ECE 2713

HW 4 SOLUTION

HAVLICEK

① $\boxed{1-1}$

$$h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$$

$$x[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$y[n] = x[n] * h[n]$$

$$= x[n] * \left(-\delta[n] + 2\delta[n-1] - \delta[n-2]\right)$$

$$= \left(\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$$
$$* \left(-\delta[n] + 2\delta[n-1] - \delta[n-2]\right)$$

$$= \left(\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]\right) * \left(-\delta[n]\right)$$
$$+ \left(\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]\right) * 2\delta[n-1]$$
$$+ \left(\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]\right) * \left(-\delta[n-2]\right)$$

$$= -\delta[n+2] - 2\delta[n+1] - 3\delta[n] - 2\delta[n-1] - \delta[n-2]$$
$$+ 2\delta[n+1] + 4\delta[n] + 6\delta[n-1] + 4\delta[n-2] + 2\delta[n-3]$$
$$- \delta[n] - 2\delta[n-1] - 3\delta[n-2] - 2\delta[n-3] - \delta[n-4]$$

$$= -\delta[n+2] + 0\delta[n+1] + 0\delta[n] + 2\delta[n-1] + 0\delta[n-2] + 0\delta[n-3] - \delta[n-4]$$

$$\boxed{y[n] = -\delta[n+2] + 2\delta[n-1] - \delta[n-4]}$$

See next page →

Here is another way to work this problem that you may like better: since $x[n]$ has the longer expression, you can solve $y[n]$ in terms of $x[n]$, and then plug in for the shifted versions of $x[n]$. Like this:

$$y[n] = x[n] * h[n] = x[n] * \left(-\delta[n] + 2\delta[n-1] - \delta[n-2]\right)$$

$$= x[n] * (-\delta[n]) + x[n] * (2\delta[n-1]) + x[n] * (-\delta[n-2])$$

$$= -x[n] * \delta[n] + 2x[n] * \delta[n-1] - x[n] * \delta[n-2]$$

$$= -x[n] + 2x[n-1] - x[n-2]$$

Now:

$$x[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$x[n-1] = \delta[n+1] + 2\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

$$x[n-2] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

Plugging these into the last expression above for $y[n]$, we get

$$y[n] = -x[n] + 2x[n-1] - x[n-2]$$

$$= -\delta[n+2] - 2\delta[n+1] - 3\delta[n] - 2\delta[n-1] - \delta[n-2]$$

$$+ 2\delta[n+1] + 4\delta[n] + 6\delta[n-1] + 4\delta[n-2] + 2\delta[n-3]$$

$$- \delta[n] - 2\delta[n-1] - 3\delta[n-2] - 2\delta[n-3] - \delta[n-4]$$

$$y[n] = -\delta[n+2] + 0\delta[n+1] + 0\delta[n] + 2\delta[n-1] + 0\delta[n-2] + 0\delta[n-3] - \delta[n-4]$$

$$\boxed{y[n] = -\delta[n+2] + 2\delta[n-1] - \delta[n-4]}$$

(same answer we got before)

② $h[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-2]$

The input signal $x[n]$ is not specified.

So we get

$$y[n] = x[n] * h[n]$$
$$= x[n] * (\delta[n] - 2\delta[n-1] + 3\delta[n-2].)$$
$$= x[n] * \delta[n] + x[n] * (-2\delta[n-1]) + x[n] * (3\delta[n-2])$$
$$= x[n] * \delta[n] - 2x[n] * \delta[n-1] + 3x[n] * \delta[n-2]$$
$$= x[n] - 2x[n-1] + 3x[n-2]$$

$$\boxed{y[n] = x[n] - 2x[n-1] + 3x[n-2]}$$

→ Once you see how this works, you can work this problem in just one or two lines like this:

$$y[n] = x[n] * h[n] = x[n] * (\delta[n] - 2\delta[n-1] + 3\delta[n-2])$$
$$= x[n] - 2x[n-1] + 3x[n-2]$$

3

$h[n] = \left(\frac{1}{3}\right)^n u[n]$

$x[n] = \left(\frac{1}{4}\right)^n u[n]$

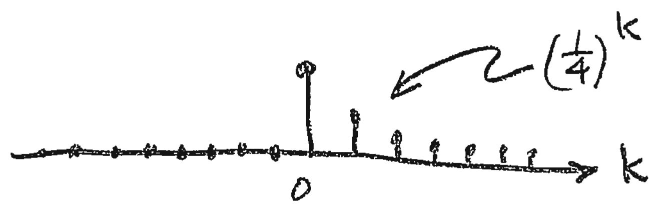
→ Since $h[n]$ and $x[n]$ both have the same form, it does not make any difference which one gets the "k" and which one gets the "$n-k$". I will put the "$n-k$" on $h[n]$. So:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

→ So we need the graphs of $x[k]$ and $h[n-k]$.

→ For $x[k]$, we just replace the "n" in $x[n]$ with "k":

$$x[k] = \left(\tfrac{1}{4}\right)^k u[k]$$



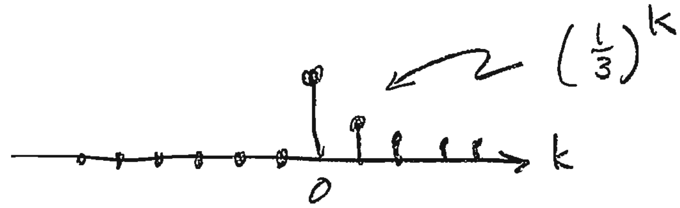
→ We think of $h[n-k]$ as $h[-k - -n]$. In other words, we have a shift right by $-n$ and a scale by -1.

→ We make the graph of $h[n-k] = h[-k - -n]$ in <u>three steps</u>.

