ECE 2713

HW 5 SOLUTION

HAVLICEK

① $h[n] = (\frac{1}{4})^n u[n]$

Table: $H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$

$x[n] = (\frac{1}{3})^n u[n]$

Table: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{3}e^{-j\omega}}$

Convolution property of DTFT:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{j\omega})}$$

- Now we need to do a partial fraction expansion (PFE) on $Y(e^{j\omega})$ to get it into a sum of terms that are in our DTFT transform table.
- The PFE must be valid for <u>all</u> choices of ω, including even complex values.
- So we can <u>simplify</u> the calculation of the PFE by doing it for some complex values of ω that make $e^{-j\omega}$ equal to some real number... call it θ.

- we don't actually care what these values of ω are... we only care that they do <u>exist</u>.

- So, for <u>working the PFE</u>, we just change $e^{-j\omega}$ to θ like this:

$$Y(\theta) = \frac{1}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)} = \frac{A}{1-\frac{1}{3}\theta} + \frac{B}{1-\frac{1}{4}\theta}$$

- Notice that the "thing" that's under A is $(1-\frac{1}{3}\theta)$.
- To find A, we multiply both sides by this "thing" and then evaluate both sides at $\theta=3$, which is the value of θ that makes $(1-\frac{1}{3}\theta)$, the "thing" under A, equal to zero:

$$\frac{\cancel{(1-\frac{1}{3}\theta)}}{\cancel{(1-\frac{1}{3}\theta)}(1-\frac{1}{4}\theta)} = \frac{A\cancel{(1-\frac{1}{3}\theta)}}{\cancel{(1-\frac{1}{3}\theta)}} + \frac{B(1-\frac{1}{3}\theta)}{(1-\frac{1}{4}\theta)}$$

- In other words:

$$\frac{1}{1-\frac{1}{4}\theta} = A + \frac{B(1-\frac{1}{3}\theta)}{(1-\frac{1}{4}\theta)}$$

- Now evaluate both sides at $\theta = 3$, which is the value that makes $(1-\frac{1}{3}\theta)$ ... the thing that was under A, equal to zero. $\boxed{1-3}$

- We get

$$\left.\frac{1}{1-\frac{1}{4}\theta}\right|_{\theta=3} = A + \left.\frac{B(1-\frac{1}{3}\theta)}{(1-\frac{1}{4}\theta)}\right|_{\theta=3}$$

$$\frac{1}{1-\frac{3}{4}} = A + \frac{B(1-1)}{1-\frac{3}{4}} \leftarrow \text{ZERO!}$$

$$\frac{1}{1/4} = A \Rightarrow \underline{\underline{A = 4}}$$

- So, in the above I wrote out every step in detail so that you could see how and why it works.

- But on a test you can write it in a much shorter way!

  - This is the "Heaviside cover up method" and it always works the same.

  - So once you understand it, you can cut some steps...

So now let's work the PFE the way you would do it on a test:

From page 1-1:

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$\frac{1}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)} = \frac{A}{1-\frac{1}{3}\theta} + \frac{B}{1-\frac{1}{4}\theta}$$

$$A = \left.\frac{1}{1-\frac{1}{4}\theta}\right|_{\theta=3} = \frac{1}{1-\frac{3}{4}} = \frac{1}{\frac{1}{4}} = 4$$

$$B = \left.\frac{1}{1-\frac{1}{3}\theta}\right|_{\theta=4} = \frac{1}{1-\frac{4}{3}} = \frac{1}{-1/3} = -3$$

$$\Rightarrow \quad Y(e^{j\omega}) = \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{3}e^{-j\omega}} - \frac{3}{1-\frac{1}{4}e^{-j\omega}}$$

Table: $\underline{\underline{y[n] = 4\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n]}}$

<u>NOTE</u>: if you have time, it's <u>always</u> a good idea to <u>check</u> your PFE... because it's easy to make a silly arithmetic mistake on a test.

- Here's how you would check the PFE for this problem:

$$\frac{A}{1-\frac{1}{3}\theta} + \frac{B}{1-\frac{1}{4}\theta} = \frac{4}{(1-\frac{1}{3}\theta)} - \frac{3}{(1-\frac{1}{4}\theta)}$$

$$= \frac{4(1-\frac{1}{4}\theta) - 3(1-\frac{1}{3}\theta)}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)} \leftarrow \text{denom is correct}\checkmark$$

$$= \frac{4-\theta-3+\theta}{\text{denom}} = \frac{1}{\text{denom}} \leftarrow \text{numerator is correct}\checkmark$$

So this verifies that

$$\frac{4}{1-\frac{1}{3}e^{-j\omega}} - \frac{3}{1-\frac{1}{4}e^{-j\omega}} = \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

... in other words, our PFE is <u>correct</u>.

