MODULE 2: SIGNALS & SYSTEMS IN THE TIME DOMAIN

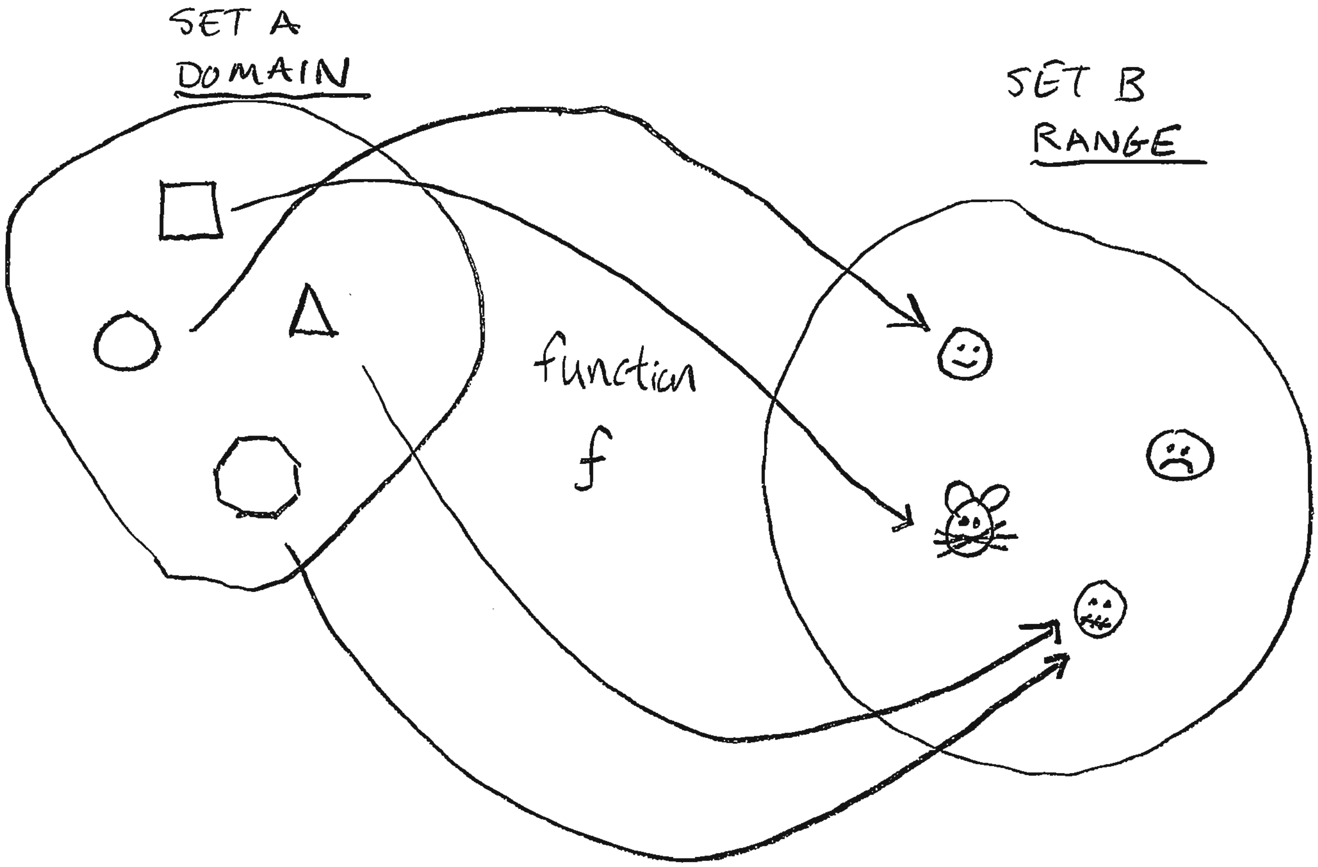
- We use <u>functions</u> to model signals.
  - So the first thing we need to do is answer the question: "what is a function?"

<u>DEF:</u> A <u>function</u> is a rule that matches each member of one set, called the <u>domain</u>, with exactly one member of a second set called the <u>range</u>.

→ In other words, the function maps every guy in the domain to one and only one "buddy" in the range.

- If it sounds really simple, that's because it is!

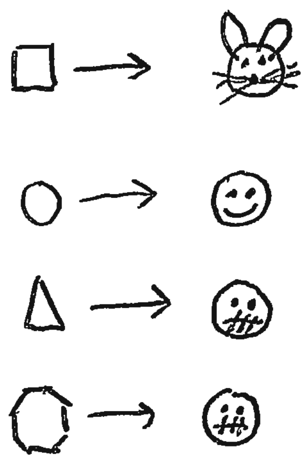
HERE IS AN example of a function:



Words: "f maps A to B"

Symbols: "$f : A \to B$"

- Here are the buddy matches made by f:



NOTES:

- Every member in set A is matched to <u>one</u> <u>and</u> <u>only</u> <u>one</u> member of B.
- Some members of B can be matched to <u>more</u> <u>than</u> <u>one</u> member of A (like in our example)
- Not everyone in B has to be a buddy... like in our example.