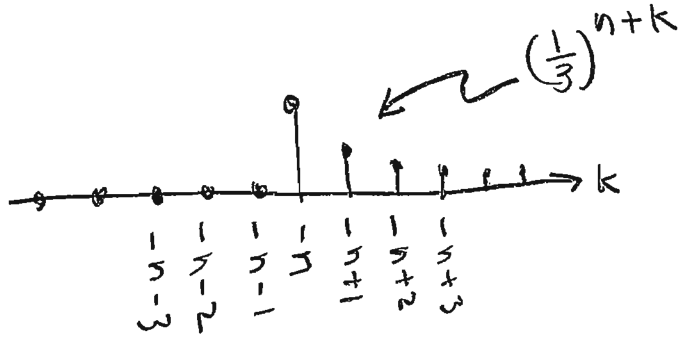
⟶

First step: make the graph of $h[k]$. Replace "n" with "k":

$$h[k] = \left(\tfrac{1}{3}\right)^k u[k]$$

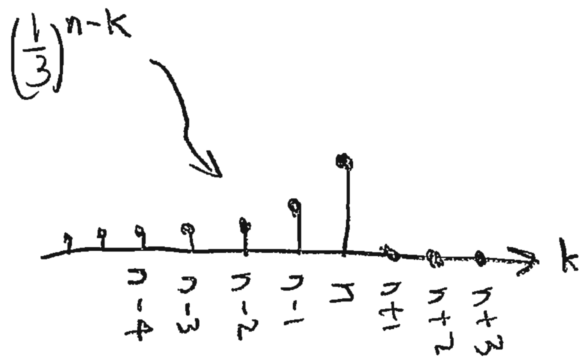


Second step: shift this graph right by $-n$ to get the graph of $h[k - -n] = h[n+k]$



Third step: flip the graph with respect to the k-axis. All the numbers on the bottom get multiplied by -1.

$$h[-k - -n] = h[n-k]$$



Now, from here on out I'm going to draw the graphs a bit like continuous-time signals because it looks clearer and saves a little bit of time.

We've got:

and $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

→ So, for each n, the number $y[n]$ is obtained by: ① multiply the two graphs above to get a product graph. Note that the product graph depends on n.
Then ② add up the product graph to get the number $y[n]$.

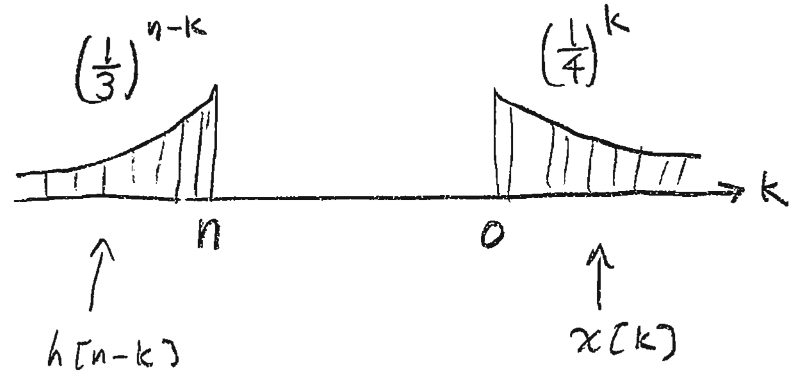
→ We have to do this for <u>all</u> of the n's from $-\infty$ to $+\infty$.

→ But we will be able to do them in batches or groups. The "batches" are called "regions" or "cases".

→

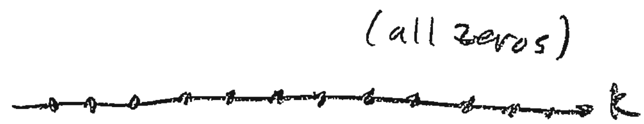
So we start by thinking of huge <u>negative</u> values of n. This gives us:

<u>Case I</u>



- when we multiply these two graphs together, we get a product graph that is <u>all</u> <u>zeros</u>.

- Here is the product graph:



- So $y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$

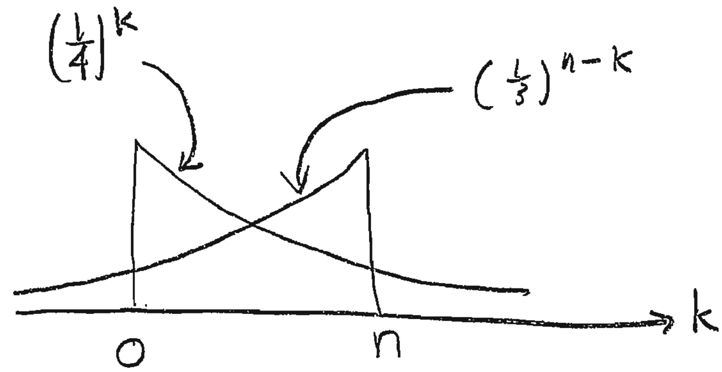
- what values of n is this good for?

  → all the n's from $-\infty$ up to $n=-1$... because for all of these n's, the product graph is <u>all</u> <u>zeros</u>.

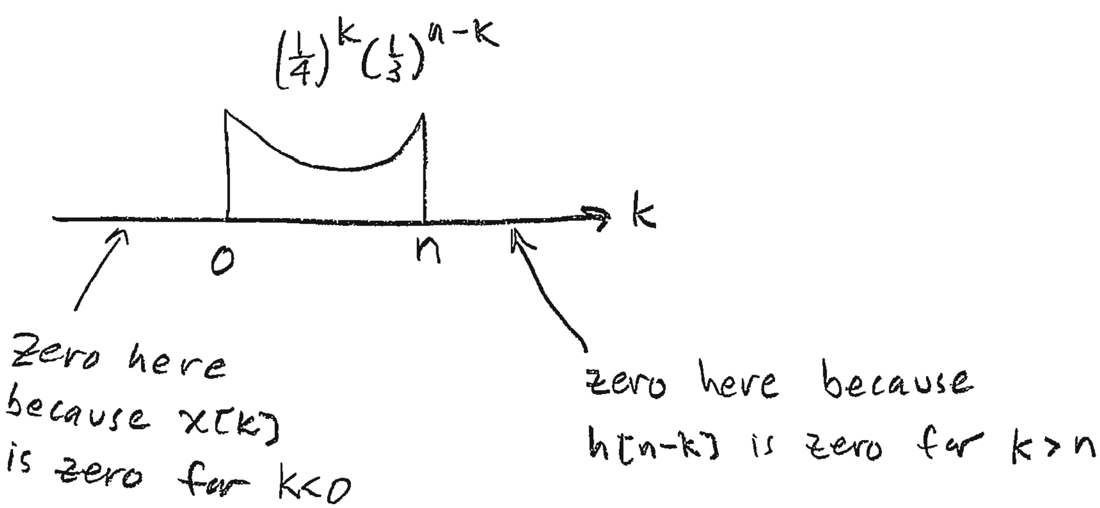
Thus: <u>Case I</u> $n<0$; $y[n]=0$

Now, as we continue to think of larger n's, the two graphs start to overlap some for $n = 0, 1, 2$ etc... . So starting at $n=0$, the product graph is <u>not</u> all zero any more.