② P-5.1

H is an FIR filter with impulse response

$$h[n] = 7\delta[n] + \delta[n-3] - 5\delta[n-4]$$

- From the convolution equation, we know that the output signal is given by

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (7\delta[n] + \delta[n-3] - 5\delta[n-4]) \\ &= 7x[n] + x[n-3] - 5x[n-4] \end{aligned}$$

- Thus, the difference equation (I/O equation) is given by :

$$\boxed{y[n] = 7x[n] + x[n-3] - 5x[n-4]}$$

(3) H is an LTI system with I/O equation (difference equation)

$$y[n] = 2x[n] - 3x[n-1] + 2x[n-2]$$

- Take the DTFT on both sides and use the time shift property $\{ x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}) \}$:

$$Y(e^{j\omega}) = 2X(e^{j\omega}) - 3e^{-j\omega} X(e^{j\omega}) + 2e^{-j2\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) = \left[2 - 3e^{-j\omega} + 2e^{-j2\omega}\right] X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{2 - 3e^{-j\omega} + 2e^{-j2\omega}}}$$

- To find $h[n]$, we simply invert this from the tables.
  - Use the transform pair $\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} \underline{1}$ and the time shift property:

Table: $\boxed{h[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-2]}$

- Since $h[n]$ has finite length, this is an <u>FIR</u> filter.
- You can also tell this because $H(e^{j\omega})$ has no denominator (and the I/O equation has <u>no shifts</u> of $y[n]$).

(4) P-6.4(a)

$$y[n] = 2x[n] - 2x[n-1] + 2x[n-2]$$

Take DTFT on both sides:

$$Y(e^{j\omega}) = 2X(e^{j\omega}) - 2e^{-j\omega}X(e^{j\omega}) + 2e^{-j2\omega}X(e^{j\omega})$$

$$Y(e^{j\omega}) = [2 - 2e^{-j\omega} + 2e^{-j2\omega}]\,X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{\underline{2 - 2e^{-j\omega} + 2e^{-j2\omega}}}$$

- The problem now asks us to write $H(e^{j\omega})$ in polar form as $H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{R(e^{j\omega}) \text{ in book}} e^{j\arg H(e^{j\omega})}$
  - In general, we would have to use Euler's formula to find $\mathrm{Re}\{H(e^{j\omega})\}$ and $\mathrm{Im}\{H(e^{j\omega})\}$.
  - Then $|H(e^{j\omega})| = \sqrt{[\mathrm{Re}\{H(e^{j\omega})\}]^2 + [\mathrm{Im}\{H(e^{j\omega})\}]^2}$
    and $\arg H(e^{j\omega}) = \arctan \dfrac{\mathrm{Im}\{H(e^{j\omega})\}}{\mathrm{Re}\{H(e^{j\omega})\}}$

$\longrightarrow$

- But, when H is an FIR filter and the impulse response has <u>symmetry</u> about the middle, there is an <u>easier</u> <u>way</u>.

  - The symmetry can happen in <u>four</u> <u>ways</u>:

    * Even symmetry about the <u>midpoint</u> . . .

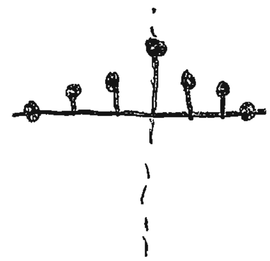
      EX

      - Even length $h[n]$
      - Even symmetry

    * Even symmetry about the <u>middle</u> <u>sample</u> . . .

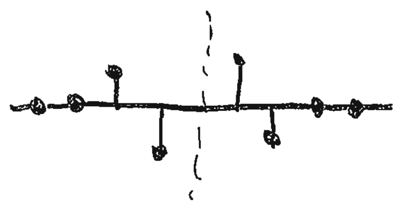
      EX

      - odd length $h[n]$
      - Even symmetry

    * odd symmetry about the <u>midpoint</u> . . .

