ECE 2713
Test 2

Thursday, April 25, 2024
12:00 PM - 1:15 PM

Spring 2024

Dr. Havlicek

Name: SOLUTION

Student Num:__________

Directions: This test is closed book and closed notes. Calculators are allowed. You may also use the formula sheet provided with the test. All work must be your own. You have 75 minutes to complete the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ______

2. (25) ______

3. (25) ______

4. (25) ______

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A discrete-time LTI system H has impulse response

$$h[n] = \left(-\frac{1}{3}\right)^n u[n]$$

and input

$$x[n] = 4\left(\frac{1}{9}\right)^n u[n].$$

Use the discrete-time Fourier transform (DTFT) to find the output signal $y[n]$.

Table: $H(e^{j\omega}) = \dfrac{1}{1+\frac{1}{3}e^{-j\omega}}$

Table: $X(e^{j\omega}) = \dfrac{4}{1-\frac{1}{9}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{4}{(1-\frac{1}{9}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}$$

PFE: $\dfrac{4}{(1-\frac{1}{9}\theta)(1+\frac{1}{3}\theta)} = \dfrac{A}{1-\frac{1}{9}\theta} + \dfrac{B}{1+\frac{1}{3}\theta}$

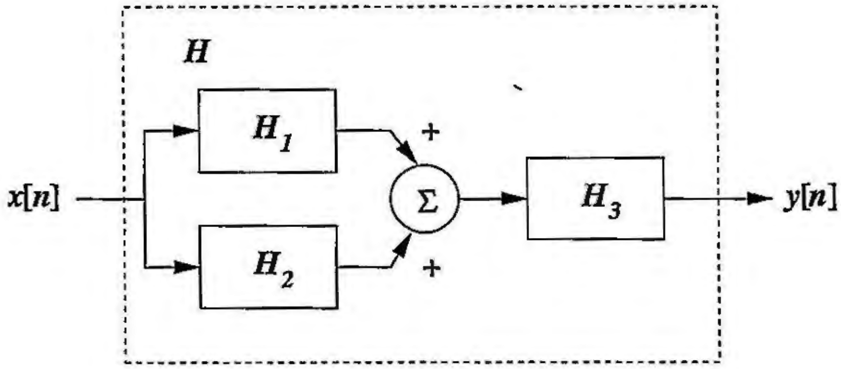
$$A = \left.\frac{4}{1+\frac{1}{3}\theta}\right|_{\theta=9} = \frac{4}{1+3} = \frac{4}{4} = 1$$

$$B = \left.\frac{4}{1-\frac{1}{9}\theta}\right|_{\theta=-3} = \frac{4}{1+\frac{1}{3}} = \frac{4}{4/3} = 4\cdot\frac{3}{4} = 3$$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{9}e^{-j\omega}} + \frac{3}{1+\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{y[n] = \left(\frac{1}{9}\right)^n u[n] + 3\left(-\frac{1}{3}\right)^n u[n]}$

2. **25 pts.** A discrete-time LTI system H is formed by connecting three discrete-time LTI systems H_1, H_2, and H_3 as shown in the figure below:



The impulse responses of the individual systems H_1, H_2, and H_3 are given by

$$h_1[n] = 3\left(-\frac{1}{2}\right)^n u[n], \qquad h_2[n] = 4\left(\frac{1}{3}\right)^{n-1} u[n-1], \qquad h_3[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[n].$$

(a) **12 pts.** Find the overall system frequency response $H(e^{j\omega})$.

Table: $H_1(e^{j\omega}) = \dfrac{3}{1+\frac{1}{2}e^{-j\omega}}$; $H_3(e^{j\omega}) = \dfrac{1/3}{1-\frac{1}{2}e^{-j\omega}}$

Table + Time Shift Property: $H_2(e^{j\omega}) = \dfrac{4e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}}$

$$H(e^{j\omega}) = \left[H_1(e^{j\omega}) + H_2(e^{j\omega})\right] H_3(e^{j\omega})$$

$$= \left[\frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{4e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}}\right]\frac{1/3}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \left[\frac{3(1-\frac{1}{3}e^{-j\omega}) + 4e^{-j\omega}(1+\frac{1}{2}e^{-j\omega})}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}\right]\frac{1/3}{1-\frac{1}{2}e^{-j\omega}}$$

$$= \frac{(3 - e^{-j\omega} + 4e^{-j\omega} + 2e^{-j2\omega})\frac{1}{3}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} = \frac{1+e^{-j\omega}+\frac{2}{3}e^{-j2\omega}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$$\boxed{H(e^{j\omega}) = \frac{1+e^{-j\omega}+\frac{2}{3}e^{-j2\omega}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}}$$

Problem 2, cont...