- our two graphs look like this:



- The product graph looks like this:



- So now we've got that $x[k]$ <u>starts</u> it at $k=0$ and $h[n-k]$ shuts it off for $k>n$.

So when we add up the product grap, we get

$$y[n] = \sum_{k=0}^{n} x[k]\, h[n-k]$$

$$= \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^{n-k} = \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k$$

Sum formula:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

provided $\alpha \neq 1$

$$= \left(\tfrac{1}{3}\right)^n \frac{\left(\tfrac{3}{4}\right)^0 - \left(\tfrac{3}{4}\right)^{n+1}}{1 - 3/4}$$

$$= \left(\tfrac{1}{3}\right)^n \frac{1 - \left(\tfrac{3}{4}\right)^{n+1}}{1/4}$$

$$= \left(\tfrac{1}{3}\right)^n \cdot 4 \cdot \left[1 - \left(\tfrac{3}{4}\right)^n \left(\tfrac{3}{4}\right)\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[4 - 3\left(\tfrac{3}{4}\right)^n\right]$$

$$= 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{3}\right)^n \left(\tfrac{3}{4}\right)^n$$

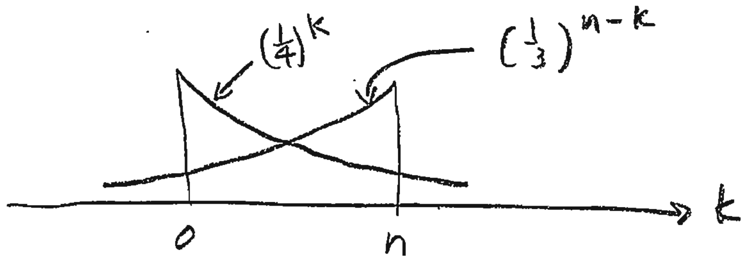
$$= 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{3} \cdot \tfrac{3}{4}\right)^n$$

$$= 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n$$

→ This is <u>case II</u>. What n's is it good for?

→ no matter how big n gets, the graphs will still look the same →

In other words, for all the n's from $n=0, 1, 2 \ldots$ all the way up to $n \to \infty$, the graphs will still look like



and the sum for $y[n]$ will go from $k=0$ to $k=n$.

→ So Case II covers all the rest of the n's... it's good for all $n \geq 0$.

→ So we've taken care of all the n's. That means we are done.

All together:

$$\boxed{y[n] = \begin{cases} 0 & , \quad n < 0 \\ 4(\frac{1}{3})^n - 3(\frac{1}{4})^n & , \quad n \geq 0 \end{cases}}$$

This can also be written as

$$y[n] = \left[4(\tfrac{1}{3})^n - 3(\tfrac{1}{4})^n\right] u[n]$$

→

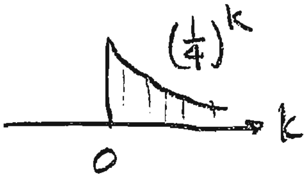
- Now, in that solution, I wrote everything out with <u>lots</u> of explanation.

- So let's do it again and show how you would actually write the solution on a test:

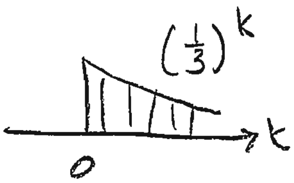
$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \qquad x[n] = \left(\tfrac{1}{4}\right)^n u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

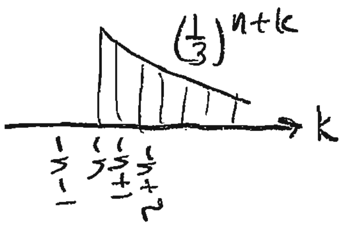
$x[k]$



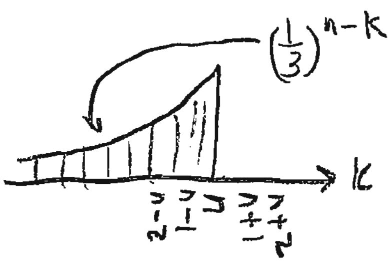
$h[k]$



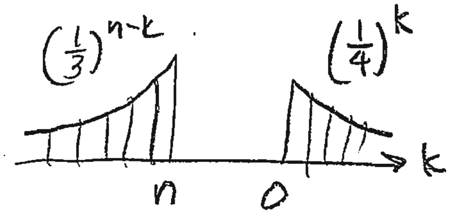
$h[n+k] = h[k - -n]$



$h[n-k] = h[-k - -n]$

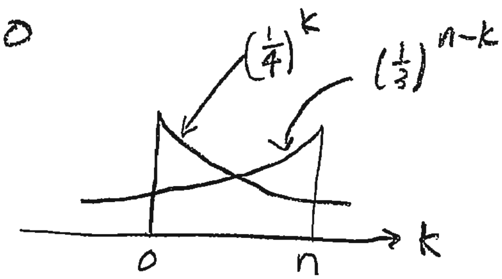


<u>Case I</u> : $n < 0$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

<u>Case II</u> : $n \geq 0$



$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^{n-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^{-k} = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k = \left(\tfrac{1}{3}\right)^n \frac{\left(\frac{3}{4}\right)^0 - \left(\frac{3}{4}\right)^{n+1}}{1 - 3/4}$$

$$= \left(\tfrac{1}{3}\right)^n \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{1/4} = 4\left(\tfrac{1}{3}\right)^n \left[1 - \tfrac{3}{4}\left(\tfrac{3}{4}\right)^n\right]$$

$$= 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{3}\right)^n\left(\tfrac{3}{4}\right)^n = 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n$$

All Together:

