4. **20 pts.** A **stable** discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by

$$y[n] + 2y[n-1] + \frac{3}{4}y[n-2] = x[n] - 2x[n-1].$$

(a) **5 pts.** Find the transfer function $H(z)$.

$z:\quad Y(z) + 2z^{-1}Y(z) + \frac{3}{4}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$

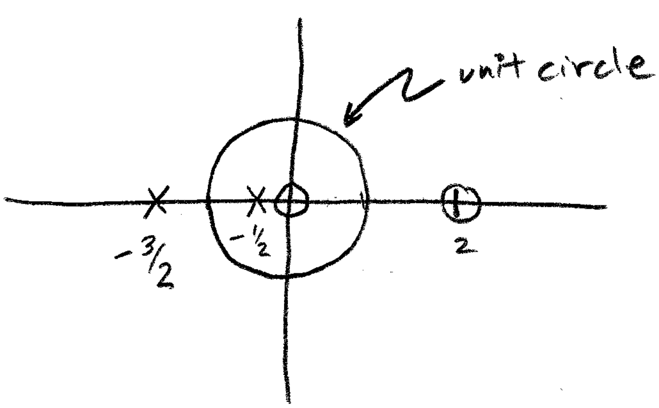
$$Y(z)\left[1 + 2z^{-1} + \frac{3}{4}z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1+2z^{-1}+\frac{3}{4}z^{-2}}$$

$$= \frac{1-2z^{-1}}{\left(1+\frac{3}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}$$

(b) **5 pts.** Give a pole-zero plot for $H(z)$ and specify the ROC.

poles: $z = -3/2,\ -1/2$ zero: $z = 2,\ z = 0.$



Because the system is stable, the ROC must include the unit circle.

ROC: $\frac{1}{2} < |z| < \frac{3}{2}$

Problem 4, cont...

(c) **10 pts**. Find the impulse response $h[n]$.

$$H(\theta) = \frac{1-2\theta}{(1+\frac{3}{2}\theta)(1+\frac{1}{2}\theta)} = \frac{A}{1+\frac{3}{2}\theta} + \frac{B}{1+\frac{1}{2}\theta}$$

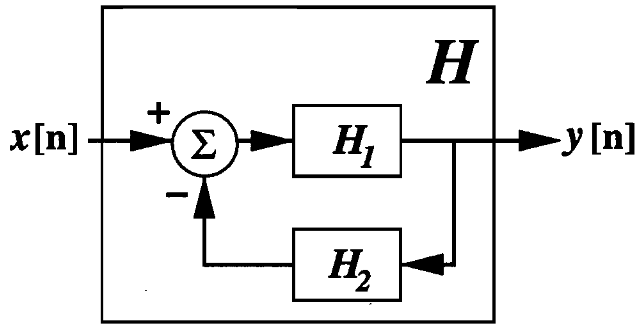
$$A = \left.\frac{1-2\theta}{1+\frac{1}{2}\theta}\right|_{\theta=\frac{-2}{3}} = \frac{1+\frac{4}{3}}{1-\frac{1}{3}} = \frac{7/3}{2/3} = \frac{7}{2}$$

$$B = \left.\frac{1-2\theta}{1+\frac{3}{2}\theta}\right|_{\theta=-2} = \frac{1+4}{1-3} = \frac{5}{-2} = -\frac{5}{2}$$

$$H(z) = \underbrace{\frac{7/2}{1+\frac{3}{2}z^{-1}}}_{\text{ROC: } |z| < 3/2} - \underbrace{\frac{5/2}{1+\frac{1}{2}z^{-1}}}_{\text{ROC: } |z| > \frac{1}{2}}$$

Table: $\boxed{h[n] = -\frac{7}{2}\left(-\frac{3}{2}\right)^n u[-n-1] - \frac{5}{2}\left(-\frac{1}{2}\right)^n u[n]}$

4. **25 pts**. The **causal** discrete-time system H is formed by connecting two discrete-time LTI systems H_1 and H_2 in a negative feedback configuration as shown in the figure below.



The impulse response of H_1 is given by

$$h_1[n] = \left(\frac{1}{\sqrt{2}}\right)^n u[n].$$

The impulse response of H_2 is given by

$$h_2[n] = \left(-\frac{1}{\sqrt{2}}\right)^n u[n].$$

(a) **2 pts**. Find the transfer function $H_1(z)$. Be sure to specify the ROC.

Table: $H_1(z) = \frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}$, $|z| > \frac{1}{\sqrt{2}}$

(b) **2 pts**. Find the transfer function $H_2(z)$. Be sure to specify the ROC.