(b) **13 pts.** Find the overall system impulse response $h[n]$.

PFE: $$\frac{1+\theta+\frac{2}{3}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1+\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{2}\theta} + \frac{C}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1+\theta+\frac{2}{3}\theta^2}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=-2} = \frac{1-2+\frac{8}{3}}{(1+1)(1+\frac{2}{3})} = \frac{\frac{-3+8}{3}}{2(5/3)} = \frac{5/3}{2(5/3)} = \frac{1}{2}$$

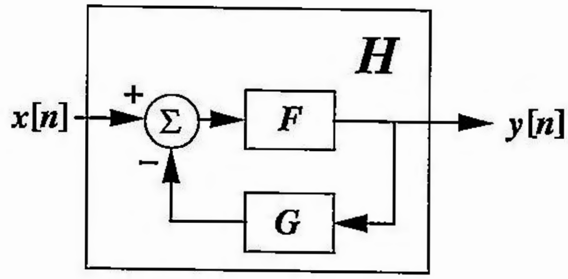
$$B = \left.\frac{1+\theta+\frac{2}{3}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=2} = \frac{1+2+8/3}{(1+1)(1-\frac{2}{3})} = \frac{\frac{9+8}{3}}{2(1/3)} = \frac{17/3}{2/3} = \frac{17}{2}$$

$$C = \left.\frac{1+\theta+\frac{2}{3}\theta^2}{(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=3} = \frac{1+3+6}{(1+\frac{3}{2})(1-\frac{3}{2})} = \frac{10}{(\frac{5}{2})(-\frac{1}{2})} = \frac{10}{-\frac{5}{4}} = \frac{-40}{5} = -8$$

$$H(e^{j\omega}) = \frac{1/2}{(1+\frac{1}{2}e^{-j\omega})} + \frac{17/2}{(1-\frac{1}{2}e^{-j\omega})} - \frac{8}{(1-\frac{1}{3}e^{-j\omega})}$$

Table: $$\boxed{h[n] = \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n] + \frac{17}{2}\left(\frac{1}{2}\right)^n u[n] - 8\left(\frac{1}{3}\right)^n u[n]}$$

3. **25 pts.** Consider the discrete-time LTI system H shown below.



F and G are both discrete-time LTI systems.

Table:

The impulse response of F is given by $f[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]$. $F(e^{j\omega}) = \dfrac{1}{(1-\frac{1}{2}e^{-j\omega})^2}$

The impulse response of G is given by $g[n] = \left(\frac{1}{4}\right)^n u[n]$. $G(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$

(a) **11 pts.** Find the overall system frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{F(e^{j\omega})}{1+F(e^{j\omega})G(e^{j\omega})} = \frac{\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2}}{1+\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2}\,\frac{1}{1-\frac{1}{4}e^{-j\omega}}}$$

$$= \frac{\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2}}{1+\frac{1}{(1-\frac{1}{2}e^{-j\omega})^2(1-\frac{1}{4}e^{-j\omega})}} \cdot \underbrace{\frac{(1-\frac{1}{2}e^{-j\omega})^2(1-\frac{1}{4}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})^2(1-\frac{1}{4}e^{-j\omega})}}_{\text{one}}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2(1-\frac{1}{4}e^{-j\omega})+1} = \frac{1-\frac{1}{4}e^{-j\omega}}{(1-e^{-j\omega}+\frac{1}{4}e^{-j2\omega})(1-\frac{1}{4}e^{-j\omega})+1}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-e^{-j\omega}+\frac{1}{4}e^{-j2\omega}+\frac{1}{4}e^{-j2\omega}-\frac{1}{16}e^{-j3\omega}+1}$$

$$\boxed{H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{2-\frac{5}{4}e^{-j\omega}+\frac{1}{2}e^{-j2\omega}-\frac{1}{16}e^{-j3\omega}}}$$

Problem 3, cont...