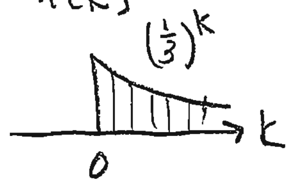
$$y[n] = \begin{cases} 0 & , n < 0 \\ 4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n & , n \geq 0 \end{cases} \quad //$$

Finally let's work this problem the "other way". $\boxed{3-9}$
In other words, let's do it with $h[k]$ and $x[n-k]$:

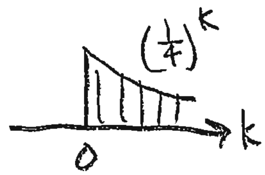
$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \qquad x[n] = \left(\tfrac{1}{4}\right)^n u[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

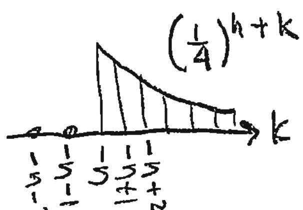
$h[k]$



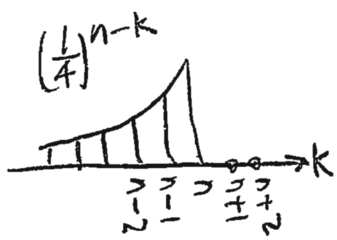
$x[k]$



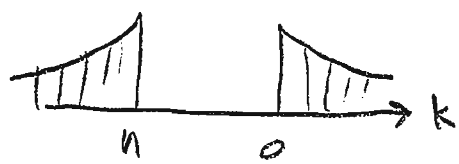
$x[n+k] = x[k - -n]$



$x[n-k] = x[-k - -n]$

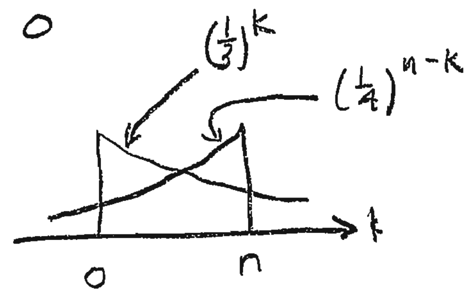


Case I : $n < 0$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II : $n \geq 0$



$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^{n-k} = \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^n \left(\tfrac{1}{4}\right)^{-k}$$

$$= \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{3}\right)^k 4^k = \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n} \left(\tfrac{4}{3}\right)^k$$

$$= \left(\tfrac{1}{4}\right)^n \frac{\left(\frac{4}{3}\right)^0 - \left(\frac{4}{3}\right)^{n+1}}{1 - 4/3} = \left(\tfrac{1}{4}\right)^n \frac{1 - \frac{4}{3}\left(\frac{4}{3}\right)^n}{-1/3}$$

$$= -3\left(\tfrac{1}{4}\right)^n \left[1 - \tfrac{4}{3}\left(\tfrac{4}{3}\right)^n\right] = -3\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{4}\right)^n\left(\tfrac{4}{3}\right)^n$$

$$= -3\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{3}\right)^n = 4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n$$

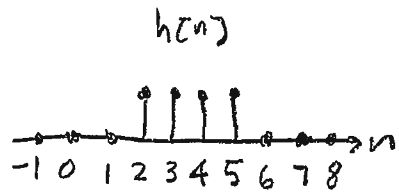
All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ 4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n, & n \geq 0 \end{cases} = \left[4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n\right] u[n]$$

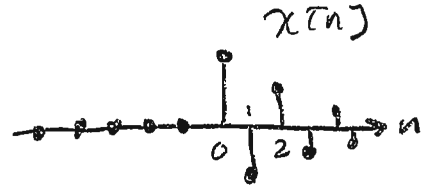
(same answer)

4 $\boxed{4-1}$

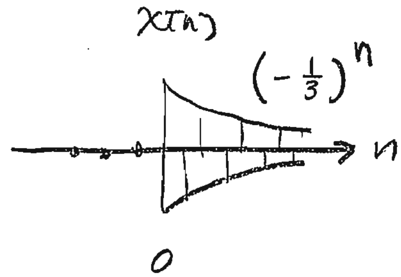
$$h[n] = \begin{cases} 1, & 2 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$



$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n]$$



→ For an alternating signal like $x[n]$, I will usually draw the graph like this:



– This makes the pictures a little bit clearer when we are figuring out the product graph.

→ In this problem, $h[n]$ has the simpler expression, since it doesn't have any exponents

→ So we'll start by putting the "$n-k$" on $h[n]$

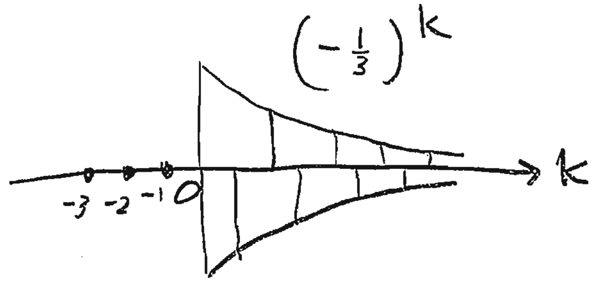
→

So $\quad y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

→ We need the graphs of $x[k]$ and $h[n-k]$

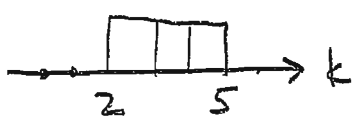
→ For $x[k]$, just change "n" to "k":

$x[k]$

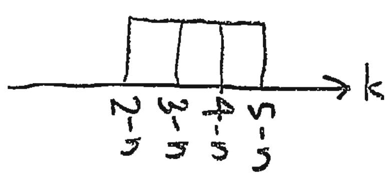


– Make the graph of $h[n-k]$ in three steps as always:

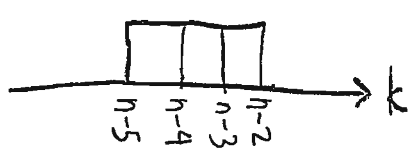
$h[k]$



$h[n+k] = h[k - -n]$ (shift right by $-n$)

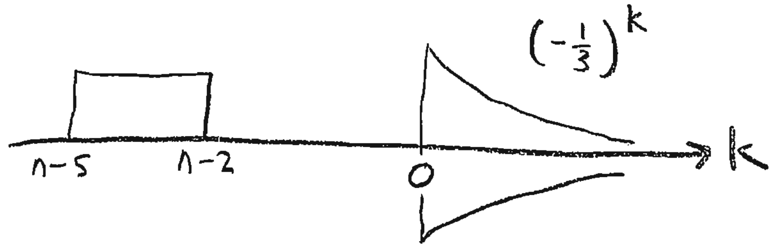


$h[n-k] = h[-k - -n]$ (flip)



Now, before I actually work the problem, let's talk through how the <u>regions</u> (cases) are going to go.

