MODULE 1: INTRODUCTION

- This course is about digital signals and filtering
  - We will talk more precisely about what a <u>signal</u> is later.
    - For now, you have an intuitive idea. Digital signals are around you everywhere.
      - The music that comes out of your MP3 player, iPad, or digital radio.
      - The pictures that come out of your TV and YouTube. The movies you see at the theater.
      - The signals that control how fast gas gets injected into the cylinders of the motor in your car.

  - <u>Filtering</u> means processing or modifying signals.
    - Removing noise from sensor data
    - Correcting transmission errors in your music and your videos
    - Monitoring your engine performance and adjusting the flow of gasoline.

- The first thing we need to do is talk some about how engineers <u>model</u> signals and filters.
  - More generally, a "thing" that inputs a signal and outputs a signal is called a <u>system</u>.
  - <u>Filters</u> are a specific class of systems.
    - The output signal of a filter is usually produced by <u>modifying</u> the input signal in a designed, predictable way.

- Electrical & computer engineers make money... and help people... by using <u>models</u> of signals and systems.
  - The models are based on <u>mathematics</u>.
  - By using the models and our mathematics, we can <u>design</u> a system or a filter using our tools, including:
    - our brains
    - our pencils, paper, and calculators
    - our computers.
  - When the models are used correctly, the mathematics will accurately predict how the designed system will behave out in the real world.
  - This is a powerful approach.
    - You can use <u>math</u> to design a system and have confidence that when the system is actually built and deployed, it will behave the way it is supposed to.
  - You can use our math models to analyze an unknown system and understand how it works.

☆ Caveat: all models have <u>assumptions</u>.

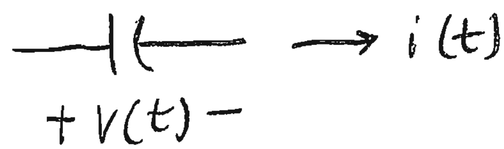
  - If the assumptions are violated, the model will generally fail to accurately describe the behavior of the real system.

Example: in Lab I you will hook up some electrical components called capacitors.

- This course is not about analog circuits, but this example is about the importance of the model assumptions.

- Capacitors usually look like a little "can" or a little "button" with two wires coming out.
  - Capacitors:

- Here is the electrical symbol for a capacitor:



  - v(t) is the voltage across the capacitor
  - i(t) is the current through the capacitor
  - v(t) and i(t) are both signals.
  - The capacitor is a system. It can be thought of as a simple filter.

- The capacitor is described by a constant number "C" called the capacitance.

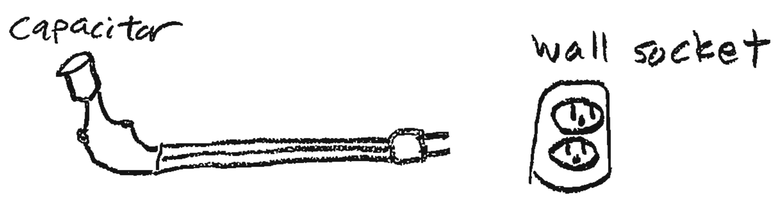
- We have a math model that relates the signals i(t) and v(t).

- The math model is given by:

$$i(t) = C \frac{d}{dt} v(t)$$

- In other words, the current <u>through</u> the capacitor is <u>proportional</u> to the <u>derivative</u> of the voltage <u>across</u> the capacitor.

- Now here is the point: the model has <u>assumptions</u>.
  - For most common capacitors, it is <u>assumed</u> that the <u>power</u>, given by the product of the current times the voltage, is less than $\frac{1}{4}$ watt (0.25).
  - As long as the assumption is satisfied, the real capacitor will behave according to the model.
- If you hook up an electrical cord to the capacitor and plug it into a wall socket:



- Then the voltage will be 120 volts AC
- In a typical house, the current could be 15 Amps
- The power is about $120 \times 15 > 1{,}000$ watts $\gg \frac{1}{4}$ watt

- In this case, the model assumptions are violated.
- The capacitor will not behave according to our math model.
    - Instead, the capacitor will explode and blow out fire.

☆ Moral of the story:
- Every engineering model has assumptions.
    - If the assumptions are satisfied, then the model will accurately describe the behavior of the real system.
    - If the assumptions are violated, then the model will fail to correctly describe the behavior of the real system.

SOME THOUGHTS ABOUT MATH

- Math is important to engineers!
- Math is at the heart of the models we use to design and analyze physical systems
    - Consumer electronics
    - Medical devices
    - Automotive systems
    - Military systems

- Sometimes math can seem confusing and hard!
  - Sometimes it may even seem like math was invented mainly to torture students!
  - But it's important to (at least try to) remember that all the math we use was actually invented to make life <u>easier</u>!!

- So why does math sometimes seem so hard?
  - Powerful & sophisticated math tools are built up by starting from first principles and taking lots of <u>small</u> simple steps.
    - If you try to "jump in" in the middle,
    - or if you try to move forward without really understanding the previous steps very well,

$\Rightarrow$ Then it will seem hard!!

$\Rightarrow$ And Confusing!!

☆ Moral of the story: it's <u>important</u> to invest the time to study the steps <u>in order</u> and to be sure that you understand each step before moving on to the next one!

MATH REVIEW

- Now it's time to start building up some math tools.
- The first thing we need are some sets of numbers.

DEF: the natural numbers are sometimes called "counting numbers."

- They are the same as the positive integers.
- $\mathbb{N}$ is the symbol for the natural numbers.

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

- They include real numbers > 0 where the decimal part (fraction part) is zero.

NOTE: some people include zero in the natural numbers. But in ECE 2713, we will generally not include zero in $\mathbb{N}$ unless otherwise specified.

DEF: the integers are the real numbers that have no "fraction" part or "decimal" part.

- The integers include all the natural numbers and their additive inverses, as well as zero.

EX: 2 is a natural number in $\mathbb{N}$.

- The additive inverse of 2 is -2, because $2 + (-2) = 0$.

- The symbol for the integers is $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

<u>DEF</u>: the <u>rational numbers</u> are the set of all real numbers that can be written as a ratio of two integers.

- In other words, every rational number can be written as a fraction p/q where p and q are integers.

<u>FACT</u>: if you write a rational number in decimal form, then the fraction part either <u>terminates</u> or <u>repeats</u>.

EX: $\frac{5}{4} = 1.25$ (terminates)

EX: $\frac{4}{3} = 1.3333333\ldots$ (repeats)

- This is sometimes written as $1.\overline{3}$

- The symbol for the rational numbers is $\mathbb{Q}$.

DEF: the irrational numbers are the set of all real numbers that can not be written as a ratio of two integers.

Examples: $\sqrt{2}$, π

FACT: when you write an irrational number in decimal form, the fraction part neither terminates nor repeats.

- The fraction part goes on forever without repeating.

EX: $\pi = 3.141592653\ldots\ldots$

NOTE: unlike the other sets we have considered so far, the irrational numbers don't have any symbol.

DEF: if you put the rational numbers $\mathbb{Q}$ together with the irrational numbers, then you get the set of real numbers.

- The real numbers are the set of all numbers that have no imaginary part (more on this later).
- The real numbers can be thought of as representing all the points on a line of infinite length, measured from some "zero point" or "origin".
    - This line is called the "real line".
- The symbol for the real numbers is $\mathbb{R}$.

Note: ∞ and $-\infty$ are numbers in a sense, but they are not part of the real numbers.

- This means that they are also not part of the naturals, not part of the integers, not part of the rationals, and not part of the irrationals.

∞ is greater than any real number.
$-\infty$ is less than any real number.

☆ The symbols $+\infty$ and $-\infty$ are often used to mean that some quantity is unbounded or fails to converge.

- Some more symbols that are sometimes used to save writing:

$\in$: "is an element of" or "in"

EX: $x \in \mathbb{R}$ read: "x is a real number" or "x is in $\mathbb{R}$"

$\exists$: (backwards "E"): "There exists"

$\forall$: (upside down "A"): "for all".

<u>EX</u>: $\forall x \in \mathbb{Q}, \exists p, q \in \mathbb{Z}$ such that $x = \frac{p}{q}$.

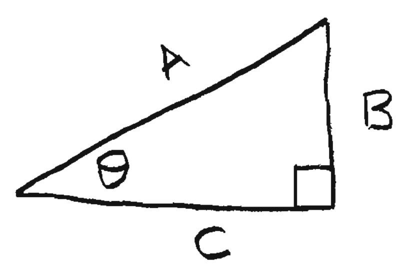
This is read in English as: "for all numbers x that are in the rationals, there exist integers p and q such that $x = p/q$."

⇒ This may seem a little bit "complicated" at first... but it is just a way to <u>save writing</u>.

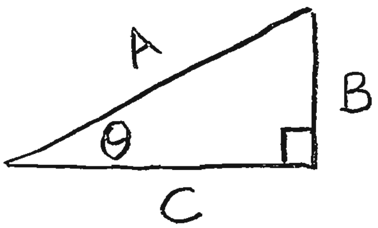
- If you practice some, then it will quickly seem easy and you will start to <u>like</u> the fact that it <u>saves writing</u>.

Some Trigonometry Review

- Suppose we have a right triangle as shown below with sides of length A, B, C and one angle equal to θ radians:



Side A is called the <u>hypotenuse</u>
Side B is called the <u>opposite</u> side
Side C is called the <u>adjacent</u> side



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{C}{A}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{B}{A}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{B}{C} = \frac{\sin\theta}{\cos\theta}$$

$$\arccos\frac{C}{A} = \cos^{-1}\left(\frac{C}{A}\right) = \theta$$

$$\arcsin\frac{B}{A} = \sin^{-1}\left(\frac{B}{A}\right) = \theta$$

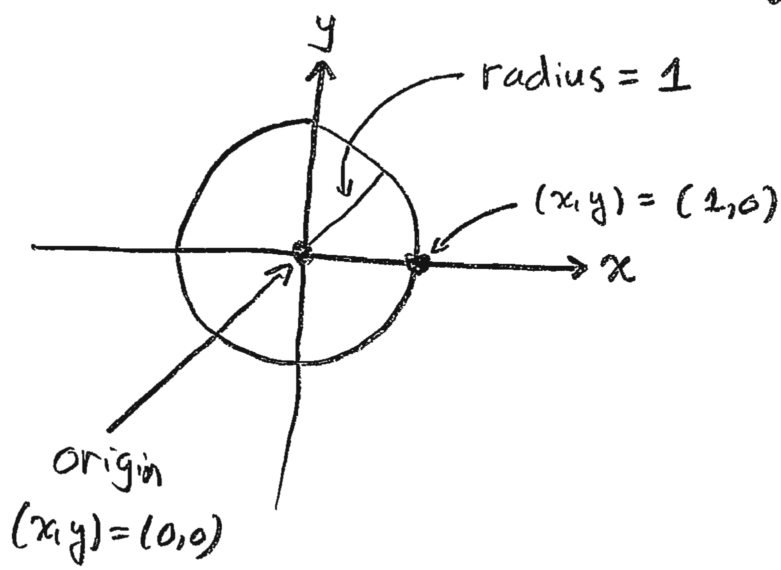
$$\arctan\frac{B}{C} = \tan^{-1}\left(\frac{B}{C}\right) = \theta$$

☆ You should briefly review the other basic trig functions:

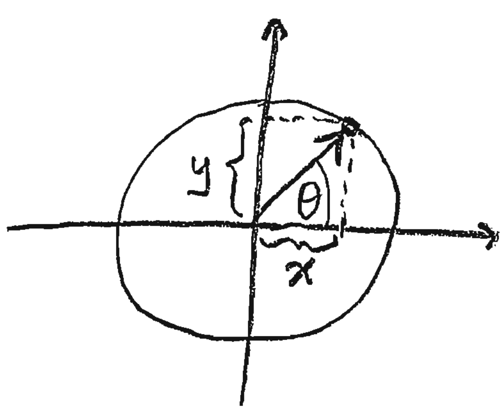
- secant: $\sec\theta = \frac{1}{\cos\theta}$
- cosecant: $\csc\theta = \frac{1}{\sin\theta}$
- cotangent: $\cot\theta = \frac{1}{\tan\theta}$