- If you have a function f, how do you tell somebody else about it?
    - In other words, how do you specify a function?

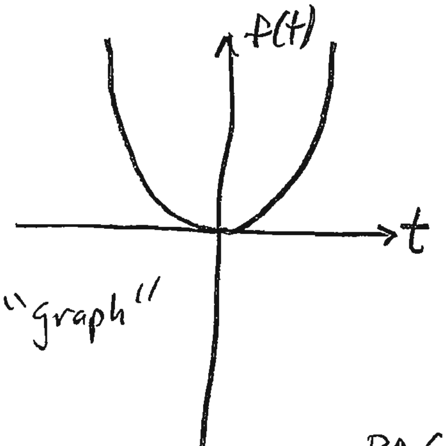
    → You have to tell what the domain and range are.

    → And you have to tell what the buddy assignments (matches) are.
    - How do you specify the matches?
        - You can draw a picture like we did on the last page
        - You can draw a graph
        - You can make a table
        - You can give a "recipe" that tells how to find the buddies... like $f(t) = t^2$.

EX: $f(t) = t^2$ domain: $\mathbb{R}$ range: $\mathbb{R}$

↑

"recipe"
or "formula"
or "equation"



"graph"

- If the domain is $\mathbb{R}$, then we call the function "continuous-time."

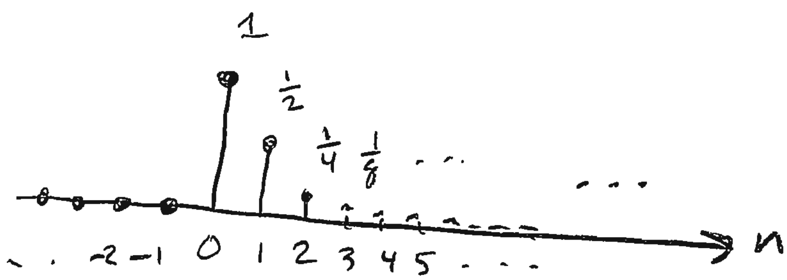
- If the domain is $\mathbb{Z}$ or a finite subset of $\mathbb{Z}$, then we call the function "discrete-time."

- We write continuous-time functions with parentheses like this: $f(t)$.

- We write discrete-time functions with square brackets like this: $f[n]$.

Here's some more examples:

<u>EX</u>: $f[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$ Domain: $\mathbb{Z}$ Range: $\mathbb{R}$



EX:

$$f(t) = \begin{cases} 2e^{(-\frac{1}{2}+j\frac{5\pi}{2})t + j\pi/5}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Domain: $\mathbb{R}$

Range: $\mathbb{C}$

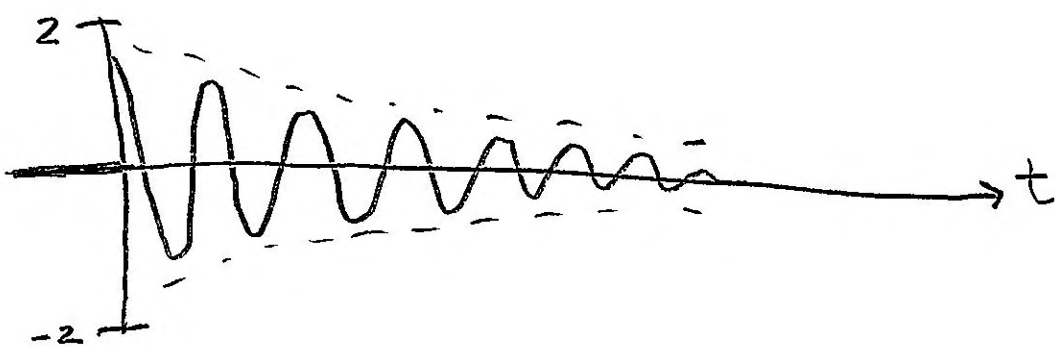
For $t \geq 0$, we have:

$$f(t) = 2\exp\left[-\tfrac{1}{2}t + j\tfrac{5\pi}{2}t + j\tfrac{\pi}{5}\right]$$