→ We start by thinking of giagantic negative n's:



<u>Case I</u>

→ The graphs don't have any overlap, so the product graph is all zeros.

→ This means $y[n]=0$.

→ That will be true as long as $n-2<0$ ... in other words, as long as $n<2$.

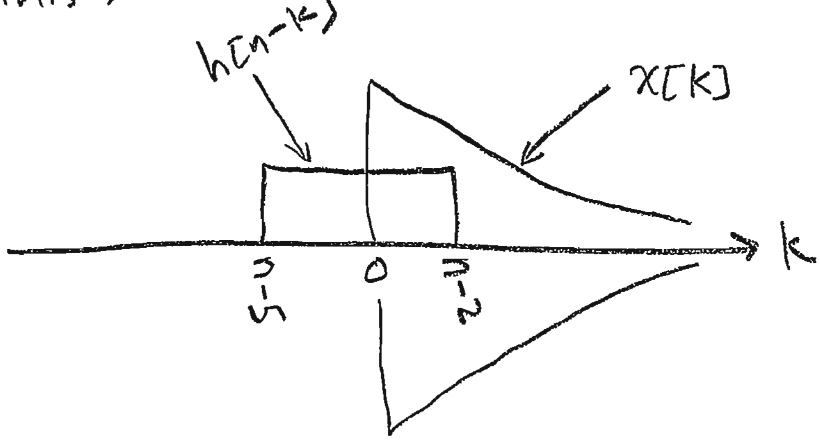
→ So our first case is $n<2$ and $y[n]=0$ for the first case.

→

- Starting at $n-2=0$ (or $n=2$), we start to get some overlap in the graphs, so the product graph is no longer all zeros.

  → as $n-2$ reaches $0, 1, 2$ the graphs look like this:

Case II



→ The graphs overlap from $k=0$ to $k=n-2$.

→ So the product graph is nonzero from $k=0$ to $k=n-2$.

→ we will get $y[n] = \sum_{k=0}^{n-2} (1)\left(-\frac{1}{3}\right)^k$

- Notice that we are getting nonzero in the product graph because $x[k]$ <u>starts</u> <u>it</u> (starts the overlap) at $k=0$ and $h[n-k]$ <u>stops</u> <u>it</u> (stops the overlap) at $k=n-2$.

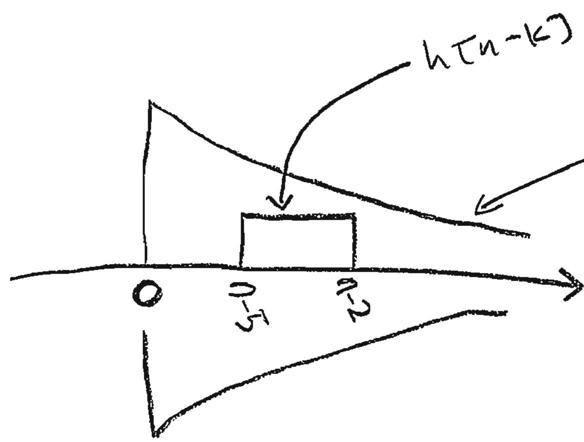
- How long does this case last?

  - In other words, as we continue to think of larger n's, how long is it true that $x[k]$ starts it and $h[n-k]$ stops it?

    → Well, it's true until $n-5$ crosses $k=0$.

      → After that, the picture will be different. It will look like this:

Case III



- In this new case (Case III), $h[n-k]$ starts the overlap and $h[n-k]$ also stops the overlap.

- So $n-5=-1$ $(n=4)$ is definitely part of case II.

- when $n-5=0$ $(n=5)$, we can consider it to be the last time that $x[k]$ starts it ...

or we can consider that $n-5=0$ $(n=5)$ is the first time that $h[n-k]$ starts it.

- In other words, we can make $n=5$ the last n in case II or we can make it the first n in case III.

  - Either choice is okay and $y[5]$ will be the same number either way... although it will be written with a different expression depending on which choice we make.

  - I will choose the 2nd way... in other words, I will make $n=5$ the first n in case III.

  - That means that Case II goes from $n-2=0$ $(n=2)$ up until $n-5=-1$ $(n=4)$.

→ So case II is for $2 \le n < 5$.

- So Case III starts when $n-5=0$ $(n=5)$.
  - How long does Case III go for?
    - For Case III, $h[n-k]$ starts the overlap and $h[n-k]$ also stops it.
    - looking back at the Case III picture on page 4-5 and considering bigger and bigger n's, we see that this will never change.
      - No matter how big n gets, it will still be true that $h[n-k]$ starts the overlap and stops it.
      - So case III covers all the rest of the n's.
      - In other words, Case III is for $5 \leq n < \infty$.
- Again looking back at the Case III picture on page 4-5, we see that the product graph is nonzero from $k=n-5$ to $k=n-2$. So we'll have

$$y[n] = \sum_{k=n-5}^{n-2} x[k]h[n-k] \quad \text{in Case III.}$$

Summarizing, we have:

Case I: $n<2$: $y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$

Case II: $2 \le n < 5$: $y[n] = \sum_{k=0}^{n-2} (1)\left(-\frac{1}{3}\right)^k$

Case III: $n \geq 5$: $y[n] = \sum_{k=n-5}^{n-2} (1)\left(-\frac{1}{3}\right)^k$

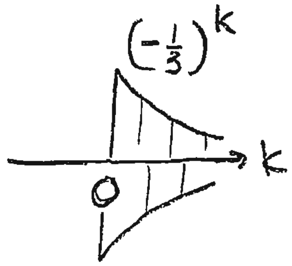
So now on the next page I'll write the solution to this problem the way you would actually want to work it on a test...