Relationship Between the Basic Trig Functions and the Unit Circle:

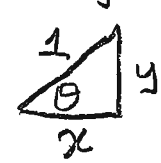
- Imagine a 2D (x, y) plane
- And imagine a circle with radius $= 1$ centered at the origin.
    - This circle is called the "unit circle"

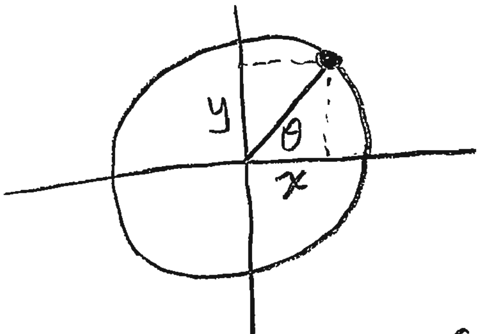


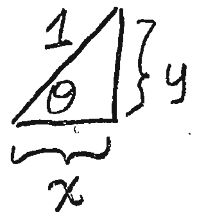
- Each point on the circle has a "horizontal" coordinate x and a "vertical" coordinate y.
- You can think of the point as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$



This defines a right triangle with hypotenuse of length 1:







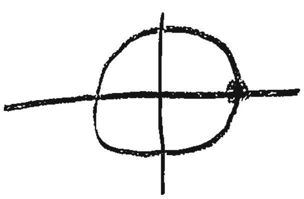
$\cos\theta$ = horizontal coordinate of point $= x$

$\sin\theta$ = vertical coordinate of point $= y$

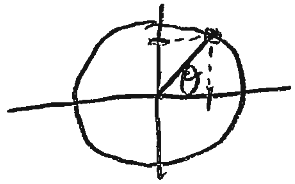
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

- This "unit circle" way of thinking about sin, cos, and tan makes it easier to understand what they mean for larger angles:

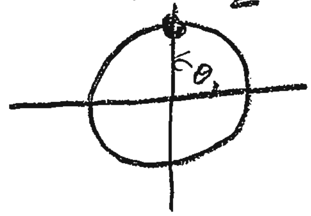
$\theta = 0$



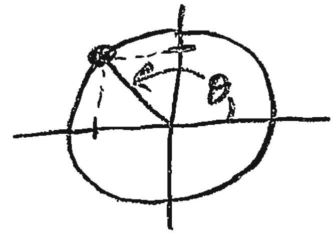
$\theta = \frac{\pi}{4}$



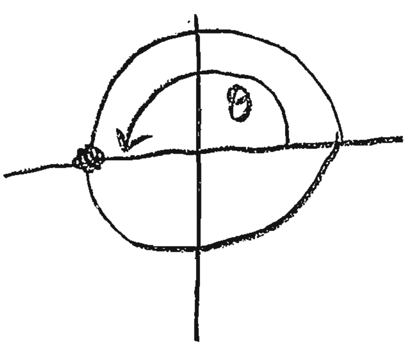
$\theta = \frac{\pi}{2}$



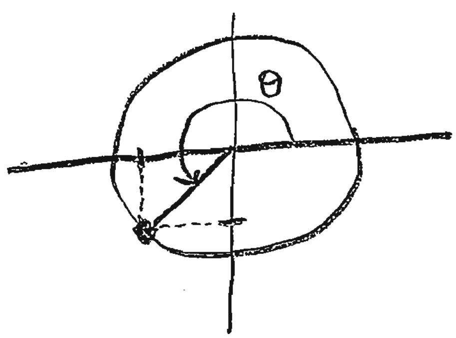
$\theta = \frac{3\pi}{4}$



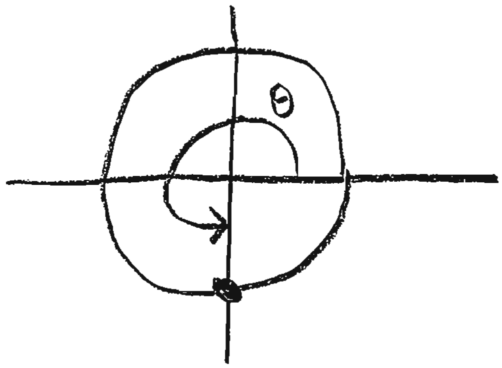
$\theta = \pi$



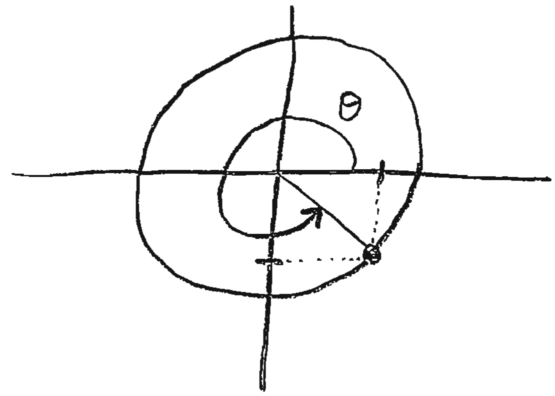
$\theta = \frac{5\pi}{4}$



$\theta = \frac{3\pi}{2}$



$\theta = \frac{7\pi}{4}$

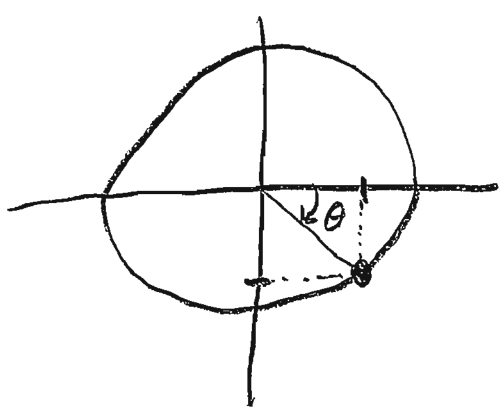


- In each case, the "x" or horizontal coordinate is equal to $\cos\theta$.

- The "y" or vertical coordinate is equal to $\sin\theta$.

<u>Note</u>: this also works for negative angles:

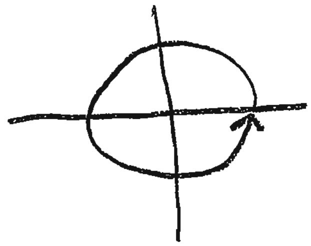
$$\theta = -\frac{\pi}{4}$$



- Has the same sine and cosine as $\theta = \frac{7\pi}{4}$.

- This "unit circle" way of thinking about sine and cosine is very important. Spend some time with it!!

- <u>Converting between radians and degrees:</u>
  - if you move the point once around the circle



  - it is one revolution or "cycle"

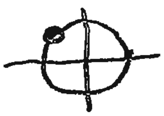
    $= 2\pi$ radians

    $= 360°$

- So 2π rad $= 360$ deg
- So $\dfrac{2\pi \text{ rad}}{360 \text{ deg}} = \dfrac{360 \text{ deg}}{2\pi \text{ rad}} = 1$

Conversion examples:

$$\frac{\pi}{4} \text{ rad} = \frac{\cancel{\pi}}{4} \cancel{\text{rad}} \times \underbrace{\frac{360 \text{ deg}}{2\cancel{\pi} \cancel{\text{rad}}}}_{\text{one}} = \frac{360}{8} \text{deg} = 45 \text{ deg} \checkmark$$



$$135 \text{ deg} = 135 \cancel{\text{deg}} \times \underbrace{\frac{2\pi \text{ rad}}{360 \cancel{\text{deg}}}}_{\text{one}} = \frac{270\pi}{360} \text{ rad} = \frac{3\pi}{4} \text{ rad} \checkmark$$

NOTE: there is a list of trigonometric identities on page 1 of the formula sheet available on the "handouts" section of the course web site.

- Spend some time reviewing it!!

RULES FOR EXPONENTS

- Let a, b, c be numbers
- These rules work for real numbers, complex numbers, and fractional numbers...
  - In other words, they work for numbers!

☆☆ Memorize these rules!!!

1) $a^{b+c} = a^b a^c$

Example: $2^{3+4} = 2^3 2^4 = 8 \cdot 16 = 128$ ✓

2) $(ab)^c = a^c b^c$

Example: $(2 \cdot 3)^4 = 2^4 3^4 = 16 \cdot 81 = 1{,}296$ ✓

3) $(a^b)^c = a^{bc}$

Example: $(4^2)^3 = 4^{2 \cdot 3} = 4^6 = 4{,}096$ ✓

4) $a^{-b} = \left(\frac{1}{a}\right)^b = \frac{1}{a^b}$

Example: $2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}$ ✓

COMPLEX NUMBERS

- The complex numbers are the set of all numbers of the form $a+jb$ where:
  - → $a, b \in \mathbb{R}$ (a and b are reals)
  - → j is a special number called the "imaginary unit".
    - → j is defined by the equation $j^2 = -1$.
- The symbol for the complex numbers is $\mathbb{C}$.

NOTE: in other fields like math and physics, the imaginary unit is written as i.
- In electrical engineering, the symbol "j" was used historically so that "i" could be reserved for electric current.

Some properties of j:

$$\rightarrow \quad j^2 = j \cdot j = (-1)(-1)(j)(j) = (-j)(-j) = (-j)^2$$

→ So $\sqrt{-1}$ has two solutions:

$$\rightarrow \sqrt{-1} = j \text{ or } \sqrt{-1} = -j \Rightarrow \sqrt{-1} = \pm j$$

☆ However, this does not mean that $+j = -j$!!

→ They are not equal

$$\rightarrow \frac{1}{j} = \frac{1}{j}\cdot 1 = \frac{1}{j}\frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$\Rightarrow$ So a j "downstairs" can be traded for a $-j$ "upstairs."

EX: $\frac{5}{j} = -j5$

- For the complex number $z = a + jb$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

  $\rightarrow$ a is called the real part of z. We write $\mathrm{Re}[z] = a$.

  $\rightarrow$ b is called the imaginary part of z. We write $\mathrm{Im}[z] = b$.

  $\Rightarrow$ Note that the real part and imaginary part are both real.

EX: $z = 2 + 3j$

$\mathrm{Re}[z] = 2$
$\mathrm{Im}[z] = 3$

Addition of complex numbers:

- let $z_1 = a + jb$ and $z_2 = c + jd$ be two complex numbers, where $a, b, c, d \in \mathbb{R}$.

- Then
$$z_1 + z_2 = (a+jb) + (c+jd)$$
$$= a + c + jb + jd$$
$$= (a+c) + j(b+d)$$

- So the real part of the sum is $a+c$ = "sum of the real parts."

- And the imaginary part of the sum is $b+d$ = "sum of imaginary parts."

Multiplication of complex numbers:

- use the "foil" rule: first, outside, inside, last:

$$z_1 z_2 = (a+jb)(c+jd)$$
$$= \underbrace{ac}_{\text{first}} + \underbrace{jad}_{\text{outside}} + \underbrace{jbc}_{\text{inside}} + \underbrace{j^2bd}_{\text{last}}$$
$$= \underbrace{(ac - bd)}_{\substack{\text{real part} \\ \text{of product} \\ = ac - bd}} + j\underbrace{(ad + bc)}_{\substack{\text{Imaginary part} \\ \text{of product} \\ = ad + bc}}$$

- So one way to think about j is that it is a special number that controls when and how there can be mixing between the real and imaginary parts:

→ Addition:

$$(a+jb) + (c+jd) = (a+c) + j(b+d)$$

$\Rightarrow$ <u>NO MIXING ALLOWED</u>

→ Multiplication:

$$(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$$

$\Rightarrow$ <u>CERTAIN KINDS OF MIXING ALLOWED</u>

<u>EX</u>: $z_1 = 1+2j \qquad z_2 = 3+j4$

$$z_1 + z_2 = (1+2j) + (3+j4) = \underline{\underline{4+j6}}$$

$$z_1 z_2 = (1+2j)(3+j4) = 3 + j4 + 6j + 8j^2$$

$$= 3 + j10 - 8$$

$$= \underline{\underline{-5+j10}}$$

Euler's Number

- Leonhard Euler was a Swiss mathematician who lived from 1707 to 1783.

- He made many important discoveries that are still important today.