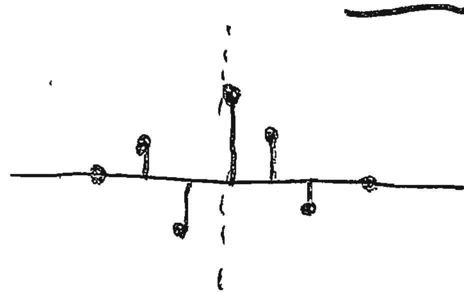
      EX

      - Even length $h[n]$
      - odd symmetry

    * odd symmetry about the <u>middle</u> <u>sample</u> . . .

      EX

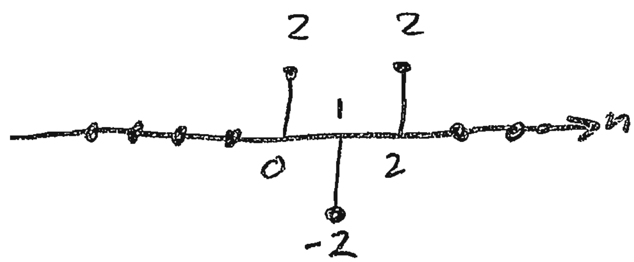
      - odd length $h[n]$
      - odd symmetry

- In <u>all</u> <u>four</u> cases, H is called a <u>linear</u> <u>phase</u> <u>FIR</u> <u>filter</u>.

- For a linear phase FIR filter, you can always do a trick to <u>easily</u> write $H(e^{j\omega})$ in polar form as $H(e^{j\omega}) = |H(e^{j\omega})| e^{j \arg H(e^{j\omega})}$
  - The <u>trick</u>: Factor out half the highest power of $e^{-j\omega}$ from $H(e^{j\omega})$. Then use Euler's formula to write the resulting terms as cosines and/or sines.
- For this problem, we have

$$H(e^{j\omega}) = 2 - 2e^{-j\omega} + 2e^{-j2\omega}$$

Table: $h[n] = 2\delta[n] - 2\delta[n-1] + 2\delta[n-2]$



→ $h[n]$ has <u>even</u> <u>symmetry</u> about the <u>middle</u> <u>sample</u>.

- So this <u>is</u> a linear phase FIR filter.
  - The trick will work!

$$H(e^{j\omega}) = 2 - 2e^{-j\omega} + 2e^{-j2\omega}$$

→ The highest power of $e^{-j\omega}$ is

$$e^{-j2\omega} = [e^{-j\omega}]^2 \quad \begin{pmatrix} \text{highest} \\ \text{power} = 2 \end{pmatrix}$$

→ half the highest power is $e^{-j\omega}$ $\begin{pmatrix} \text{half the} \\ \text{highest} \\ \text{power} = 1 \end{pmatrix}$

→ So factor out $e^{-j\omega}$ from $H(e^{j\omega})$ and then apply Euler's formula:

$$H(e^{j\omega}) = 2 - 2e^{-j\omega} + 2e^{-j2\omega}$$

$$= 2\underbrace{e^{-j\omega}e^{j\omega}}_{\text{one}} - 2e^{-j\omega} + 2e^{-j\omega}e^{-j\omega}$$

$$= e^{-j\omega}\left[2e^{j\omega} - 2 + 2e^{-j\omega}\right]$$

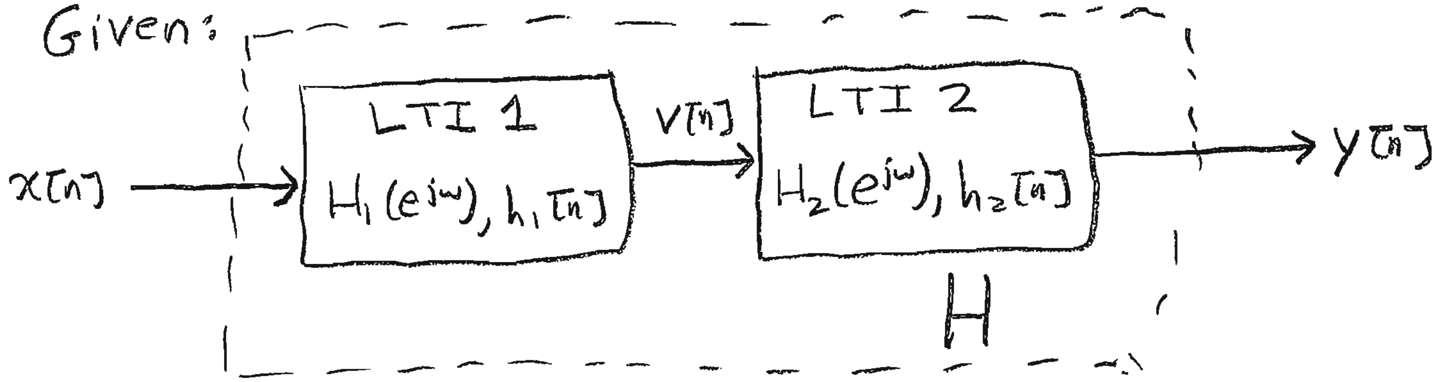
$$= \left[2(e^{j\omega} + e^{-j\omega}) - 2\right]e^{-j\omega}$$