$$h[n] = \begin{cases} 1, & 2 \le n \le 5 \\ 0, & \text{otherwise} \end{cases} \qquad x[n] = \left(-\tfrac{1}{3}\right)^n u[n]$$

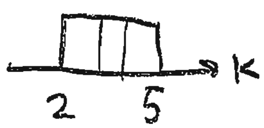
4-9

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

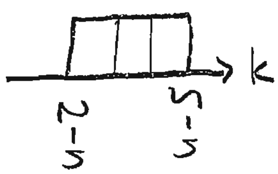
$x[k]$



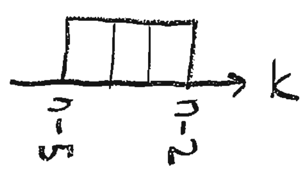
$h[k]$



$h[n+k] = h[k - -n]$

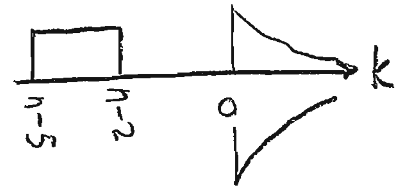


$h[n-k] = h[-k - -n]$



Case I: $n-2 < 0$

$n < 2$

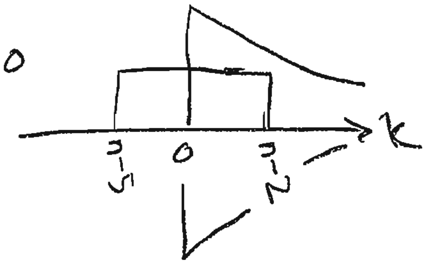


$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II: $n-2 \geq 0$ and $n-5<0$

$n \geq 2$ and $n<5$

$2 \le n < 5$



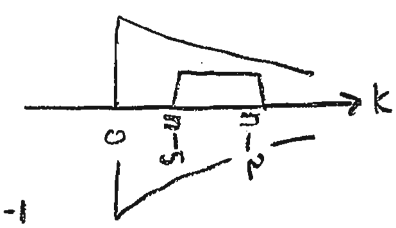
$$y[n] = \sum_{k=0}^{n-2} (1)\left(-\tfrac{1}{3}\right)^k$$

$$= \sum_{k=0}^{n-2} \left(-\tfrac{1}{3}\right)^k = \frac{\left(-\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{n-2+1}}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{1 - \left(-\frac{1}{3}\right)^{n-1}}{4/3} = \frac{3}{4}\left[1 - \left(-\tfrac{1}{3}\right)^{-1}\left(-\tfrac{1}{3}\right)^n\right]$$

$$= \frac{3}{4}\left[1 - (-3)\left(-\tfrac{1}{3}\right)^n\right] = \frac{3}{4} + \frac{9}{4}\left(-\tfrac{1}{3}\right)^n$$

Case III: $n-5 \geq 0$: $n \geq 5$



$$y[n] = \sum_{k=n-5}^{n-2} (1)\left(-\tfrac{1}{3}\right)^k$$

$$= \sum_{k=n-5}^{n-2} \left(-\tfrac{1}{3}\right)^k = \frac{\left(-\frac{1}{3}\right)^{n-5} - \left(-\frac{1}{3}\right)^{n-1}}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{\left(-\frac{1}{3}\right)^n\left(-\frac{1}{3}\right)^{-5} - \left(-\frac{1}{3}\right)^n\left(-\frac{1}{3}\right)^{-1}}{4/3} = \frac{3}{4}\left(-\tfrac{1}{3}\right)^n\left[(-3)^5 - (-3)\right]$$

$$= \frac{3}{4}\left(-\tfrac{1}{3}\right)^n\left[-243+3\right] = \frac{3}{4}\left(-\tfrac{1}{3}\right)^n(-240)$$

$$= -180\left(-\tfrac{1}{3}\right)^n$$

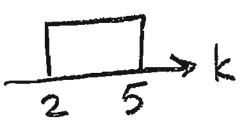
All Together:

$$y[n] = \begin{cases} 0 & , \ n < 2 \\ \frac{3}{4} + \frac{9}{4}\left(-\frac{1}{3}\right)^n & , \ 2 \le n < 5 \\ -180\left(-\frac{1}{3}\right)^n & , \ n \geq 5 \end{cases}$$

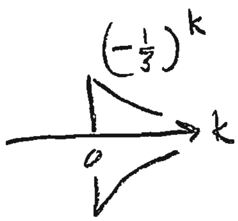
→ Now let's work this one the other way:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

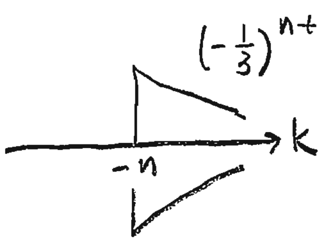
$h[k]$



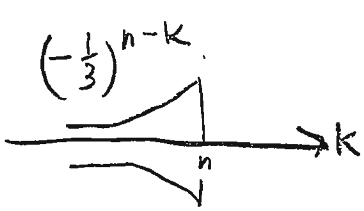
$x[k]$



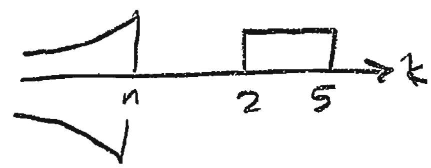
$x[n+k] = x[k - -n]$



$x[n-k] = x[k - -n]$



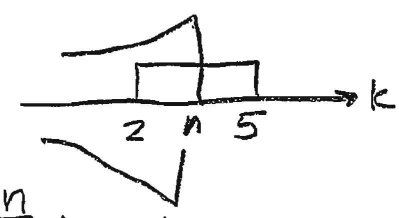
Case I: $n < 2$:



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II: $n \geq 2$ and $n < 5$

$2 \le n < 5$



$$y[n] = \sum_{k=2}^{n} \left(-\frac{1}{3}\right)^{n-k} = \left(-\frac{1}{3}\right)^n \sum_{k=2}^{n} \left(-\frac{1}{3}\right)^{-k}$$

$$= \left(-\frac{1}{3}\right)^n \sum_{k=2}^{n} (-3)^k = \left(-\frac{1}{3}\right)^n \frac{(-3)^2 - (-3)^{n+1}}{1-(-3)}$$