- The "Eu" in his last name is pronounced like the "oi" in the word "oil"... not like the "Eu" in "Europe".

  → So it's pronounced "oil-er"... not "you-ler".

- Although it was actually first discoverd by Jacob Bernoulli, the number "e" is called "Euler's number" in honor of Euler

  → It is an irrational real number.

  → $e = 2.71828\ldots$

A FACT ABOUT NUMBERS

- Any number can be written in more than one way.

- For example,

$$10 = 4+6$$
$$10 = 8+2$$
$$10 = 2 \cdot 5$$

Some Basic Results on Series

- At some point in a math class that you have taken or will take, you have learned or will learn about series.

- In ECE 2713, we aren't really interested in proving anything about the convergence of series, but we will need to use some results about them.

- Here is one:

$$e = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

Taylor Series and Maclaurin Series

- A Taylor series is a certain kind of power series (you have or will study them in some math class).

- A Maclaurin series is a certain kind of Taylor series (one that is centered around zero).

- Here are the Maclaurin series that are most important to us in ECE 2713:
  - For any real or complex number x, (in other words: $\forall x \in \mathbb{C}$)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$\Rightarrow$ In fact, these formulas are actually the formal <u>definitions</u> of e^x, $\cos x$, and $\sin x$ when x is complex.

Capital Sigma "do loops"

- suppose you have an array "data" with 100 integer elements.
- Suppose you have to write a program to compute the sum.
- You could do it like this:

```
sum= data[0];
sum= sum + data[1];
sum= sum + data[2];
   ⋮
sum= sum + data[99];
```

- what a pain!!!
- So how do you really do it?

→ You use a "do loop": (or "for loop")

```
sum=0;
for (i=0; i<100; i++) {
    sum+= data[i];
}
```

- Easy, right? And helpfull!!

- Mathematicians actually figured out how to do this centuries before there were computers
  → They did it to <u>save</u> <u>writing</u> and make life easier.

- when you are writing math, you can make a do loop by using the capital Greek letter "Σ":

$$\text{sum} = \sum_{i=0}^{99} \text{data}[i]$$

- This saves <u>tons</u> of writing for the series we just looked at:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Good for any real <u>or</u> complex number x

- How should you think about This?
  - In each of the last four equations on PAGE 1.26,

    → The left side and right side are just <u>two</u> <u>different ways</u> of writing <u>the</u> <u>same</u> <u>number</u>.

    → If this seems confusing, look back at the top of PAGE 1.23 now.

More on Complex Numbers

- To <u>conjugate</u> a complex number means to negate the imaginary part.

  - In other words, you replace the imaginary part with its additive inverse.

→ The conjugate of $a+jb$ is $a-jb$.
→ The conjugate of $2+j3$ is $2-j3$.
→ The conjugate of $-7-j5$ is $-7+j5$

- In electrical and computer engineering, the conjugate is written with a superscript asterisk like this:

$$z^*.$$

→ So if $z = 6-j3$, then $z^* = 6+j3$.

- You can always compute the conjugate by multiplying every j by -1.

  - This works even if the complex number is written in a complicated way where there is more than one j.

For example,

$$z = 1+2j + (3+4j)(5-6j) + e^{5j}$$

$$z^* = 1-2j + (3-4j)(5+6j) + e^{-5j}$$

→ In other words, just "minus" every j... i.e., replace every j with $(-j)$.

<u>Note:</u> Every <u>real</u> number is also a complex number.

- A real number is a complex number with an imaginary part that is zero.

- For example, notice that 5 is equal to $5+j0$.

- Since $0 = -0$, conjugating a <u>real</u> number doesn't actually do anything:

$$5^* = (5+j0)^* = 5-j0 = 5$$

Note: zero is the only number that is its own additive inverse:

This is not true for other numbers $\left\{ \begin{array}{l} -0 = 0 \\ \text{and } \quad 0 + (-0) = 0 + 0 = 0. \end{array} \right.$

$\Rightarrow$ So if you know that $x = -x$, then $x = 0$.

Note: in some other fields, different notation may be used to mean "conjugate."

- For example, sometimes in math you might use an "overbar" and write $\bar{z}$ for the conjugate.
- But in ECE we almost always use the asterisk and write z^*.

Division of Complex Numbers

- When you write your complex numbers in the form $z = a + jb$, $a, b \in \mathbb{R}$,
  - it is called "rectangular" or "cartesian" form.
  - there is another form called "polar form" that we will talk about in a few minutes.
- when you write your complex numbers in rectangular form, division will generally give you j's downstairs.
  - To work the quotient and simplify the number, you have to get the j's out of the denominator.
- To do that, you multiply the quotient (i.e., the fraction) times <u>ONE</u> in this tricky form;

$$1 = \frac{\text{conjugate of denominator}}{\text{conjugate of denominator}}$$

<u>Example:</u> let $z_1 = 1-2j$ and $z_2 = 3+4j$

- Then $\frac{z_1}{z_2} = \frac{1-2j}{3+4j} = \frac{1-2j}{3+4j} \cdot \underbrace{\frac{3-4j}{3-4j}}_{\text{one}}$

$$= \frac{(1-2j)(3-4j)}{(3+4j)(3-4j)} = \frac{3-4j-6j+8j^2}{9-12j+12j-16j^2}$$

use "foil" rule upstairs and downstairs

$$= \frac{3-10j-8}{9+16} = \frac{-5-10j}{25}$$

$$= -\frac{5}{25} - j\,\frac{10}{25} = \underline{\underline{-\frac{1}{5} - j\,\frac{2}{5}}}$$

- You can use the same trick to invert a complex number... in other words to compute $z^{-1} = \frac{1}{z}$.

- Keeping z_1 as above ($z_1 = 1-2j$), we have

$$z_1^{-1} = \frac{1}{z_1} = \frac{1}{1-2j} = \frac{1}{1-2j} \cdot \underbrace{\frac{1+2j}{1+2j}}_{\text{one}}$$

$$= \frac{1+2j}{1+2j-2j-4j^2} = \frac{1+2j}{1+4} = \frac{1}{5} + \frac{2}{5}j$$

- Now that we have talked about complex addition, subtraction, conjugation, multiplication, and division,
  - here are some more examples of doing arithmetic on complex numbers in rectangular form:

- Let $z = 2 + 3j$ and $w = 5 - 2j$.
- Then

$$z + w = (2 + j3) + (5 - j2) = (2+5) + j(3-2)$$
$$= \underline{\underline{7 + j}}.$$

$$w - z = (5 - 2j) - (2 + 3j) = (5-2) + j(-2-3)$$
$$= 3 - j5 \quad //$$

$$zw = (2+3j)(5-2j) = 10 - 4j + 15j - 6j^2$$
$$= 10 + 11j + 6 = 16 + j11 \quad //$$

$$z^* = (2+3j)^* = 2 - 3j \quad //$$
$$w^* = (5-2j)^* = 5 + 2j \quad //$$

$$\frac{w}{z} = \frac{5-2j}{2+3j} = \frac{5-2j}{2+3j} \cdot \frac{2-3j}{2-3j} = \frac{10 - 15j - 4j + 6j^2}{4 - 6j + 6j - 9j^2}$$
$$= \frac{10 - 19j - 6}{4 + 9} = \frac{4 - 19j}{13} = \frac{4}{13} - j\frac{19}{13} \quad //$$

- There is a second way to write complex numbers.
  - It is called "polar form."

- Let z be any complex number
  - Then there exist real numbers a and b such that $z = a + jb$.

    (Recall: we could save time and pencil lead by writing the above statement this way:

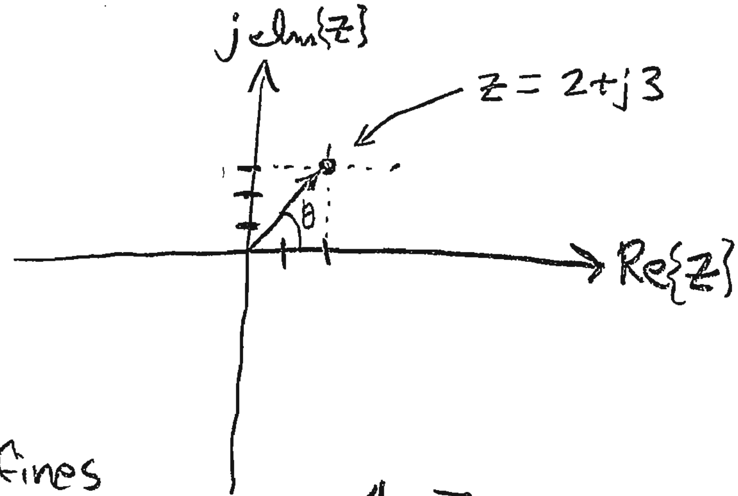
    $\forall z \in \mathbb{C}, \ \exists \ a, b \in \mathbb{R}$ such that $z = a + jb$)

- If you make a 2D plane with:
  - horizontal axis $= \mathrm{Re}\{z\} = a$
  - vertical axis $= j\,\mathrm{Im}\{z\} = jb$,

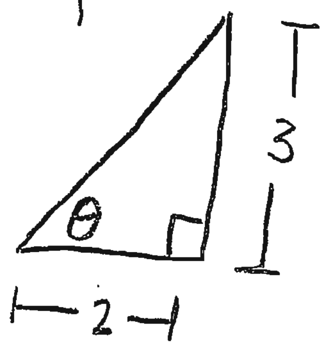
- Then we can graph the complex number z as a <u>point</u> or a <u>vector</u> in this plane.

- This plane is called <u>the complex plane</u>.

<u>EX</u>: $z = 2 + j3$



Notice that this defines a right triangle:



- The hypotenuse of this triangle has length $r = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$.
- This is called the <u>magnitude</u> or <u>modulus</u> of the complex number.
  - We write the magnitude using "absolute value" notation like this:

$$r = |2 + j3| = \sqrt{13}$$

- By using some geometry along with the "unit circle" way of thinking about sine, cosine, and tangent that we discussed back on pages 1.13 through 1.15,

  - It can be shown more generally that:
    - for any complex number $z = a + jb$

$$|z| = \sqrt{a^2 + b^2}$$

$$= \sqrt{(\mathrm{Re}\{z\})^2 + (\mathrm{Im}\{z\})^2}$$

$$= \left[a^2 + b^2\right]^{1/2}$$

Note: $|z| \geq 0$ because it is a length.

    - we often use the symbol "r" for the magnitude of a complex number

FACT: for any complex number $z = a + jb$,

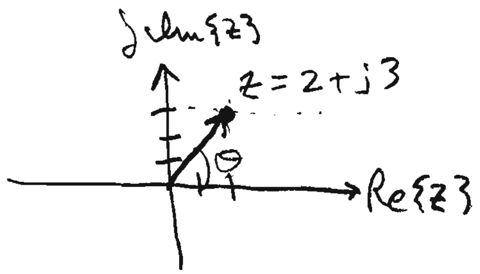
$$zz^* = |z|^2$$

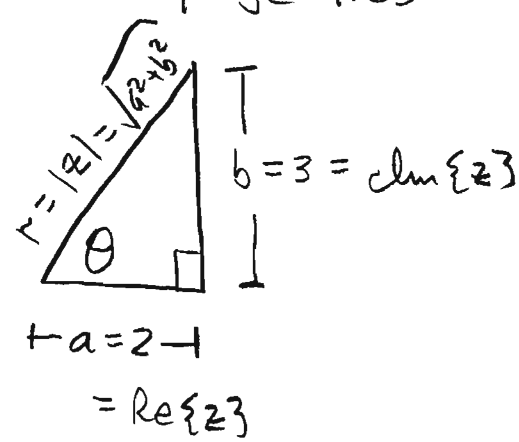
$\longrightarrow$

- It's not too hard to show this:

$$zz^* = (a+jb)(a-jb)$$
$$= a^2 - jab + jab - b^2 j^2$$
$$= a^2 + b^2$$
$$= |z|^2. \qquad \left(\text{because } |z| = r = \sqrt{a^2+b^2}\right)$$

- Back to our right triangle from page 1.35:





Notice that:

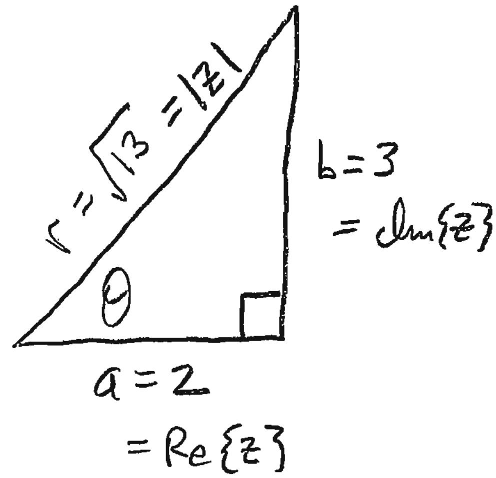
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{r} \quad \left(= \frac{\mathrm{Re}\{z\}}{|z|}\right)$$

→ multiply both sides by r:

$$r\cos\theta = a = \mathrm{Re}\{z\}. \quad ☆☆$$

- Similarly,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{r}$$



- Multiply both sides by r:

$$r\sin\theta = b = \mathrm{Im}\{z\}$$

- Also, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} \quad \left(= \frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}} \right)$

$$\Rightarrow \theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}\right)$$

- By using some geometry along with the "unit circle" way of thinking about sine, cosine, tangent from pages 1.13 through 1.15,

$\Rightarrow$ It can be shown that all of this works for <u>any</u> complex number z,

- even if the angle θ is outside of the first quadrant.