$$= \left[4\cos\omega - 2\right]e^{-j\omega} \leftarrow \left\{ \begin{matrix} \text{This is} \\ R(e^{j\omega})e^{-j\omega n_0} \\ \text{in the book} \end{matrix} \right.$$

$R(e^{j\omega}) = |H(e^{j\omega})| = 4\cos\omega - 2$; $\arg H(e^{j\omega}) = -\omega$ $(n_0 = 1)$

(5) P-6.19

Given:



$$H_1(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$h_2[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$$

a) Find $H(e^{j\omega})$:

Table: $H_2(e^{j\omega}) = 1 - e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}$

- For the overall series connection, we have

$$h[n] = h_1[n] * h_2[n]$$

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

$\Rightarrow$ To see this, let $x[n] = \delta[n]$ in the figure above. Then $y[n]$ is the overall impulse response $h[n]$.

- Because the first system is LTI, we get $v[n] = \delta[n] * h_1[n] = h_1[n]$.
- Because the second system is LTI, we get $y[n] = h_1[n] * h_2[n] = h[n]$

So

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

$$= \left[1 + 2e^{-j\omega} + e^{-j2\omega}\right]\left[1 - e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}\right]$$

$$\begin{aligned} = 1 &- e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega} \\ &+ 2e^{-j\omega} - 2e^{-j2\omega} + 2e^{-j3\omega} - 2e^{-j4\omega} \\ &\quad + e^{-j2\omega} - e^{-j3\omega} + e^{-j4\omega} - e^{-j5\omega} \end{aligned}$$

$$= 1 + e^{-j\omega} + 0e^{-j2\omega} + 0e^{-j3\omega} - e^{-j4\omega} - e^{-j5\omega}$$

$$\boxed{H(e^{j\omega}) = 1 + e^{-j\omega} - e^{-j4\omega} - e^{-j5\omega}}$$

– Next, part (a) asks us to simplify $H(e^{j\omega})$ "as much as possible."

- We are not asked to find $h[n]$ until part (b).
- But if you look ahead to the solution for part (b), you will see that $h[n]$ has odd symmetry about the midpoint.
- This means that the "linear phase trick" will work for simplifying $H(e^{j\omega})$.

$\longrightarrow$

- The highest power of $e^{-j\omega}$ that appears in $H(e^{j\omega})$ is $e^{-j5\omega} = \left[e^{-j\frac{5}{2}\omega}\right]^2$.

- So factor out $e^{-j\frac{5}{2}\omega}$ (half the highest power $= \frac{5}{2}$).

  - Then apply Euler's formula to combine exponentials.

- we get:

$$H(e^{j\omega}) = 1 + e^{-j\omega} - e^{-j4\omega} - e^{-j5\omega}$$

$$= 1 + e^{-j\frac{2}{2}\omega} - e^{-j\frac{8}{2}\omega} - e^{-j\frac{10}{2}\omega}$$

$$= \underbrace{1\, e^{-j\frac{5}{2}\omega}e^{+j\frac{5}{2}\omega}}_{\text{one}} + e^{-j\frac{5}{2}\omega}e^{j\frac{3}{2}\omega} - e^{-j\frac{5}{2}\omega}e^{-j\frac{3}{2}\omega} - e^{-j\frac{5}{2}\omega}e^{-j\frac{5}{2}\omega}$$

$$= e^{-j\frac{5}{2}\omega}\left[e^{j\frac{5}{2}\omega} + e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega} - e^{-j\frac{5}{2}\omega}\right]$$

$$= e^{-j\frac{5}{2}\omega}\left[\left(e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}\right) + \left(e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}\right)\right]$$