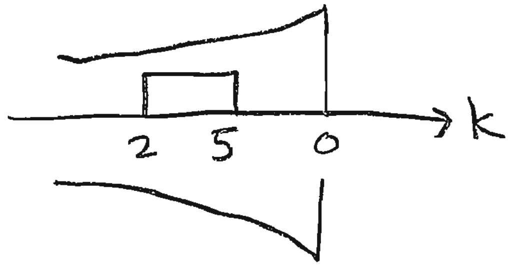
$$= \left(-\frac{1}{3}\right)^n \frac{9 - (-3)(-3)^n}{4} = \frac{1}{4}\left(-\frac{1}{3}\right)^n \left[9 + 3(-3)^n\right]$$

$$= \frac{9}{4}\left(-\frac{1}{3}\right)^n + \frac{3}{4}\left(-\frac{1}{3}\right)^n(-3)^n = \frac{9}{4}\left(-\frac{1}{3}\right)^n + \frac{3}{4}(1)^n$$

$$= \frac{3}{4} + \frac{9}{4}\left(-\frac{1}{3}\right)^n$$

→

Case III : $n \geq 5$:



$$y[n] = \sum_{k=2}^{5} \left(-\tfrac{1}{3}\right)^{n-k}$$

$$= \left(-\tfrac{1}{3}\right)^{n} \sum_{k=2}^{5} \left(-\tfrac{1}{3}\right)^{-k} = \left(-\tfrac{1}{3}\right)^{n} \sum_{k=2}^{5} (-3)^{k}$$

$$= \left(-\tfrac{1}{3}\right)^{n} \frac{(-3)^2 - (-3)^6}{1-(-3)} = \left(-\tfrac{1}{3}\right)^{n} \frac{9 - 729}{4}$$

$$= \left(-\tfrac{1}{3}\right)^{n} \frac{-720}{4} = -180\left(-\tfrac{1}{3}\right)^{n}$$

All Together;

$$y[n] = \begin{cases} 0 & , \; n < 2 \\ \frac{3}{4} + \frac{9}{4}\left(-\frac{1}{3}\right)^{n} & , \; 2 \leq n < 5 \\ -180\left(-\frac{1}{3}\right)^{n} & , \; n \geq 5 \end{cases}$$

(Same answer we got before)

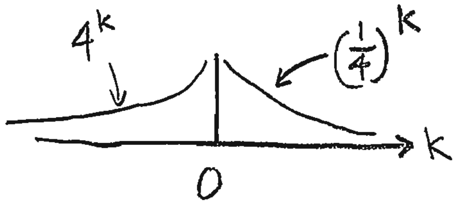
(5) $\boxed{5-1}$

$$h[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

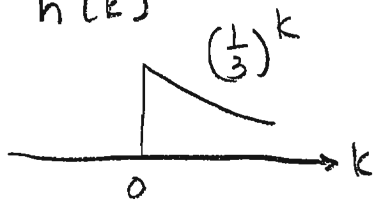
$$x[n] = \left(\tfrac{1}{4}\right)^{|n|} = \begin{cases} \left(\tfrac{1}{4}\right)^n, & n \geq 0 \\ 4^n, & n < 0 \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

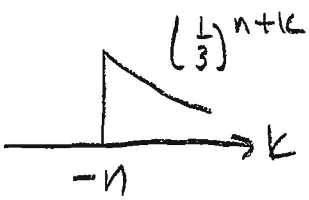
$x[k]$



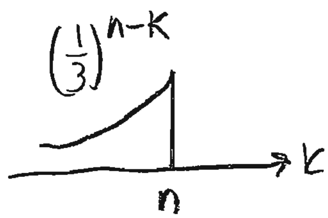
$h[k]$



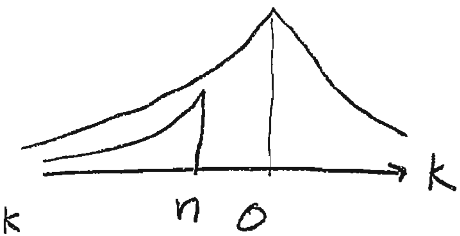
$h[n+k] = h[k - -n]$



$h[n-k] = h[-k - -n]$



<u>Case I</u> : $n < 0$



$$y[n] = \sum_{k=-\infty}^{n} 4^k \left(\tfrac{1}{3}\right)^{n-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=-\infty}^{n} 4^k \left(\tfrac{1}{3}\right)^{-k} = \left(\tfrac{1}{3}\right)^n \sum_{k=-\infty}^{n} 4^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=-\infty}^{n} 12^k = \lim_{A \to -\infty} \left(\tfrac{1}{3}\right)^n \sum_{k=A}^{n} 12^k$$

$$= \lim_{A \to -\infty} \left(\tfrac{1}{3}\right)^n \frac{12^A - 12^{n+1}}{1-12}$$

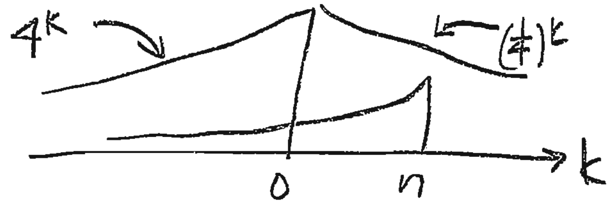
$$= \left(\tfrac{1}{3}\right)^n \left(\tfrac{-1}{11}\right) \left[0 - 12(12)^n\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left(-\tfrac{1}{11}\right)(-12)\, 12^n = \tfrac{12}{11} \left(\tfrac{12}{3}\right)^n$$

$$= \tfrac{12}{11} 4^n.$$

$\longrightarrow$

Case II : $n \geq 0$



$\boxed{5-2}$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{-1} x[k]h[n-k] + \sum_{k=0}^{n} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{-1} 4^k \left(\tfrac{1}{3}\right)^{n-k} + \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^{n-k}$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=-\infty}^{-1} 4^k \left(\tfrac{1}{3}\right)^{-k} + \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{3}\right)^{-k}$$