SUMMARY: any complex number z can be written in rectangular form as

$$z = a + jb$$

where $a, b \in \mathbb{R}$,

→ Or in polar coordinates as

$$z = r \measuredangle \theta$$

where $\quad r = \sqrt{a^2+b^2} \geq 0$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

⇒ r is called the magnitude or modulus of z and is written as $|z|$

⇒ θ is called the angle or "argument" of z and is written $\measuredangle z$ or $\arg z$.

⇒ The number z is sometimes written in all of these ways:

rectangular:	Polar:
$a+jb$	$r \measuredangle \theta$
(a,b)	(r,θ)

- Here are the equations for converting between polar and rectangular coordinates for complex numbers:

$$z = a + jb = r\angle\theta$$

$$a = \mathrm{Re}\{z\} = r\cos\theta$$

$$b = \mathrm{Im}\{z\} = r\sin\theta$$

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{zz^*}$$

$$z = r\cos\theta + jr\sin\theta$$

$$= r\{\cos\theta + j\sin\theta\}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

MEMORIZE THESE EQUATIONS !!!!!

$\Rightarrow$ MEMORIZE THESE EQUATIONS

memorize these equations!!

memorize these equations!

- While the book sometimes writes (r, θ) or $r\measuredangle\theta$ to represent a complex number in polar form,

  → There is another <u>better</u> way that is much more widely used.

  → It is based on the power series that we saw on pages 1.24 and 1.26 and on the exponent formulas from page 1.17.

- We have not yet talked very much about how exponents work with complex numbers, but it should be clear to you that, for any complex number z,

$$z^2 = z \cdot z$$

$$z^3 = z \cdot z \cdot z$$

$$z^4 = z \cdot z \cdot z \cdot z$$

etc...

$$z^{-1} = \frac{1}{z}$$

$$z^{-3} = \frac{1}{z \cdot z \cdot z} = \frac{1}{z^3}$$

⇒ It would be a pain, but given any complex number z and any integer n, you <u>could</u> calculate the number $w = z^n$ if you had to.

- use the foil rule to do the multiplying,
- use the "conjugate trick" to clear any j's out of the denominator if $n < 0$.

- Now, for any complex number z, e^z is also a complex number.
  - what does e^z mean when $z \in \mathbb{C}$?
  - How should we think about this?

→ e^z is a number that is equal to: (see pages 1.24, 1.26)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

⇒ You could write a computer program to evaluate the right side if you had to.

→ The right side is a number that's not too hard to think about

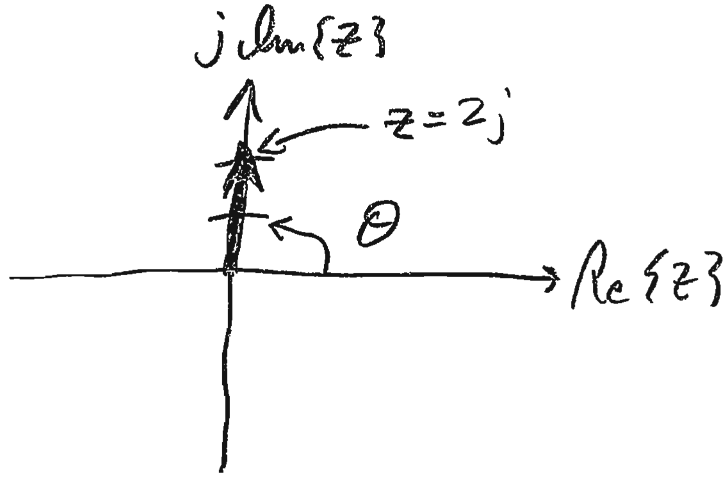
→ And that number is e^z.

DEF: if z is a complex number and the real part is zero, then z is called "pure imaginary."

- A pure imaginary number can be written as $z = 0 + jb = jb$, where $b \in \mathbb{R}$.

- Polar form of a pure imaginary number:
  - The easiest way to write a pure imaginary number in polar form is to graph the number in the complex plane and simply read the magnitude and angle off of the graph.

EX: $z = 2j$



$r = |z| = 2$

$\theta = \measuredangle z = \pi/2$

FACT: if you add an integer multiple of 2π to any angle,

- It does not change the cosine
- It does not change the sine
- It does not change tangent

- So we could also use

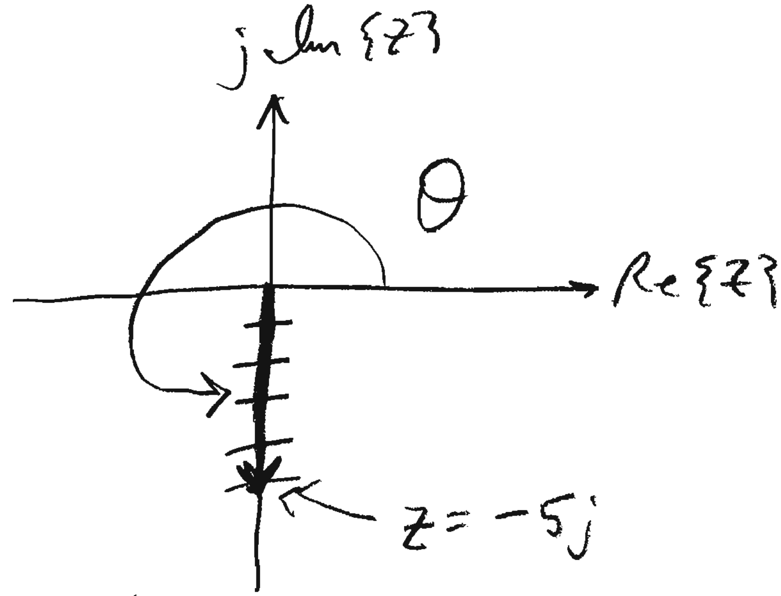
$$r = |z| = 2$$

$$\theta = \measuredangle z = \frac{\pi}{2} - 2\pi = -\frac{3\pi}{2}$$

<u>EX</u>: $z = -5j$

$$r = |z| = 5$$

$$\theta = \measuredangle z = \frac{3\pi}{2}$$



→ you could alternatively use $\theta = \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$.

- To work these same two examples using the formulas on page 1.40, you need to look at the graph of arctan and realize that:

$$\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

So, doing them again:

<u>EX</u>: $z = 2j$

$a = \mathrm{Re}\{z\} = 0$

$b = \mathrm{Im}\{z\} = 2$

$r = |z| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$

$$\theta = \measuredangle z = \arctan \frac{b}{a}$$

$$= \lim_{a \to 0} \arctan \frac{2}{a}$$

$$= \lim_{x \to \infty} \arctan x$$

$$= \pi/2$$

$\longrightarrow$

<u>EX</u>: $z = -5j$

$$a = \text{Re}\{z\} = 0$$

$$b = \text{Im}\{z\} = -5$$

$$r = |z| = \sqrt{0^2 + (-5)^2} = \sqrt{25} = 5$$

$$\theta = \measuredangle z = \arctan \frac{b}{a}$$

$$= \lim_{a \to 0} \arctan \frac{-5}{a}$$

$$= \lim_{x \to -\infty} \arctan x$$

$$= -\pi/2 .$$

- Now we are going to do something that is very very important!

  → You need to make sure to understand every step!!

Let $\theta \in \mathbb{R}$.

- then $j\theta$ is a pure imaginary number.

step (1) : Use the series on pages 1.24 and and 1.26 to write $\cos\theta$ as a power series:

$$(*) \qquad \cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{2n!}$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots$$

Step ② : use the formulas (pages 1.24, 1.26) to write $\sin\theta$ in a power series:

(**)

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

Step ③ : Multiply the series (**) for $\sin\theta$ times j:

(***)

$$j\sin\theta = j\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= j\theta - j\frac{\theta^3}{6} + j\frac{\theta^5}{5!} - j\theta^7/7! + j\frac{\theta^9}{9!} - \cdots$$

Step 4: Let $z = e^{j\theta}$ be a complex number.

NOTE: θ is still the <u>same</u> real number as in steps 1 – 3.

$\rightarrow$ write $z = e^{j\theta}$ in a power series using the formulas on pages 1.24 and 1.26:

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{(j\theta)^n}{n!}$$

$$= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

$$= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \cdots$$

$$= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j(j^2)\theta^3}{3!}$$

$$+ \frac{(j^2)(j^2)\theta^4}{4!} + \frac{j(j^2)(j^2)\theta^5}{5!} + \ldots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{(-1)(-1)\theta^4}{4!}$$

$$+ \frac{j(-1)(-1)\theta^5}{5!} + \ldots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} + \ldots$$

$\Rightarrow$ So:

(****)

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{6}$$

$$+ \frac{\theta^4}{24} + j\frac{\theta^5}{5!} + \ldots$$

Step 5 : Add together the series for $\cos\theta$ from step 1 and the series for $j\sin\theta$ from step 3 :

$$\cos\theta + j\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n+1)!} + j \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \left\{ 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots \right\}$$

$$+ \left\{ j\theta - j\frac{\theta^3}{6} + j\frac{\theta^5}{5!} - j\frac{\theta^7}{7!} + j\frac{\theta^9}{9!} - \cdots \right\}$$

$$= 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{6} + \frac{\theta^4}{24} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j\frac{\theta^7}{7!}$$

$$+ \frac{\theta^8}{8!} + j\frac{\theta^9}{9!} - \cdots$$

$\Rightarrow$ Compare this to the series (****) we got for $e^{j\theta}$ on page 1.50.

⇒ They are the same number!!!

⇒ For <u>any</u> real number θ,

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

- This is called <u>Euler's formula</u>.
- It is <u>MEGA</u> <u>important</u>.

⇒ MEMORIZE THIS EQUATION!!!!

- Let $\theta \in \mathbb{R}$ be any real number.
  - Then $e^{j\theta} = \cos\theta + j\sin\theta$
  - Note that $(-\theta)$ is also a real number.
  - So $e^{j(-\theta)} = e^{-j\theta} = \cos(\theta) + j\sin(-\theta)$

$$= \cos\theta - j\sin\theta$$

(because sine is odd)

Then :

$$e^{j\theta} + e^{-j\theta} = \cos\theta + j\sin\theta + \cos\theta - j\sin\theta$$

$$= 2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

And :

$$e^{j\theta} - e^{-j\theta} = \cos\theta + j\sin\theta - \cos\theta + j\sin\theta$$

$$= 2j\sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- These are alternate ways of writing Euler's formula. The book calls them "inverse Euler formulas", but that doesn't actually make any sense.

Summary of Euler's formula:

$\forall \theta \in \mathbb{R}$:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

→ These are good for <u>any</u> real number θ.

→ MEGA MEGA MEGA IMPORTANT!!!

→ MEMORIZE!!!!