Table: $H_2(z) = \frac{1}{1+\frac{1}{\sqrt{2}}z^{-1}}$, $|z| > \frac{1}{\sqrt{2}}$

Problem 4, cont...

(c) **6 pts.** Find the transfer function $H(z)$ of the overall system H.

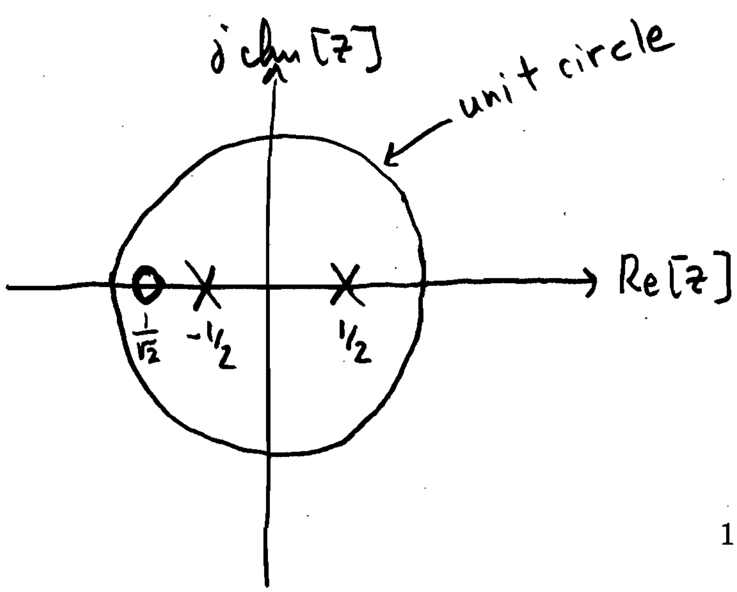
$$H(z) = \frac{H_1(z)}{1+H_1(z)H_2(z)} = \frac{\frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}}{1+\frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}}\,\frac{1}{1+\frac{1}{\sqrt{2}}z^{-1}}} \cdot \frac{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})}{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})}$$

$$= \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{(1-\frac{1}{\sqrt{2}}z^{-1})(1+\frac{1}{\sqrt{2}}z^{-1})+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{1-\frac{1}{2}z^{-2}+1} = \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2-\frac{1}{2}z^{-2}}$$

$$= \frac{1/2}{1/2} \cdot \frac{1+\frac{1}{\sqrt{2}}z^{-1}}{2-\frac{1}{2}z^{-2}} = \frac{\frac{1}{2}(1+\frac{1}{\sqrt{2}}z^{-1})}{1-\frac{1}{4}z^{-2}}$$

$$H(z) = \frac{\frac{1}{2}(1+\frac{1}{\sqrt{2}}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})}$$

(d) **3 pts.** Give a pole-zero plot for $H(z)$ and specify the ROC. **Note:** it was given above that the system H is **causal**.



zero: $-\frac{1}{\sqrt{2}}$

poles: $\pm\frac{1}{2}$

Since the system is causal, the ROC is exterior.

ROC: $|z| > \frac{1}{2}$

Problem 4, cont...

(e) **2 pts**. Is the system H BIBO stable? Justify your answer.

Yes. Because it has a rational transfer function and the ROC contains the unit circle.

Alternate Answer: Yes, because it's causal and the poles are inside the unit circle.

(f) **10 pts**. Find the overall system impulse response $h[n]$.

$$H(z) = \frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1-\frac{1}{2}\theta}\right|_{\theta=-2} = \frac{\frac{1}{2}\left(1+\frac{-2}{\sqrt{2}}\right)}{1+1} = \frac{1}{4}\left(1-\sqrt{2}\right)$$

$$B = \left.\frac{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\theta\right)}{1+\frac{1}{2}\theta}\right|_{\theta=2} = \frac{\frac{1}{2}\left(1+\frac{2}{\sqrt{2}}\right)}{1+1} = \frac{1}{4}\left(1+\sqrt{2}\right)$$

$$H(z) = \underbrace{\frac{1}{4}\left(1-\sqrt{2}\right)\frac{1}{1+\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{1}{4}\left(1+\sqrt{2}\right)\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>1/2}$$

Table: $$\boxed{h[n] = \frac{1}{4}\left(1-\sqrt{2}\right)\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{4}\left(1+\sqrt{2}\right)\left(\frac{1}{2}\right)^n u[n]}$$

4. **25 pts**. A discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by the linear constant coefficient difference equation

$$y[n] + \frac{1}{2}y[n-1] - \frac{3}{16}y[n-2] = x[n-1] - \frac{1}{2}x[n-2].$$

(a) **6 pts**. Find the transfer function $H(z)$.