$$= \left(\tfrac{1}{3}\right)^n \left[\lim_{A \to -\infty} \sum_{k=A}^{-1} 4^k (3)^k + \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k (3)^k\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\lim_{A \to -\infty} \sum_{k=A}^{-1} 12^k + \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\lim_{A \to -\infty} \frac{12^A - 12^0}{1-12} + \frac{\left(\frac{3}{4}\right)^0 - \left(\frac{3}{4}\right)^{n+1}}{1 - 3/4}\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{0-1}{-11} + \frac{1 - \frac{3}{4}\left(\frac{3}{4}\right)^n}{1/4}\right] = \left(\tfrac{1}{3}\right)^n \left\{\tfrac{1}{11} + 4\left[1 - \tfrac{3}{4}\left(\tfrac{3}{4}\right)^n\right]\right\}$$

$$= \tfrac{1}{11}\left(\tfrac{1}{3}\right)^n + 4\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{3}\right)^n \cdot 4 \cdot \tfrac{3}{4} \cdot \left(\tfrac{3}{4}\right)^n$$

$$= \left(4 + \tfrac{1}{11}\right)\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{3} \cdot \tfrac{3}{4}\right)^n = \tfrac{45}{11}\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n$$

$\longrightarrow$

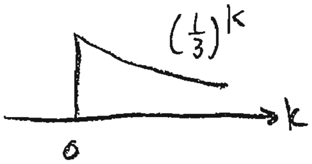
All Together:

$$y[n] = \begin{cases} \frac{12}{11} 4^n , & n<0 \\ \frac{45}{11}\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n , & n \geq 0 \end{cases}$$

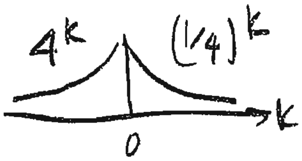
- Now let's work this one the other way:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

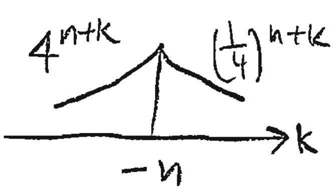
h[k]



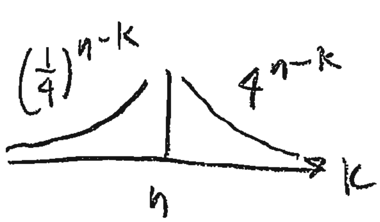
x[k]



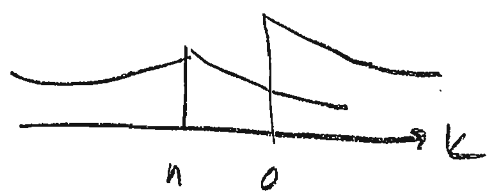
$x[n+k] = x[k--n]$



$x[n-k] = x[-k--n]$



<u>Case I: $n<0$</u>



$$y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k 4^{n-k}$$

$$= 4^n \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k 4^{-k} = 4^n \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{4}\right)^k$$

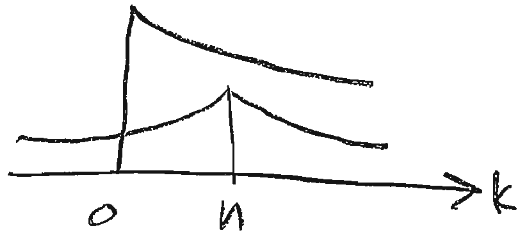
$$= 4^n \lim_{A\to\infty} \sum_{k=0}^{A} \left(\frac{1}{12}\right)^k$$

$$= 4^n \lim_{A\to\infty} \frac{\left(\frac{1}{12}\right)^0 - \left(\frac{1}{12}\right)^{A+1}}{1-\frac{1}{12}} = 4^n \frac{1-0}{11/12}$$

$$= \frac{12}{11} 4^n$$

$\longrightarrow$

Case II : $n \geq 0$



5-4

$$y[n] = \sum_{k=0}^{\infty} h[k]\,x[n-k]$$

$$= \sum_{k=0}^{n-1} h[k]\,x[n-k] + \sum_{k=n}^{\infty} h[k]\,x[n-k]$$

$$= \sum_{k=0}^{n-1} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^{n-k} + \sum_{k=n}^{\infty} \left(\tfrac{1}{3}\right)^k 4^{n-k}$$

$$= \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{3}\right)^k 4^k + 4^n \sum_{k=n}^{\infty} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^k$$

$$= \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{4}{3}\right)^k + \lim_{A\to\infty} 4^n \sum_{k=n}^{A} \left(\tfrac{1}{12}\right)^k$$

$$= \left(\tfrac{1}{4}\right)^n \frac{(4/3)^0 - (4/3)^n}{1 - 4/3} + \lim_{A\to\infty} 4^n \frac{\left(\tfrac{1}{12}\right)^n - \left(\tfrac{1}{12}\right)^{A+1}}{1 - 1/12}$$

$$= \left(\tfrac{1}{4}\right)^n \frac{1 - (4/3)^n}{-1/3} + 4^n \frac{\left(\tfrac{1}{12}\right)^n - 0}{11/12} = -3\left(\tfrac{1}{4}\right)^n \left[1 - \left(\tfrac{4}{3}\right)^n\right] + \tfrac{12}{11} 4^n \left(\tfrac{1}{12}\right)^n$$

$$= -3\left(\tfrac{1}{4}\right)^n + 3\left(\tfrac{1}{4}\right)^n \left(\tfrac{4}{3}\right)^n + \tfrac{12}{11}\left(\tfrac{4}{12}\right)^n$$

$$= -3\left(\tfrac{1}{4}\right)^n + 3\left(\tfrac{1}{3}\right)^n + \tfrac{12}{11}\left(\tfrac{1}{3}\right)^n = -3\left(\tfrac{1}{4}\right)^n + \tfrac{45}{11}\left(\tfrac{1}{3}\right)^n$$

All Together:

$$y[n] = \begin{cases} \tfrac{12}{11} 4^n, & n < 0 \\ -3\left(\tfrac{1}{4}\right)^n + \tfrac{45}{11}\left(\tfrac{1}{3}\right)^n, & n \geq 0 \end{cases}$$

(same answer as before)