- How to think about it:
  - The left side of each equation above is a number.
  - The right side is just a different way of writing that exact same number.

Now: here is the better way that we briefly mentioned back on page 1.41:

- Let z be any complex number.
  - Then you can write z in rectangular form as $z = a + jb$ where $a, b \in \mathbb{R}$.
  - You can write z in polar form as

$$z = re^{j\theta} = r\{\cos\theta + j\sin\theta\} = r\cos\theta + jr\sin\theta$$

- The relationships between (a, b) and (r, θ) are:

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{zz^*}$$

$$\theta = \measuredangle z = \arg z = \arctan \frac{b}{a}$$

$$\mathrm{Re}\{z\} = a = r\cos\theta$$

$$\mathrm{Re}\{z\} = \frac{z + z^*}{2}$$

$$\mathrm{Im}\{z\} = b = r\sin\theta$$

$$\mathrm{Im}\{z\} = \frac{z - z^*}{2j}$$

$\Rightarrow$ You MUST memorize everything on this page!

EX: $z = (3-j4)^* = 3+j4$

$a = \mathrm{Re}\{z\} = 3$

$$a = \mathrm{Re}\{z\} = \frac{z+z^*}{2} = \frac{3+j4+(3-j4)}{2}$$

$$= \frac{3+3+j4-j4}{2} = \frac{6}{2} = 3$$

$b = \mathrm{Im}\{z\} = 4$

$$b = \mathrm{Im}\{z\} = \frac{z-z^*}{2j} = \frac{3+4j-(3-4j)}{2j}$$

$$= \frac{3-3+4j+4j}{2j} = \frac{8j}{2j} = 4$$

$$r = |z| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$r = \sqrt{zz^*} = \{(3+j4)(3-j4)\}^{1/2}$$

$$= \{9-j12+j12-(j^2)16\}^{1/2}$$

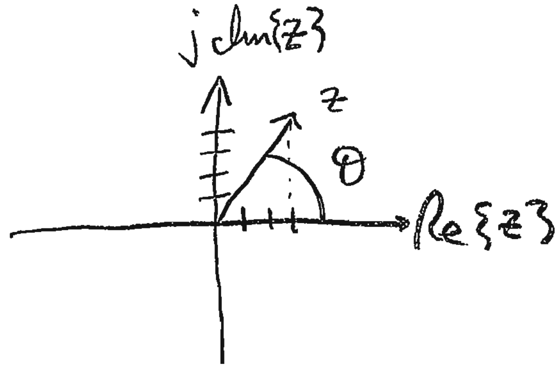
$$= \{9+16\}^{1/2} = \sqrt{25} = 5$$

$\longrightarrow$

EX ... $z = 3 + j4 \qquad z^* = 3 - j4$

$$\theta = \arctan \frac{b}{a} = \arctan \frac{4}{3}$$

→ θ is a first quadrant angle, so we can just use "$\tan^{-1}$" on our calculator to evaluate



$$\theta = \arctan \frac{4}{3} = 0.927295 \text{ rad}$$

$$\text{Re}\{z\} = r\cos\theta = 5\cos(0.927295) = 3 \checkmark$$

$$\text{Im}\{z\} = r\sin\theta = 5\sin(0.927295) = 4 \checkmark$$

$$z = 3 + j4 \quad \text{(rectangular form)}$$

$$= 5e^{j0.927295} \quad \text{(polar form)}$$

EX: $z = -2 + j3$ $\qquad z^* = -2 - j3$

$$a = \mathrm{Re}\{z\} = -2$$

$$a = \mathrm{Re}\{z\} = \frac{z + z^*}{2} = \frac{-2 + j3 + (-2 - j3)}{2}$$

$$= \frac{-2 + (-2) + j3 - j3}{2} = \frac{-4}{2} = -2$$

$$b = \mathrm{Im}\{z\} = 3$$

$$b = \mathrm{Im}\{z\} = \frac{z - z^*}{2j} = \frac{-2 + j3 - (-2 - j3)}{2j}$$

$$= \frac{-2 - (-2) + j3 - (-j3)}{2j}$$

$$= \frac{-2 + 2 + j3 + j3}{2j} = \frac{j6}{2j} = 3$$

$$r = |z| = \sqrt{(-2)^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

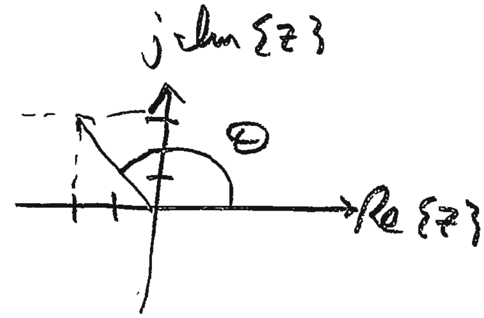
$$r = \sqrt{zz^*} = \sqrt{(-2 + j3)(-2 - j3)}$$

$$= \sqrt{4 + j6 - j6 - (j^2)9} = \sqrt{4 + 9} = \sqrt{13}$$

EX. . . – $z = -2 + j3$ $\quad z^* = -2 - j3$

$\theta = \arctan \frac{b}{a} = \arctan \frac{3}{-2}$

→ θ is in the 2nd quadrant.

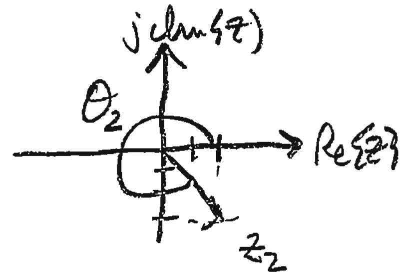


→ $\tan^{-1}$ will not give the right answer.

→ "atan" and $\tan^{-1}$ always give an angle between $-\pi/2$ and $\pi/2$.

→ In this case they will give you the angle for $z_2 = 2 - j3$

$\theta_2 = \arctan \frac{-3}{2}$



– On the computer, you can find θ using

$\theta = \text{atan2}(3, -2)$

– when working the problem by hand, you have to find θ_2 and then add or subtract π (either will work)

EX...

$z = -2 + j3 \qquad z^* = -2 - j3$

$\theta_2 = \arctan(-1.5) = -0.982794$

$\theta = \theta_2 + \pi = 2.15880$

in polar, $z = \sqrt{13}\, e^{j2.15880} \quad (= re^{j\theta})$

Converting back to rectangular:

$$\begin{aligned} Re\{z\} = a &= r\cos\theta \\ &= \sqrt{13}\cos(2.15880) = -2 \checkmark \end{aligned}$$

$$\begin{aligned} Im\{z\} = b &= r\sin\theta \\ &= \sqrt{13}\sin(2.15880) = 3 \checkmark \end{aligned}$$

- For adding and subtracting complex numbers, it is best to write them in rectangular form:

$$z_1 = a + jb$$

$$z_2 = c + jd$$

$$z_1 + z_2 = (a+c) + j(b+d)$$

$$z_1 - z_2 = (a-c) + j(b-d)$$

- We have already seen how to multiply and divide complex numbers in rectangular form.
  - This works pretty well when the real and imaginary parts are "nice" numbers.
  - But if they are messy, it's usually better to write the complex numbers in polar form.

- Multiplication in polar form:

$$z_1 = r_1 e^{j\theta_1}$$

$$z_2 = r_2 e^{j\theta_2}$$

$$z_1 z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2})$$

$$= r_1 r_2 e^{j\theta_1} e^{j\theta_2}$$

$$= r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

magnitude of product $= r_1 r_2$

angle of product $= \theta_1 + \theta_2$

$$\frac{z_1}{z_2} = z_1 (z_2)^{-1} = r_1 e^{j\theta_1} (r_2 e^{j\theta_2})^{-1}$$

$$= r_1 e^{j\theta_1} (r_2^{-1} e^{-j\theta_2})$$

$$= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

magnitude: $\frac{r_1}{r_2}$

angle: $\theta_1 - \theta_2$

- For raising complex numbers to powers, use the exponent rules on page 1.17.

- You will sometimes need to calculate e^z where $z \in \mathbb{C}$. Here's how to handle that:

$$z = a + jb = re^{j\theta}$$

$$e^z = e^{a+jb} = e^a e^{jb}$$

$$= e^a \{\cos b + j\sin b\}$$

$$= e^a \cos b + je^a \sin b$$

<u>Note</u>: for any $\theta \in \mathbb{R}$,

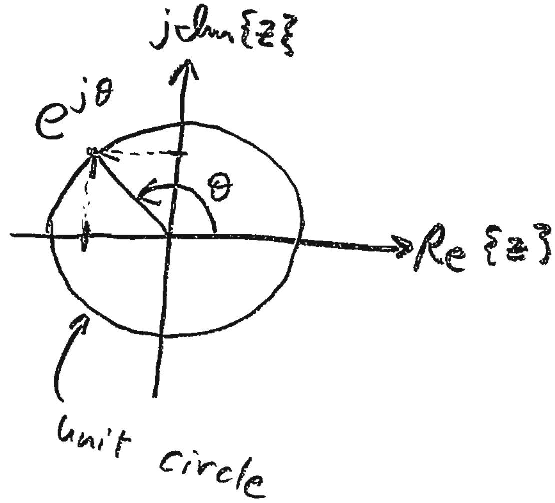
$$|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \sqrt{1} = 1.$$

- So $e^{j\theta}$, $\theta \in \mathbb{R}$, is a complex number that has unit length. If you graph this number in the complex plane, it lies <u>on the unit circle</u>.

  - The horizontal coordinate is the real part of the number. It is equal to $\cos\theta$.

  - The vertical coordinate is the imaginary part of the number. It is equal to $\sin\theta$.



- If you imagine θ running through the real numbers from big negative numbers to big positive numbers in order...

  - Then the point $e^{j\theta}$ spins around the unit circle counterclockwise.

    - Each time θ goes through 2π radians, the point goes around the unit circle one time.

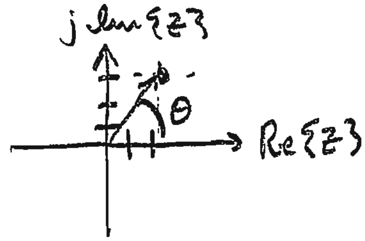
- How to Raise a Complex number to a real power:
  - Let $z = re^{j\theta}$ be a complex number where r and θ are real and $r \geq 0$.
  - Let x be a real number.
  - Then
$$z^x = (re^{j\theta})^x$$
$$= (r)^x (e^{j\theta})^x = r^x e^{j\theta x}$$
$$= r^x \left[\cos(\theta x) + j\sin(\theta x)\right]$$
$$= \underbrace{r^x \cos(\theta x)}_{\text{real part}} + j\,\underbrace{r^x \sin(\theta x)}_{\text{imaginary part}}$$

<u>EX</u>: Let $z = 2 + j3$ and let $x = -\frac{1}{7}$

- To find the number z^x, we need to first write z in polar form:

$$r = |z| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$

$$\theta = \arg z = \arctan\left(\tfrac{3}{2}\right)$$



→ It's a first quadrant angle, so atan will give the right answer.

$$\theta = \arctan(1.5) = 0.982794 \text{ rad}$$

→

– So $z = 2 + j3 = \sqrt{13}\, e^{j0.982794}$

Now, using the formula we derived on page 1.65, we get

$$z^x = (2+j3)^{-\frac{1}{7}}$$

$$= r^x \cos(\theta x) + j r^x \sin(\theta x)$$

$$= (\sqrt{13})^{-1/7} \cos[(0.982794)(-1/7)]$$

$$+ j(\sqrt{13})^{-1/7} \sin[(0.982794)(-1/7)]$$

$$= 0.832593 \cos[-0.140399]$$

$$+ j(0.832593) \sin[-0.140399]$$

$$= (0.832593)(0.99016) + j(0.832593)(-0.139938)$$

$$= 0.82440 - j0.11651$$

- Before we talk about how to raise a complex number to a complex power, we need to briefly review natural logarithms for real numbers.

- If $y \in \mathbb{R}$, then

$$x = \ln y$$

is a number. It is the power to which you have to raise the number e in order to get y.

$\Rightarrow$ In other words, $x = \ln y$ is the number x such that $e^x = y$.