(b) **11 pts.** Find the difference equation (I/O equation) that relates the input $x[n]$ and output $y[n]$ of the overall system H.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1-\frac{1}{4}e^{-j\omega}}{2-\frac{5}{4}e^{-j\omega}+\frac{1}{2}e^{-j2\omega}-\frac{1}{16}e^{-j3\omega}}$$

$$Y(e^{j\omega})\left[2-\frac{5}{4}e^{-j\omega}+\frac{1}{2}e^{-j2\omega}-\frac{1}{16}e^{-j3\omega}\right] = X(e^{j\omega})\left[1-\frac{1}{4}e^{-j\omega}\right]$$

$$2Y(e^{j\omega})-\frac{5}{4}e^{-j\omega}Y(e^{j\omega})+\frac{1}{2}e^{-j2\omega}Y(e^{j\omega})-\frac{1}{16}e^{-j3\omega}Y(e^{j\omega})$$
$$= X(e^{j\omega})-\frac{1}{4}e^{-j\omega}X(e^{j\omega})$$

IDTFT:

$$\boxed{2y[n]-\frac{5}{4}y[n-1]+\frac{1}{2}y[n-2]-\frac{1}{16}y[n-3] = x[n]-\frac{1}{4}x[n-1]}$$

(c) **3 pts.** Is H an FIR digital filter or an IIR digital filter? Briefly justify your answer.

$\Rightarrow$ IIR because the I/O equation has shifts of $y[n]$.

$\Rightarrow$ IIR because $H(e^{j\omega})$ has a nontrivial denominator

[either answer is full credit]

4. **25 pts.** Consider a digital filter H with difference equation (I/O equation) given by

$$y[n] = -\frac{1}{4}x[n] + \frac{1}{2}x[n-1] - x[n-2] + 4x[n-3] - x[n-4] + \frac{1}{2}x[n-5] - \frac{1}{4}x[n-6].$$

(a) **3 pts.** Is H an FIR digital filter or an IIR digital filter? Briefly justify your answer.

FIR because the difference equation has no shifts of $y[n]$.

(b) **6 pts.** Find the system impulse response $h[n]$.

$$\boxed{h[n] = -\tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] - \delta[n-2] + 4\delta[n-3] - \delta[n-4] + \tfrac{1}{2}\delta[n-5] - \tfrac{1}{4}\delta[n-6]}$$

(c) **6 pts.** Use the DTFT to find the filter frequency response $H(e^{j\omega})$.

Table:

$$\boxed{H(e^{j\omega}) = -\tfrac{1}{4} + \tfrac{1}{2}e^{-j\omega} - e^{-j2\omega} + 4e^{-j3\omega} - e^{-j4\omega} + \tfrac{1}{2}e^{-j5\omega} - \tfrac{1}{4}e^{-j6\omega}}$$

Problem 4, cont...

(d) **10 pts.** Find the filter magnitude response $|H(e^{j\omega})|$ and phase response $\arg H(e^{j\omega})$.

Hint: use the "linear phase trick" to write $H(e^{j\omega})$ in polar form and then simply read off the magnitude response and phase response.

Highest Power $= e^{-j6\omega}$. Half the highest power $= e^{-j3\omega}$

$\rightarrow$ Factor out half the highest power

$$H(e^{j\omega}) = \left[-\tfrac{1}{4}e^{j3\omega} + \tfrac{1}{2}e^{j2\omega} - e^{j\omega} + 4 - e^{-j\omega} + \tfrac{1}{2}e^{-j2\omega} - \tfrac{1}{4}e^{-j3\omega}\right]e^{-j3\omega}$$

$$= \left[-\tfrac{1}{4}(e^{j3\omega} + e^{-j3\omega}) + \tfrac{1}{2}(e^{j2\omega} + e^{-j2\omega}) - (e^{j\omega} + e^{-j\omega}) + 4\right]e^{-j3\omega}$$

$$= \left[-\tfrac{1}{4}\cdot 2\cos(3\omega) + \tfrac{1}{2}\cdot 2\cos(2\omega) - 2\cos\omega + 4\right]e^{-j3\omega}$$

$$= \underbrace{\left[4 - 2\cos\omega + \cos(2\omega) - \tfrac{1}{2}\cos(3\omega)\right]}_{|H(e^{j\omega})|}\ \underbrace{e^{-j3\omega}}_{e^{j\arg H(e^{j\omega})}}$$

$$\boxed{\begin{aligned} |H(e^{j\omega})| &= 4 - 2\cos\omega + \cos(2\omega) - \tfrac{1}{2}\cos(3\omega) \\ \arg H(e^{j\omega}) &= -3\omega \end{aligned}}$$