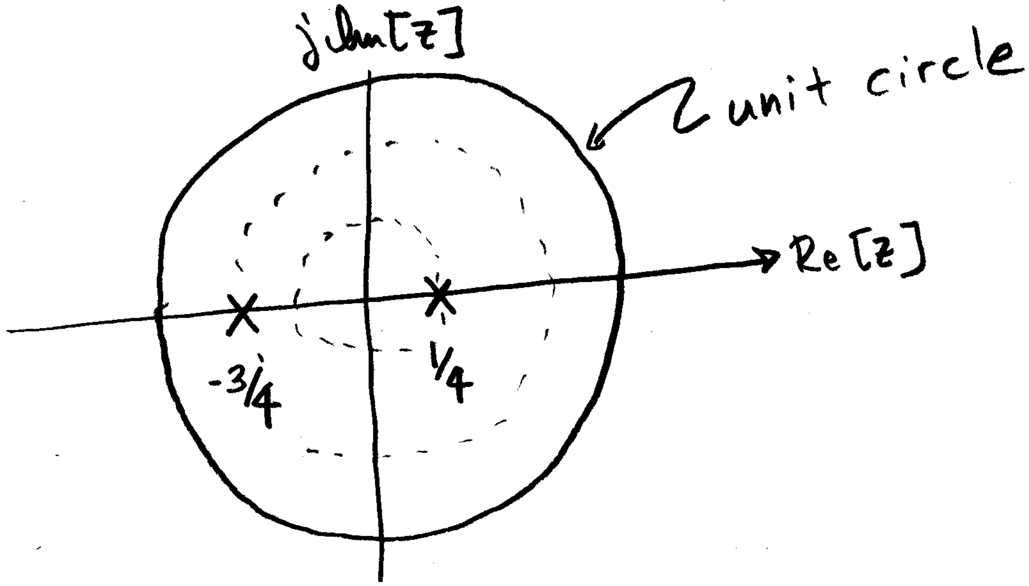
$$\mathcal{Z}: \quad Y(z) + \frac{1}{2}z^{-1}Y(z) - \frac{3}{16}z^{-2}Y(z) = z^{-1}X(z) - \frac{1}{2}z^{-2}X(z)$$

$$\left[1 + \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}\right]Y(z) = \left[z^{-1} - \frac{1}{2}z^{-2}\right]X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{2}z^{-1} - \frac{3}{16}z^{-2}} = \frac{z^{-1}(1 - \frac{1}{2}z^{-1})}{(1 + \frac{3}{4}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

$$\boxed{H(z) = \frac{z^{-1}(1 - \frac{1}{2}z^{-1})}{(1 + \frac{3}{4}z^{-1})(1 - \frac{1}{4}z^{-1})}}$$

(b) **3 pts**. Give a pole-zero plot for $H(z)$.



Problem 4, cont...

(c) **6 pts.** Assume that the system frequency response $H(e^{j\omega})$ exists. Give the ROC for $H(z)$ and find the system impulse response $h[n]$.

ROC must contain the unit circle: $|z| > \frac{3}{4}$

PFE on the proper fraction $\dfrac{1-\frac{1}{2}z^{-1}}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{4}z^{-1})}$:

$$\frac{1-\frac{1}{2}\theta}{(1+\frac{3}{4}\theta)(1-\frac{1}{4}\theta)} = \frac{A}{1+\frac{3}{4}\theta} + \frac{B}{1-\frac{1}{4}\theta}$$

$$A = \left.\frac{1-\frac{1}{2}\theta}{1-\frac{1}{4}\theta}\right|_{\theta=-\frac{4}{3}} = \frac{1+\frac{4}{6}}{1+\frac{1}{3}} = \frac{1+\frac{2}{3}}{4/3} = \frac{5/3}{4/3} = \frac{5}{4}$$

$$B = \left.\frac{1-\frac{1}{2}\theta}{1+\frac{3}{4}\theta}\right|_{\theta=4} = \frac{1-2}{1+3} = \frac{-1}{4}$$

$\Rightarrow z^{-1}$ gives a time shift

$$h[n] = \left.\frac{5}{4}\left(-\frac{3}{4}\right)^m u[m]\right|_{m=n-1} - \left.\frac{1}{4}\left(\frac{1}{4}\right)^m u[m]\right|_{m=n-1}$$

$$H(z) = \underbrace{\frac{5}{4}\,\frac{z^{-1}}{1+\frac{3}{4}z^{-1}}}_{|z|>3/4} - \underbrace{\frac{1}{4}\,\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}}$$

$$\boxed{h[n] = \frac{5}{4}\left(-\frac{3}{4}\right)^{n-1} u[n-1] - \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1]}$$

(d) **4 pts.** Under the assumption of part (c) — that $H(e^{j\omega})$ exists — is the system causal? Is it BIBO stable? Justify your answers.