- From this definition, it follows that:

① $\ln(e^x) = x$ for any real number x.

② if $x > 0$, then $x = e^{\ln x}$

$\rightarrow$ we need to use the <u>second</u> <u>one</u>.

---

side Note: if $y < 0$, then the number $\ln y$ is <u>complex</u>.

- Raising a complex number to a complex power:

  - Suppose we have two complex numbers z_1 and z_2.
  - We want to compute the complex number

$$(z_1)^{z_2}$$

    - in other words, z_1 to the power z_2.

  → write z_1 in polar form as $z_1 = r_1 e^{j\theta_1}$

  → write z_2 in rectangular form as $z_2 = a_2 + jb_2$.

  ⇒ Note: $r_1 \geq 0$, so $r_1 = e^{\ln r_1}$

- We have:

$$
\begin{aligned}
(z_1)^{z_2} &= (r_1 e^{j\theta_1})^{a_2+jb_2} \\
&= (r_1)^{a_2+jb_2} (e^{j\theta_1})^{a_2+jb_2} \\
&= r_1^{a_2} r_1^{jb_2} e^{j\theta_1 a_2} e^{(j^2)\theta_1 b_2} \\
&= r_1^{a_2} \cdot r_1^{jb_2} \cdot e^{j\theta_1 a_2} \cdot e^{-\theta_1 b_2}
\end{aligned}
$$

complex in general

real and ≥ 0

So $(z_1)^{z_2} = \left[r_1^{a_2} e^{-\theta_1 b_2}\right] r_1^{jb_2} e^{j\theta_1 a_2}$

$\rightarrow$ but $r_1 = e^{\ln r_1}$

$$= \left[r_1^{a_2} e^{-\theta_1 b_2}\right] \left(e^{\ln r_1}\right)^{jb_2} e^{j\theta_1 a_2}$$

$$= \left[r_1^{a_2} e^{-\theta_1 b_2}\right] e^{jb_2 \ln r_1} e^{j\theta_1 a_2}$$

$$= \left[r_1^{a_2} e^{-\theta_1 b_2}\right] \exp\left[j\left(b_2 \ln r_1 + \theta_1 a_2\right)\right]$$

$\rightarrow$ magnitude $= r_1^{a_2} e^{-\theta_1 b_2}$

$\rightarrow$ angle $= b_2 \ln r_1 + \theta_1 a_2$

In rectangular form, we get (by Euler's formula):

$$(z_1)^{z_2} = r_1^{a_2} e^{-\theta_1 b_2} \cos\left[\theta_1 a_2 + b_2 \ln r_1\right] + j r_1^{a_2} e^{-\theta_1 b_2} \sin\left[\theta_1 a_2 + b_2 \ln r_1\right]$$

- At some point in one of your math classes, you will run into De Moivre's formula:

$$[\cos\theta + j\sin\theta]^n = \cos n\theta + j\sin n\theta.$$

→ It is very easy to derive using Euler's formula:

$$(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta}$$

$$= \cos n\theta + j\sin n\theta. \quad //$$

<u>FACT</u>: if two complex numbers are equal, then:

- Their real parts must be equal
- Their imaginary parts must be equal
- Their magnitudes must be equal
- Their angles must be equal up to adding or subtracting integer multiples of 2π.

→ The last one follows because adding an integer multiple of 2π to an angle

- Does not change the cosine
- Does not change the sine
- Does not change the tangent

→ In other words, if $r \geq 0$ and θ are real numbers and k is any integer,

then $re^{j(\theta + 2\pi k)}$

$$= r\cos(\theta + 2\pi k) + jr\sin(\theta + 2\pi k)$$

$$= r\cos\theta + jr\sin\theta$$

$$= r\{\cos\theta + j\sin\theta\}$$

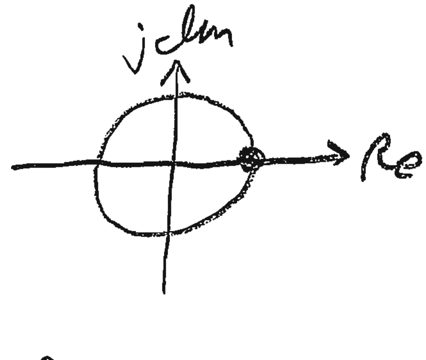
$$= re^{j\theta}.$$

— You can use this to solve for the so called "N^{th} roots of unity."

- They are the solutions of the equation

$$z = \sqrt[N]{1}. \qquad (*)$$

- There are N unique solutions.
- To find them, begin by writing 1 as a complex number in polar form:



- The magnitude is $\underline{1}$
- The angle is zero, or $\pm 2\pi$, or $\pm 4\pi$, or $\pm 6\pi$, ... or $2\pi k$ for any integer k.

- So, as a complex number in polar form,

$$1 = 1e^{j2\pi k}, \quad k = \text{any integer.}$$

- Now, raise both sides of eq. (*) on page 1.71 to the power N:

$$z^N = 1$$

- write both sides in polar form:

$$(re^{j\theta})^N = 1e^{j2\pi k} \quad (k \in \mathbb{Z})$$

$$r^N e^{jN\theta} = 1e^{j2\pi k}$$

→ Magnitudes must be equal:

$$r^N = 1 \longmapsto r = 1.$$

(this solution is unique because r must be real and ≥ 0).

$\longrightarrow$

→ Angles must be equal:

$$N\theta = 2\pi k \quad , \quad k \in \mathbb{Z}$$

$$\theta = \frac{2\pi}{N} k$$

– So the solutions are complex numbers equal to $z = re^{j\theta} = 1e^{j\frac{2\pi}{N}k}$

$$= e^{j\frac{2\pi}{N}k} \quad , \quad k \in \mathbb{Z}.$$

– You get N unique roots for $k = 0, 1, 2, \ldots N-1$

– For choices k outside this range, you just get the same numbers over again as you got for $k = 0, 1, \ldots, N-1$.

– These numbers all have magnitude 1.
 – They all lie on the unit circle of the complex plane.
 – Their angles go in steps of $\frac{2\pi}{N}$ rad.
 – They all satisfy $z^N = 1$.

– See Fig. A-17 on p. 487 of the book.

MATRICES

- A matrix is an array of numbers
- EX:

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

  - This is called a "2×3" ("two by three") matrix because it has two rows and three columns.
    - Note: although it's not shown in this example, the entries could be complex numbers.
    - We call the numbers "entries." Entries are usually written $a_{i,j}$. This means the number on row i and column j. We usually call it the "i,j^{th}" entry.
      - EX: For the matrix A above,

$$a_{2,3} = 9.$$

- To transpose a matrix, you turn the rows into columns. The first row becomes the first column, the second row becomes the second column, and so on...

- The transpose operation is written with a superscript "T".

<u>EX</u>: using the matrix A from the previous page, we get

$$A^T = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ 1 & 5 \\ 4 & 9 \end{bmatrix}$$

→ Notice that A^T is a 3×2 matrix in this case.

- Scalar multiplication: To multiply a scalar times a matrix, you must multiply the scalar with every element of the matrix: (still using our example matrix A from page 1.74):

$$5A = 5 \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 15 & 5 & 20 \\ 5 & 25 & 45 \end{bmatrix}$$

- To conjugate a matrix, you must conjugate every entry.
  - if $B = \begin{bmatrix} j & 2+j \\ 3-j & 5 \end{bmatrix}$
  - Then $B^* = \begin{bmatrix} j & 2+j \\ 3-j & 5 \end{bmatrix}^* = \begin{bmatrix} -j & 2-j \\ 3+j & 5 \end{bmatrix}$

- Note that vectors can also be thought of as matrices:

  - if $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then we can think of $\vec{v}$ as a 3×1 matrix.

  - And then $\vec{v}^T = [1 \ 2 \ 3]$ is a 1×3 matrix (and is also a vector).

  - NOTE: the entries of a vector are numbers. As with matrices, they can be complex numbers in general.
  - NOTE: Transposition, scalar multiplication, and conjugation of vectors works just like for matrices.

- Vector product: there are three main ways to multiply vectors

  - inner product (a.k.a. "dot product")
  - outer product
  - cross product

  - we will only use the inner product

- The "dot" or "inner" product of two vectors is computed as follows:

1. line up the two vectors beside each other.

2. Conjugate the entries of the second vector (if they are complex)

3. Multiply the entries that are beside each other

4. Add it up down the vectors.

$\Rightarrow$ You get a <u>number</u>, not a vector.

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}^*$$

$$4 + 10 + 18 = 32$$

- Notice that, for two vectors $\vec{v}$, $\vec{w}$, this can be written as

$$\vec{v}^T \vec{w}^* \qquad [\cdot \cdot \cdot]\begin{bmatrix} \vdots \end{bmatrix}$$

- You will sometimes see the dot product written with a "dot" "$\cdot$" as in $\vec{v} \cdot \vec{w}$.
- But I will usually write it with angle brackets as in $\langle \vec{v}, \vec{w} \rangle$. This is how it is more often written in math, physics, and IEEE Transactions on Signal Processing.

<u>EX</u>:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 4+j \\ 5 \\ -6j \end{bmatrix}$$

$$\underbrace{\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle}_{\text{just two different ways of writing it}} = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4+j \\ 5 \\ -6j \end{bmatrix} \right\rangle = \vec{v}^T \vec{w}^*$$

$$= 1(4-j) + 2(5) + 3(6j)$$

$$= 4 - j + 10 + 18j$$

$$= 14 + 17j \quad \text{(a number)}$$

- Matrix product
  - Multiplying matrices is kind of like taking dot products between the rows of the first matrix and the columns of the second matrix
    - Except that you don't conjugate (at least not with the "usual" definition of matrix multiplication)
  - If A and B are matrices, then the product

$$C = AB$$

is also a matrix.

A and B are called "conformable" if this is true

  - To do this multiplication, it is required that the number of columns in A must be the same as the number of rows in B.
  - The number of rows in C is the same as the number of rows in A.
  - The number of columns in C is the same as the number of columns in B.

- The entry C_{ij} on row i and column j of matrix C is found by taking a "sort of like" inner product between the i^{th} row of A and the j^{th} column of B.
  - like a dot product <u>without</u> the conjugation.

- Technically, it is written like this.

$$C_{ij} = \sum_{k=1}^{\text{No. cols in } A} A_{ik} B_{kj}$$

- Sounds very complicated but it's actually easy:

<u>EX</u>: $A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 5 & 3 \\ 3 & 6 & -4 \end{bmatrix}$

$C = AB \longrightarrow$

C_{11} is found by taking the first row of A with the first column of B:

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & -2 \\ 2 & 5 & 3 \\ 3 & 6 & -4 \end{bmatrix}$$

$$C_{11} = 3\cdot 1 + 1\cdot 2 + 4\cdot 3 = 17$$

- For C_{23}, it's the second row of A with the third column of B:

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & -2 \\ 2 & 5 & 3 \\ 3 & 6 & -4 \end{bmatrix}$$

$$\begin{aligned} C_{23} &= 1(-2) + 5(3) + 9(-4) \\ &= -2 + 15 - 36 \\ &= -23 \end{aligned}$$

- And here's the whole thing:

$$C = AB = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 2 & 5 & 3 \\ 3 & 6 & -4 \end{bmatrix}$$

$$c_{11} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3(1) + 1(2) + 4(3)$$
$$= 3 + 2 + 12 = 17$$

$$c_{12} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 3(4) + 1(5) + 4(6)$$
$$= 12 + 5 + 24 = 41$$

$$c_{13} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = 3(-2) + 1(3) + 4(-4)$$
$$= -6 + 3 - 16 = -19$$

$$c_{21} = \begin{bmatrix} 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1(1) + 5(2) + 9(3)$$
$$= 1 + 10 + 27 = 38$$

$$c_{22} = \begin{bmatrix} 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1(4) + 5(5) + 9(6)$$
$$= 4 + 25 + 54 = 83$$

$$c_{23} = \begin{bmatrix} 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = 1(-2) + 5(3) + 9(-4)$$
$$= -2 + 15 - 36 = -23$$

$$C = \begin{bmatrix} 17 & 41 & -19 \\ 38 & 83 & -23 \end{bmatrix}$$

- When someone says "matrix multiplication", by default it means the matrix product we have just been discussing.