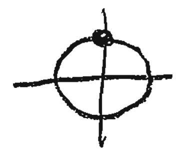
$$= e^{-j\frac{5}{2}\omega}\left[2j\left(\frac{e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}}{2j}\right) + 2j\left(\frac{e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}}{2j}\right)\right]$$

$$= \left[2j\sin\left(\tfrac{3}{2}\omega\right) + 2j\sin\left(\tfrac{5}{2}\omega\right)\right]e^{-j\frac{5}{2}\omega}$$

$$= \left[2\sin\tfrac{3}{2}\omega + 2\sin\tfrac{5}{2}\omega\right] j e^{-j\frac{5}{2}\omega}$$

$\longrightarrow$

But $j = e^{j\pi/2}$



So,

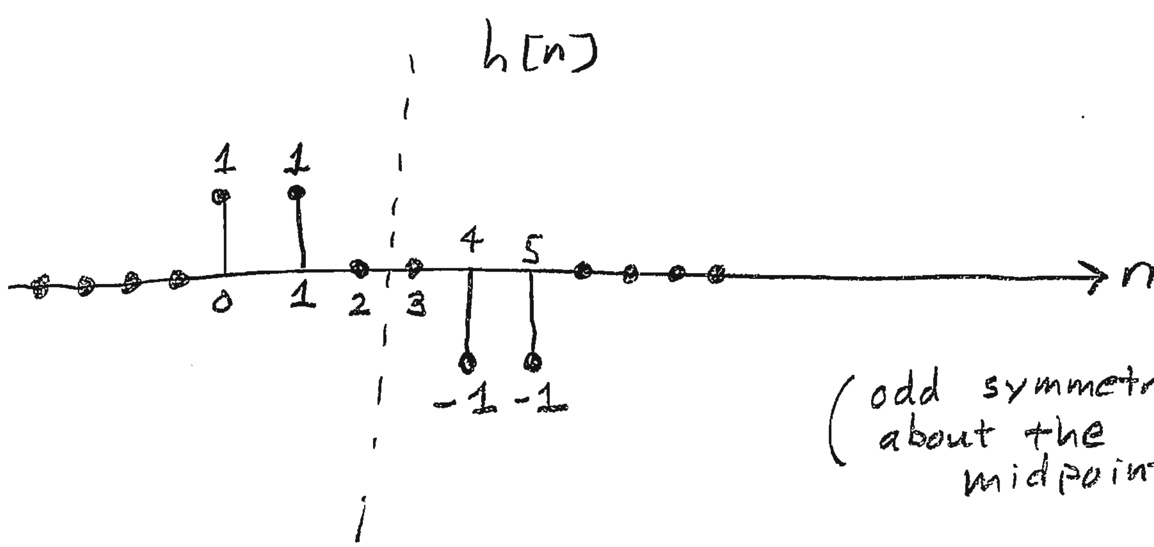
$$H(e^{j\omega}) = \left[2\sin\tfrac{3}{2}\omega + 2\sin\tfrac{5}{2}\omega\right] e^{j\pi/2} e^{-j\frac{5}{2}\omega}$$

$$= 2\left[\sin\tfrac{3}{2}\omega + \sin\tfrac{5}{2}\omega\right] e^{-j\left(\frac{5}{2}\omega - \frac{\pi}{2}\right)}$$

b) Go back to our original expression for $H(e^{j\omega})$ on page 5-2:

$$H(e^{j\omega}) = 1 + e^{-j\omega} - e^{-j4\omega} - e^{-j5\omega}$$

Table: $\boxed{h[n] = \delta[n] + \delta[n-1] - \delta[n-4] - \delta[n-5]}$



(odd symmetry about the midpoint)

c) Again go back to our original expression for $H(e^{j\omega})$ on p. 5-2:

$$H(e^{j\omega}) = 1 + e^{-j\omega} - e^{-j4\omega} - e^{-j5\omega} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega}) = X(e^{j\omega})\left[1 + e^{-j\omega} - e^{-j4\omega} - e^{-j5\omega}\right]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - e^{-j4\omega}X(e^{j\omega}) - e^{-j5\omega}X(e^{j\omega})$$

Take the inverse DTFT on both sides... use the time shift property:

$$\boxed{y[n] = x[n] + x[n-1] - x[n-4] - x[n-5]}$$