Yes, it is causal, because $h[n]=0\ \forall n<0$.

Yes, it is BIBO stable, because it is causal and the poles of $H(z)$ are all inside the unit circle.

Problem 4, cont...

(e) **6 pts.** Now assume that the system H is unstable and that the impulse response $h[n]$ is two-sided. Give the ROC for $H(z)$ and find the impulse response $h[n]$.

Because $h[n]$ is two-sided, the ROC of $H(z)$ must be $\frac{1}{4} < |z| < 3/4$.

$$H(z) = \underbrace{\frac{5}{4} z^{-1} \frac{1}{1+\frac{3}{4}z^{-1}}}_{|z| < \frac{3}{4}} - \underbrace{\frac{1}{4} z^{-1} \frac{1}{1-\frac{1}{4}z^{-1}}}_{|z| > \frac{1}{4}}$$

Table, plus recognizing that multiplication by z^{-1} is a time shift by 1:

$$h[n] = \frac{5}{4}(-1)\left(-\frac{3}{4}\right)^m u[-m-1]\Big|_{m=n-1}$$

$$- \frac{1}{4}\left(\frac{1}{4}\right)^m u[m]\Big|_{m=n-1}$$

$$\boxed{h[n] = -\frac{5}{4}\left(-\frac{3}{4}\right)^{n-1} u[-n] - \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u[n-1]}$$

5. **25 pts**. Consider a discrete-time LTI system H. When the system input is the unit step sequence $u[n]$, the output is

$$y[n] = \left[2 - \left(\tfrac{1}{2}\right)^n\right] u[n].$$

Find the system transfer function $H(z)$. Is the system causal? Is it BIBO stable? *(justify your answers)*

$$x[n] = u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$y[n] = 2u[n] - \left(\tfrac{1}{2}\right)^n u[n]$$

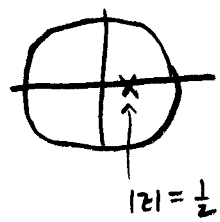
$$2u[n] \xleftrightarrow{\mathcal{Z}} \frac{2}{1-z^{-1}}, \quad |z| > 1$$

$$-\left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} -\frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$

$$Y(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2 - z^{-1} - 1 + z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \cdot \frac{1-z^{-1}}{1}, \quad |z| > 1$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$



The system is causal and stable because the ROC of $H(z)$ is exterior and includes the unit circle.

5. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by the difference equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n] - 4x[n-1].$$

(a) **10 pts.** Assuming that the system is **stable**, find the impulse response $h[n]$.

$$Y(z)\left[1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}\right] = X(z)\left[1-4z^{-1}\right]$$

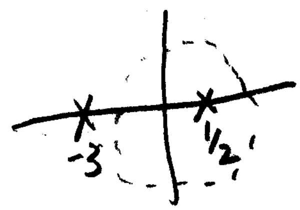
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-4z^{-1}}{1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}} = \frac{1-4z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+3z^{-1}}$$

$$A = \left.\frac{1-4\theta}{1+3\theta}\right|_{\theta=2} = \frac{-7}{7} = -1$$

$$B = \left.\frac{1-4\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}} = \frac{1+\frac{4}{3}}{1+\frac{1}{6}} = \frac{7/3}{7/6} = \frac{6}{3} = 2$$

$$X(z) = \frac{2}{1+3z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$



There are <u>three</u> possible ROC's:

$|z|>3$, $\frac{1}{2}<|z|<3$, and $|z|<\frac{1}{2}$.

For the system to be stable, the ROC must include the unit circle: $\frac{1}{2}<|z|<3$.

$$\frac{2}{1+3z^{-1}},\ |z|<3 \longleftrightarrow -2(-3)^n u[-n-1]$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}},\ |z|>\frac{1}{2} \longleftrightarrow -\left(\tfrac{1}{2}\right)^n u[n]$$

$$\underline{\underline{h[n] = -\left(\tfrac{1}{2}\right)^n u[n] - 2(-3)^n u[-n-1]}}$$

(b) **10 pts**. Now assume that the system is **causal**, but not necessarily stable. Find the impulse response $h[n]$.