- But there are other types of matrix products (just like there's more than one way to multiply vectors).

- Another important one is the pointwise product.

    - It is also called the "Hadamard product"
    - It is also called the "Schur product"

- It is pretty simple:

    - The matrices have to be the same size.
    - The pointwise product is also a matrix and it is the same size too.
    - the pointwise product $C = A \circ B$ is defined by $c_{ij} = (a_{ij})(b_{ij})$

        → In other words, take the product "point by point"

EX:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$C = A \circ B = \begin{bmatrix} 1 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 7 & 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix}$$

- Note: the entries could also be complex. You still take the pointwise product the same way... just multiply corresponding entries.

NOTE: The pointwise product works for vectors too... since every vector can also be considered as a matrix.

EX:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\vec{v} \circ \vec{w} = \begin{bmatrix} 1 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}$$

Note: - in Matlab, every variable is a matrix.
("Matlab" stands for "Matrix Laboratory")

- if you type $x=5;$ in matlab, then x is a 1×1 matrix... which is the same as a scalar.

- if you type `C = A*B;` in Matlab, it means <u>matrix multiplication</u>.

- Often in signal processing, we want the pointwise product instead.

- In matlab, the pointwise product is written as ".*", (e.g. "point product"), like this:

`C = A.*B;`

MORE ON VECTORS, VECTOR SPACES, ETC

- YOU MAY TAKE a course called Math 3333 that will be all about vector spaces and linear algebra.

- We will not cover all the technical details in ECE 2713, but we will talk about the basics and try to develop some intuition.

- To have a "vector space", you need a few things:

① A big bag full of vectors:

- The vectors can be <u>very</u> abstract mathematical objects.

← The big bag of vectors, call it "$\mathcal{V}$"

- But we will not need anything too abstract.

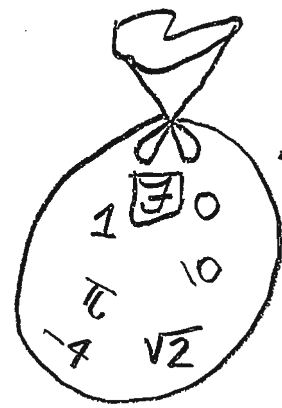
- For us, it will be good enough to think of vectors as ordered n-tuples of numbers.

- Could be real numbers or complex numbers.

- The only abstract thing we will need is that the number of entries in our vectors will sometimes need to be infinite.

(2) A second bag full of "scalars."

- The scalars are numbers.
- They can be quite abstract in general,
  - but we won't need that.
  - For us, the scalars will allways be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.



Big bag of Scalars, call it $\mathcal{F}$

- The scalars can be combined with each other using the normal rules of arithmetic like addition, subtraction, multiplication, division...
- <u>Most Importantly</u>, the scalars can be multiplied times vectors.

<u>EX:</u>

$$5\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5\\10\\15\end{bmatrix}$$

- Technically, the bag of scalors is referred to as the scalor "field". We say that the vector space $\mathcal{V}$ is "defined over" the field $\mathcal{F}$.

③ An "addition" operation that combines two vectors from the big bag $\mathcal{V}$ to get another guy in the bag $\mathcal{V}$.

- For us, it will be good enough to think of "normal" vector addition

EX: $\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -j \\ 2-j \end{bmatrix} = \begin{bmatrix} 5 \\ 2-j \\ 5-j \end{bmatrix}$$

- These three things have to be defined such that certain rules are satisfied.
    - Depending on how you count them, there are about 10 or 12 rules that have to be satisfied.
    - The details aren't important to us in ECE 2713, but here are some examples of the rules:
        - Closure of addition:

          $\forall \vec{u}, \vec{v} \in \mathcal{V} \quad \vec{u}+\vec{v} \in \mathcal{V}$

          → In other words, when you add two vectors from the big bag, you get a vector that's also in the big bag.
        - Existence of additive inverses:

          $\forall \; \vec{v} \in \mathcal{V}, \; \exists \; \vec{w} \in \mathcal{V}$ such that $\vec{v}+\vec{w} = \vec{0}$

          → In other words, for every vector $\vec{v}$ in the big bag, there is another guy $\vec{w}$ in the big bag such that $\vec{v}+\vec{w}$ is the zero vector

- Distributivity of scalar multiplication over vector addition:

$$c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$$

- Etc...

BASIS

- A <u>spanning set</u> is a small bag of vectors, all taken from the big bag, such that <u>any guy</u> from the big bag can be written as a linear combination of guys from the small bag using scalars from the field.

<u>EX:</u> The big bag: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the set of all vectors like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with two real entries.

The small bag: $\left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \end{bmatrix} \right\}$

→ Any guy $\vec{v} \in \mathbb{R}^2$ can be written as a linear combination of these three

- A basis is a spanning set with the fewest possible guys in it.

EX: $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

- The number of vectors in a basis is called the dimension of the vector space.
  - A basis for $\mathbb{R}^2$ has two vectors in it, so the dimension of the vector space $\mathbb{R}^2$ is 2.
    - While this is the technical definition of dimension, you can think of it intuitively as the number of entries in a vector.

      → For $\mathbb{R}^2$, you've got vectors like $\vec{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

      → Two entries

      → Dimension = 2

- Two vectors are orthogonal if their dot product is zero.

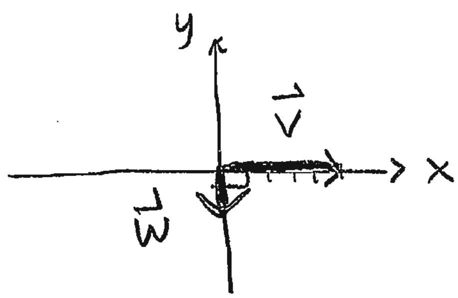
EX: $\vec{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = \left\langle \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\rangle$$

$$= 5 \cdot 0^* + 0 \cdot (-2)^*$$

$$= 5 \cdot 0 + 0 \cdot (-2)$$

$$= 0 + 0 = \underline{\underline{0}} \rightarrow \text{orthogonal}$$



"perpendicular"

- If all the vectors in a basis are mutually orthogonal, then it is called an orthogonal basis.

EX: Basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \end{bmatrix} \right\}$

$$\left\langle \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \end{bmatrix} \right\rangle = 2(-4)^* + 2(4)^*$$

$$= 2(-4) + 2(4)$$

$$= -8 + 8 = 0 \checkmark$$

→ Since all (both) vectors in the basis are mutually orthogonal to each other, it is an orthogonal basis.

Norm: the norm is the length of a vector.

- It can be defined in abstract ways, but we won't need that for ECE 2713.
- For us, it will be good enough to say that the norm of a vector $\vec{v}$ is the square root
  - of the dot product
    - of $\vec{v}$ with itself.
- In other words, the norm of $\vec{v}$, which is written as $\|\vec{v}\|$, is given by:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^T \vec{v}^*}$$

- For vectors with real entries, this is the same as "add up the squares of the entries and take the square root", which you are probably familiar with.

EX:
$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\|\vec{v}\| = \sqrt{(1)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{1 + 4 + 9} = \sqrt{14}$$

→ But notice that this is exactly the same thing that you get if you take the dot product of $\vec{v}$ with himself and then take the square root:

$$\langle \vec{v}, \vec{v} \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle$$

$$= 1 \cdot 1^* + 2 \cdot 2^* + 3 \cdot 3^*$$
$$= 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3$$
$$= 1 + 4 + 9 = 14$$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{14} \checkmark$$

- For vectors with complex entries, you should use the definition to compute the norm.

EX: Let $\vec{v} = \begin{bmatrix} 1+2j \\ 2-3j \end{bmatrix}$

$$\vec{v}^* = \begin{bmatrix} 1-2j \\ 2+3j \end{bmatrix}$$

$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle &= (1+2j)(1-2j) + (2-3j)(2+3j) \\ &= (1-2j+2j+4) + (4+3j-3j+9) \\ &= (1+4) + (4+9) \\ &= 5 + 13 = 18 \end{aligned}$$

$$\begin{aligned} \text{Norm of } \vec{v} &= \|\vec{v}\| \\ &= \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{18} \end{aligned}$$

- A vector that has length (norm) 1 is said to be "unit norm".

For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a vecter that has "unit norm".

Orthonormal Basis

- if you have an orthogonal basis and all of the basis vectors have unit norm,
  → then it's called an **orthonormal basis**.

EX: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

EX: $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ is also an orthonormal basis for $\mathbb{R}^2$.

Check:

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = \sqrt{1} = 1 \checkmark$$ unit norm

$$\left\| \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\| = \sqrt{\left(\tfrac{-1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}}$$

$$= \sqrt{1} = 1 \checkmark$$ unit norm

$$\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)^* + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^*$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{2} + \frac{1}{2} = 0 \quad \checkmark$$

mutually orthogonal

<u>EX</u>: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is an orthonormal basis for $\mathbb{R}^3$.

- Orthonormal means:

    - The dot product of any basis vector with itself is <u>one</u>.

    - The dot product of any basis vector with a <u>different</u> basis vector is <u>zero</u>.

<u>Representation of vectors in $\mathbb{R}^2$</u>

- "Representation" means writing a vector as a linear combination of a basis.

- For an orthonormal basis, this can always be done as follows:

    - Step 1: take the dot product of your vector with each basis vector.

        → This gives you a number for each basis vector

        → These numbers are called the "coordinates" of your vector with respect to the basis.

- Step 2 : Add up the dot products (numbers) times the basis vectors

→ That sum, or "linear combination", will be your vector.

EX : in $\mathbb{R}^2$, the two vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form an orthonormal basis.

NOTE : here, $\vec{j}$ means the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It's got nothing to do with the imaginary unit $j = \sqrt{-1}$.

NOTE : We define the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$. This means that the entries of all the vectors are real numbers (since it's $\mathbb{R}^2$) and also all the scalars are real numbers (because the field is $\mathbb{R}$).

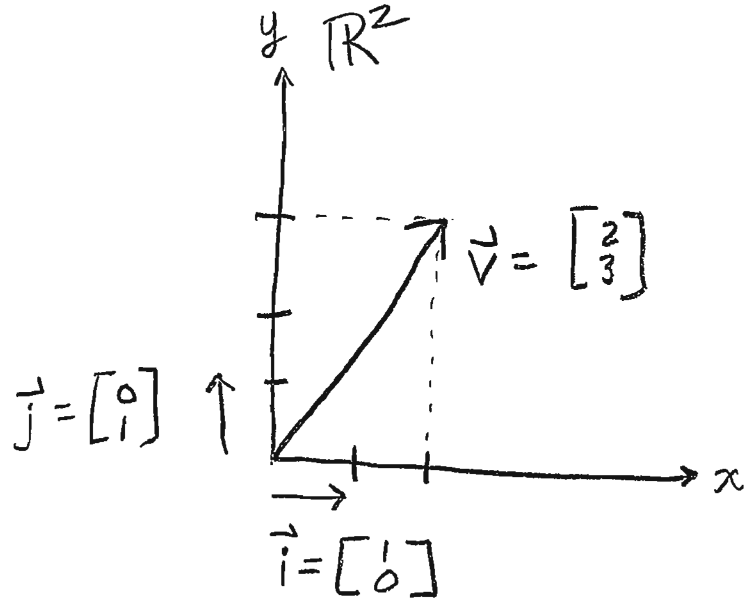
- → So it won't make any difference whether or not we remember to conjugate the second vector when we take dot products.
  - → because conjugating a real number doesn't actually do anything.
- Nevertheless, we will keep writing the conjugations to help train ourselves <u>not</u> to forget when the vectors have complex entries.

<u>FACT:</u> Any vector $\vec{v} \in \mathbb{R}^2$ can be written as a linear combination of the basis $\{\vec{i}, \vec{j}\}$ as follows:

$$\underbrace{\vec{v}}_{\text{a vector}} = \underbrace{\langle \vec{v}, \vec{i} \rangle}_{\text{a number}} \underbrace{\vec{i}}_{\text{a vector}} + \underbrace{\langle \vec{v}, \vec{j} \rangle}_{\text{a number}} \underbrace{\vec{j}}_{\text{a vector}}$$

EX :

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



- write $\vec{v}$ as a linear combination of the basis:

$$\vec{v} = \langle \vec{v}, \vec{i} \rangle \vec{i} + \langle \vec{v}, \vec{j} \rangle \vec{j}$$

$$= \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (2 \cdot 1^* + 3 \cdot 0^*) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2 \cdot 0^* + 3 \cdot 1^*) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \checkmark$$

- Now, all of this is practically obvious when the basis is $\{\vec{i}, \vec{j}\}$.

