the impulse response $h_2[n]$ of system H_2.

PFE on $H_2(e^{j\omega})$:

$$\frac{1-\frac{1}{4}\theta}{(1-\frac{3}{4}\theta)(1-\frac{1}{2}\theta)} = \frac{A}{1-\frac{3}{4}\theta} + \frac{B}{1-\frac{1}{2}\theta}$$

$$A = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=4/3} = \frac{1-\frac{1}{3}}{1-2/3} = \frac{2/3}{1/3} = 2$$

$$B = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{3}{4}\theta}\right|_{\theta=2} = \frac{1-1/2}{1-3/2} = \frac{1/2}{-1/2} = -1$$

$$H_2(e^{j\omega}) = \frac{2}{1-\frac{3}{4}e^{-j\omega}} - \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Table: $\boxed{h_2[n] = 2\left(\frac{3}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n]}$

3. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] + \frac{1}{2}x[n-1].$$

(a) **10 pts.** Use the DTFT to find the system frequency response $H(e^{j\omega})$.

DTFT: $Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-j2\omega}Y(e^{j\omega})$

$= X(e^{j\omega}) + \frac{1}{2}e^{-j\omega}X(e^{j\omega})$

$$Y(e^{j\omega})\left[1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{2}e^{-j\omega}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}}$$

(b) **5 pts.** Is the system H an IIR digital filter or an FIR digital filter? Give a brief explanation to justify your answer.

IIR $\longrightarrow$ because the difference equation has shifts of $y[n]$.

OR: because $H(e^{j\omega})$ has a nontrivial denominator polynomial.

OR: because in part (c) we see that $h[n]$ is not finite length.

Problem 3, cont...

(c) **10 pts.** Find the system impulse response $h[n]$.

$$H(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-j2\omega}} = \frac{1+\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

PFE:

$$\frac{1+\frac{1}{2}\theta}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1+\frac{1}{2}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{1+1}{1-2/3} = \frac{2}{1/3} = 3\cdot 2 = 6$$

$$B = \left.\frac{1+\frac{1}{2}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{1+\frac{3}{2}}{1-\frac{3}{2}} = \frac{5/2}{-1/2} = -5$$

$$H(e^{j\omega}) = \frac{6}{1-\frac{1}{2}e^{-j\omega}} - \frac{5}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{h[n] = 6(\tfrac{1}{2})^n u[n] - 5(\tfrac{1}{3})^n u[n]}$

4. **25 pts.** A **linear phase** FIR digital filter H has impulse response

$$h[n] = \frac{1}{4}\delta[n] - \frac{1}{2}\delta[n-1] + 2\delta[n-2] - \frac{1}{2}\delta[n-3] + \frac{1}{4}\delta[n-4].$$

(a) **10 pts.** Use the DTFT to find the filter frequency response $H(e^{j\omega})$.

Table: (plus time shift property)

$$\boxed{H(e^{j\omega}) = \frac{1}{4} - \frac{1}{2}e^{-j\omega} + 2e^{-j2\omega} - \frac{1}{2}e^{-j3\omega} + \frac{1}{4}e^{-j4\omega}}$$

Problem 4, cont...

(b) **15 pts.** Find the filter magnitude response $|H(e^{j\omega})|$ and phase response $\arg H(e^{j\omega})$.

Hint: use the FIR "linear phase trick" to write $H(e^{j\omega})$ in polar form and then simply read off the magnitude response and phase response.

Highest power of the character: $e^{-j4\omega}$

Half the highest power: $e^{-j2\omega} \rightarrow$ Factor out $e^{-j2\omega}$

$$H(e^{j\omega}) = \left[\tfrac{1}{4}e^{j2\omega} - \tfrac{1}{2}e^{j\omega} + 2 - \tfrac{1}{2}e^{-j\omega} + \tfrac{1}{4}e^{-j2\omega}\right]e^{-j2\omega}$$

$$= \left[2 - \tfrac{1}{2}(e^{j\omega} + e^{-j\omega}) + \tfrac{1}{4}(e^{j2\omega} + e^{-j2\omega})\right]e^{-j2\omega}$$

$$= \underbrace{\left[2 - \cos\omega + \tfrac{1}{2}\cos(2\omega)\right]}_{|H(e^{j\omega})|}\ \underbrace{e^{-j2\omega}}_{e^{j\arg H(e^{j\omega})}}$$

$$\boxed{\begin{aligned} |H(e^{j\omega})| &= 2 - \cos\omega + \tfrac{1}{2}\cos(2\omega) \\ \arg H(e^{j\omega}) &= -2\omega \end{aligned}}$$