For the system to be causal, the ROC must be exterior: $|z|>3$.

$$\frac{2}{1+3z^{-1}},\ |z|>3 \longleftrightarrow 2(-3)^n u[n]$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}},\ |z|>\frac{1}{2} \longleftrightarrow -\left(\tfrac{1}{2}\right)^n u[n]$$

$$h[n] = 2(-3)^n u[n] - \left(\tfrac{1}{2}\right)^n u[n]$$

(c) **5 pts**. Does a system exist that is causal **and** stable **and** has its input and output related by this difference equation? Justify your answer.

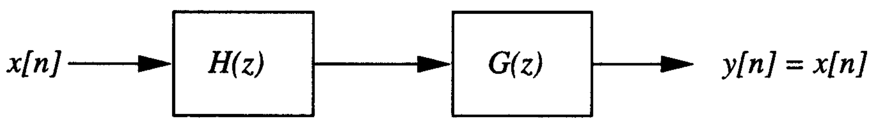
NO. The $h[n]$ in part (a) is stable but not causal. The $h[n]$ in part (b) is causal but not stable. The third possible $h[n]$ corresponds to the interior ROC $|z|<\frac{1}{2}$ and is neither stable nor causal.

4. **25 pts.** Suppose that Joebob uses his cellular phone to call up Hidea on her cellular phone and ask for a date. The call gets digitized, and Joebob's voice ends up as a discrete-time signal $x[n]$. Because of interference being put out by Joebob's microwave oven and electric toilet paper dispenser, his voice is distorted during transmission so that the signal $r[n]$ received by Hidea's phone is not exactly equal to $x[n]$. In fact, the received signal is given by

$$r[n] = \frac{1}{2}\left(\frac{3}{4}r[n-1] + x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]\right).$$

Since this date is **really important** to Joebob, you have been hired to design an LTI channel equalizer that will "undo" the distortion. The equalizer is required to be both **causal** and **stable**.

Let $G(z)$ be the transfer function of the distortion and $H(z)$ be the transfer function of the equalizer. The equalizer will be placed in Joebob's phone so that it is in series with the distortion:



Note that, since the order does not matter when two linear systems are connected in series, this is equivalent to processing the received signal $r[n]$ above with an identical equalizer installed in Hidea's phone instead of in Joebob's phone.

(a) **10 pts.** Find $G(z)$, the transfer function of the distortion.

$$2R(z) = \frac{3}{4}z^{-1}R(z) + X(z) - \frac{3}{4}z^{-1}X(z) + \frac{1}{8}z^{-2}X(z)$$

$$\left[2 - \frac{3}{4}z^{-1}\right]R(z) = \left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right]X(z)$$

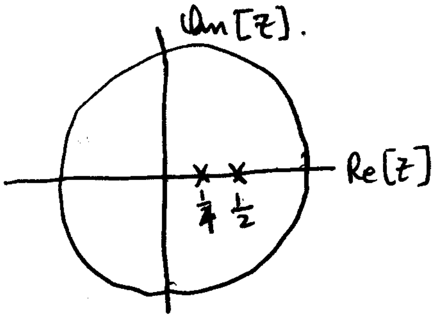
$$G(z) = \frac{R(z)}{X(z)} = \frac{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}{2 - \frac{3}{4}z^{-1}}$$

Problem 4, cont...

(b) **5 pts.** Find the equalizer transfer function $H(z) = 1/G(z)$ so that the signal received by Hidea's phone will equal $x[n]$. Be sure to specify the region of convergence so that the equalizer will be causal and stable.

$$H(z) = \frac{1}{G(z)} = \frac{2-\frac{3}{4}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \underline{\underline{\frac{2-\frac{3}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}}}$$

$H(z)$ has poles at $z=\frac{1}{2}$ and $z=\frac{1}{4}$



For the system to be causal and stable, the ROC must be exterior and the poles must be inside the unit circle.

ROC: $|z| > \frac{1}{2}$.

(c) **10 pts.** Find the equalizer impulse response $h[n]$.

$$H(z) = \frac{2-\frac{3}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{2-\frac{3}{4}\theta}{1-\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2-\frac{3}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$B = \left.\frac{2-\frac{3}{4}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2-3}{1-2} = \frac{-1}{-1} = 1$$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{ROC}: |z| > \frac{1}{2}$$

$$\underline{\underline{h[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{4}\right)^n u[n]}}$$