- It's fairly obvious that $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

DEF: the natural basis in any vector space is an orthonormal basis where:

- each basis vector has only one entry that is nonzero
- the nonzero entry is equal to one.

$\Rightarrow$ so, in fact, $\{\vec{i}, \vec{j}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is the natural basis in $\mathbb{R}^2$.

$\Rightarrow$ With the natural basis, representation is easy and you don't really have to go through computing dot products...

- Because it's pretty obvious that

$$\begin{bmatrix}2\\3\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}.$$

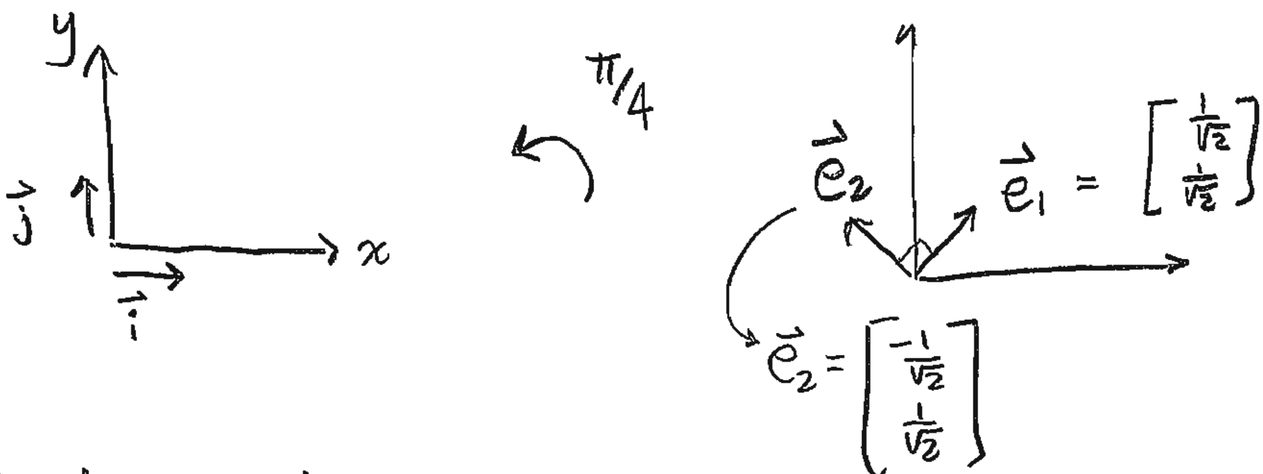
$\Rightarrow$ But the point is that our procedure,

- take dot product of $\vec{v}$ with each basis vector
- add up dot products times basis vectors

$\rightarrow$ works for any orthonormal basis.

- And things are a lot <u>less</u> obvious when it's <u>not</u> the natural basis.

<u>EX</u>: if you take the natural basis $\{\vec{i}, \vec{j}\}$ in $\mathbb{R}^2$ and rotate it counterclockwise by $\pi/4$ rad, then you get a new orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$



- We already showed on pages 1.96 and 1.97 that this basis is orthonormal:

  - each basis vector has unit norm:

$$\|\vec{e}_1\| = \|\vec{e}_2\| = 1$$

  - They are mutually orthogonal:

$$\langle \vec{e}_1, \vec{e}_2 \rangle = 0$$

- But compared to the natural basis, representation is a lot less obvious with this basis!
- To write $\vec{v}$ as a linear combination of this basis, we will have to actually compute the dot products.

EX: $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad \vec{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\vec{v} = \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2$$

$$= \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \left(2 \cdot \frac{1}{\sqrt{2}}^* + 3 \cdot \frac{1}{\sqrt{2}}^* \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left(2 \left(\frac{-1}{\sqrt{2}} \right)^* + 3 \cdot \frac{1}{\sqrt{2}}^* \right) \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \left(\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left(\frac{-2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \right) \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{5}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} \\ \frac{6}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \checkmark$$

It Works!!

- Some observations about the rotated basis:
  - with the natural basis, everything is easy and obvious because each basis vector is turned on in only one place, and it is equal to <u>one</u> in that one place.
  - with the rotated basis, each basis vector is generally turned on all over the place.
  - We see that the entries of the basis vectors start to pick up some sign changes... or "oscillation."

$$\vec{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- In fact, to rotate a basis you multiply the basis vectors times a rotation matrix.
  - The entries of the rotation matrix are sines and cosines.
  - So we <u>expect</u> the entries of a rotated basis vector to follow a pattern like sine and cosine.

→ This idea will be important later.

What happens if the basis is orthogonal, but not orthonormal?

- we will always assume that all the basis vectors have the same norm ...
  - It's just that this norm will be some other number... not one.

EX: the vectors $\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^2$.

- It's not an orthonormal basis because each basis vector has length 3, not one.

- This will mess things up for our method.
- If we try to write

$$\vec{v} \stackrel{?}{=} \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2$$

it will not work.

→ The dot products will both be too big by 3 (the length of a basis vector)

- These "too big" dot products will get multiplied by basis vectors that are <u>too long</u>... by a factor of 3 (the length of a basis vector).
- So, overall, we get the length of a basis vector <u>too much</u> twice... in other words, our representation is too big by the length of a basis vector squared.
- Let's see all of this in an example:

$$\vec{v} = \begin{bmatrix}2\\3\end{bmatrix} \qquad \vec{e}_1 = \begin{bmatrix}3\\0\end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\vec{v} \stackrel{?}{=} \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2$$

$$= \left\langle \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix} \right\rangle \begin{bmatrix}3\\0\end{bmatrix} + \left\langle \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix} \right\rangle \begin{bmatrix}0\\3\end{bmatrix}$$

$$= (2 \cdot 3^* + 3 \cdot 0^*) \begin{bmatrix}3\\0\end{bmatrix} + (2 \cdot 0^* + 3 \cdot 3^*) \begin{bmatrix}0\\3\end{bmatrix}$$

$$= 6 \begin{bmatrix}3\\0\end{bmatrix} + 9 \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}18\\0\end{bmatrix} + \begin{bmatrix}0\\27\end{bmatrix} = \begin{bmatrix}18\\27\end{bmatrix} \quad X$$

dot product too big by 3

basis vector too long by 3

dot product too big by 3

basis vector too long by 3

- To fix this up, we've got to divide by the squared length of a basis vector somewhere:

$$\frac{1}{3^2}\begin{bmatrix}18\\27\end{bmatrix} = \frac{1}{9}\begin{bmatrix}18\\27\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \checkmark$$

- we can do this fix when we take the dot products,

- or when we add up the dot products times the basis vectors,

  - or we can do a mix of both: divide by 3 when we take the dot products and then divide by 3 again when we add up the dot products times the basis vectors.

- This will all be very important later.

How about $\mathbb{R}^3$?

- The natural basis:

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Any vector $\vec{v} \in \mathbb{R}^3$ can be written as

$$\vec{v} = \langle \vec{v}, \vec{i} \rangle \vec{i} + \langle \vec{v}, \vec{j} \rangle \vec{j} + \langle \vec{v}, \vec{k} \rangle \vec{k}$$

EX: $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\langle \vec{v}, \vec{i} \rangle = \vec{v}^T \vec{i}^* = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^* = 4$$

$$\langle \vec{v}, \vec{j} \rangle = \vec{v}^T \vec{j}^* = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^* = 5$$

$$\langle \vec{v}, \vec{k} \rangle = \vec{v}^T \vec{k}^* = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^* = 6$$

$$\vec{v} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \checkmark$$

→ of course, this is all obvious when we use the natural basis.

- But if I give you <u>another</u> orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$,

    - like maybe a rotated basis,

- Then you would have to go ahead and compute the dot products explicitly:

$$\vec{v} = \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2 + \langle \vec{v}, \vec{e}_3 \rangle \vec{e}_3$$

$\Rightarrow$ The beauty of this is that it <u>always</u> works and it always works <u>the same way</u>.

1. Take the dot product of your vector with each basis vector... this gives you a <u>number</u> for each basis vector.

2. Add up the dot products times the basis vectors... that will give you your vector $\vec{v}$ as a linear combination of the basis.

- Don't forget to <u>conjugate</u> the second vector in a dot product if the entries are complex numbers!

- Don't forget to divide by the length of a basis vector squared if the basis is orthogonal but <u>not</u> orthonormal!!

What about higher dimensional spaces?

- Like what about $\mathbb{R}^{100}$?
- The good news is: everything still works exactly like it did in $\mathbb{R}^2$.
  - But the vectors are harder to visualize when the number of dimensions is greater than 3.
  - But have no fear... the dot product math will not fail you.
- A vector in $\mathbb{R}^{100}$ is just an ordered n-tuple of 100 numbers:

$$\vec{v} = [v_1 \;\; v_2 \;\; v_3 \;\; \dots \;\; v_{100}]^T$$

- The dot product still works just like before:
  - you line up the two vectors beside each other
  - You conjugate the entries of the second vector
  - you multiply the entries that are beside each other
  - you add it up down the vector to get a number.

- If $\vec{v} = [v_1 \ v_2 \ v_3 \ \ldots \ v_{100}]^T$

  and $\vec{w} = [w_1 \ w_2 \ w_3 \ \ldots \ w_{100}]^T$

  where the v_k and w_k are numbers,

we usually write these vectors as transposes so that we can write them horizontally instead of vertically. saves space on the page.

- Then the dot product is

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1^* + v_2 w_2^* + \ldots + v_{100} w_{100}^*$$

(a number)

- In higher dimensional spaces like $\mathbb{R}^{100}$, you can save a <u>lot</u> of writing by using capital Σ do loops:

$$\langle \vec{v}, \vec{w} \rangle = \sum_{k=1}^{100} v_k w_k^*$$

- If I give you an orthonormal basis

$$\{ \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_{100} \}$$

- Then you can write your vector $\vec{v} \in \mathbb{R}^{100}$ as a linear combination of the basis just like we always do:
  - take dot product of $\vec{v}$ with each basis vector to get a number.
  - Add up these numbers times the basis vectors.

$$\vec{v} = \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2 + \cdots + \langle \vec{v}, \vec{e}_{100} \rangle \vec{e}_{100}$$

- Capital Σ do loops can save a lot of writing:

$$\vec{v} = \sum_{k=1}^{100} \langle \vec{v}, \vec{e}_k \rangle \vec{e}_k$$

$$= \sum_{k=1}^{100} \underbrace{\left[\sum_{n=1}^{100} v_n \vec{e}^{\,*}_{k,n} \right]}_{\text{here, I have written the dot product using a do loop... with loop counter "n", just like we did on page 1.112.}} \vec{e}_k$$

- When you take a dot product, don't forget to conjugate the entries of the second vector if they are complex numbers.

- When you add up the dot products times the basis vectors, don't forget to divide by the length of a basis vector squared if the basis is orthogonal but <u>not</u> orthonormal.

- The natural basis in $\mathbb{R}^{100}$ works just like in $\mathbb{R}^2$ and $\mathbb{R}^3$.
    - Each basis vector has "all zeros" except for one nonzero entry
        - The nonzero entry is equal to <u>one</u>.
    - In the basis, you get one basis vector that is turned on in each "place."

Natural basis in $\mathbb{R}^{100}$:

$$\vec{e}_1 = [1\ 0\ 0\ 0\ \ldots\ 0]^T$$

$$\vec{e}_2 = [0\ 1\ 0\ 0\ \ldots\ 0]^T$$

$$\vec{e}_3 = [0\ 0\ 1\ 0\ \ldots\ 0]^T$$

$$\vdots$$

$$\vec{e}_{100} = [0\ 0\ 0\ 0\ \ldots\ 1]^T$$

$\boxed{\text{We are done with chapter 1!}}$