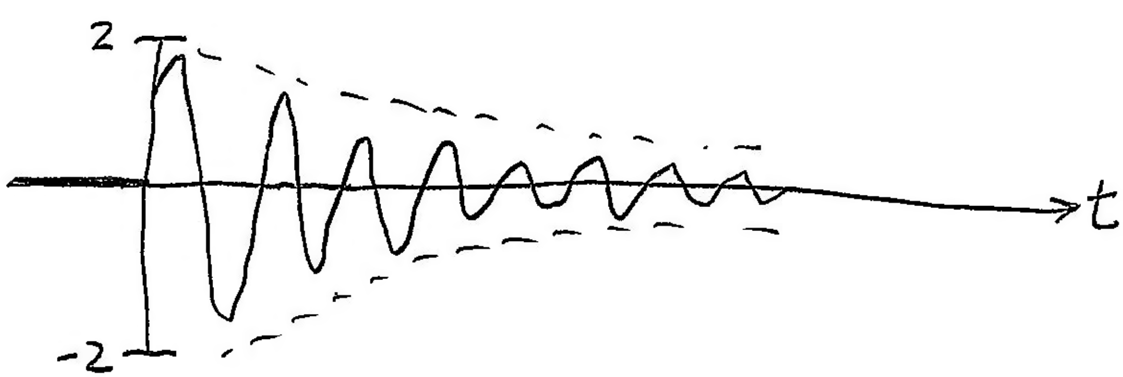
$$= 2e^{-\frac{1}{2}t} e^{j\left[\frac{5\pi}{2}t + \pi/5\right]}$$

$$= 2e^{-\frac{1}{2}t}\left[\cos\left(\tfrac{5\pi}{2}t + \pi/5\right) + j\sin\left(\tfrac{5\pi}{2}t + \tfrac{\pi}{5}\right)\right]$$

Real Part:



Imaginary Part:

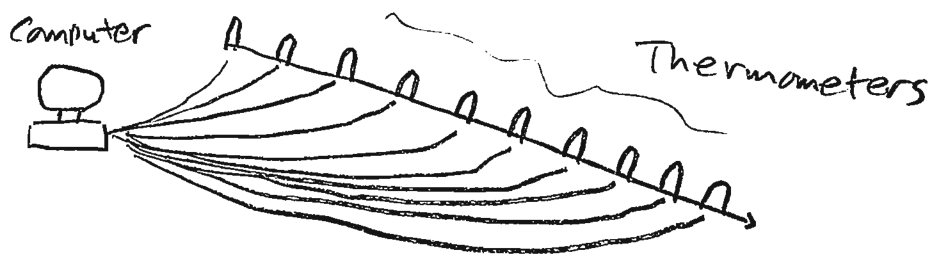


DEF: A **signal** is a manifestation of the variation of one physical quantity with respect to another.

EX: The voltage in a circuit measured with respect to time.

EX: The current in a circuit measured with respect to time.

EX: We set up a row of digital thermometers.



- They are all hooked up to the computer.
- When we hit "return", they all take a temperature measurement at the same time.

→ This makes a signal: Variation of temperature with respect to space (or displacement).

EX: We have one thermometer. Every hour, we take it outside and measure the temperature.

→ This makes a signal: variation of temperature with respect to time.

- We use functions to model signals.
- For example, we can model the voltage coming out of a wall socket as $v(t) = 120\cos(2\pi \cdot 60t)$.
- Technically, the signal and the function are <u>not</u> the same thing.
  - The signal is a manifestion of the variation in a physical quantity.
  - The function is a math object that we use to model the signal.
- But in ECE 2713, we won't have to worry too much about the difference between the signal and the function that models the signal.
  - We will often refer to them interchangeably.

For example, we will refer to the voltage signal coming out of a wall socket as:

"the signal $x(t) = 120\cos(2\pi 60t)$"

- Even though we all understand that $x(t)$ is actually a function that we are using to model the signal (rather than being the signal itself).

- A continuous-time signal $x(t)$ has domain $\mathbb{R}$.
- A discrete-time signal $x[n]$ has domain $\mathbb{Z}$ or a finite subset of $\mathbb{Z}$.

- When we write $x(t)$ ... with parentheses ... it implies that the domain is $\mathbb{R}$ unless otherwise specified. The range could be $\mathbb{R}$ or $\mathbb{C}$.

- When we write $x[n]$ ... with square brackets ... it implies that the domain is $\mathbb{Z}$ unless otherwise specified. The range could be $\mathbb{R}$ or $\mathbb{C}$.

- So what is a "digital signal"?
    - After all, this course is supposed to be about digital signals and filtering!
  - A digital signal is a discrete-time signal where the range is a finite set.

- This means that the signal values (numbers)

$$\ldots \; x[-2],\; x[-1],\; x[0],\; x[1],\; x[2],\; \ldots$$

can only take a finite number of values

→ This means that they can be represented in the memory of a computer or DSP chip using a <u>finite</u> number of bits for each one.

- The mathematics that is needed to deal with true digital signals is a lot harder and more cumbersome than the math for discrete-time signals.

- For this reason, we usually do analysis and design using discrete-time signals.

  - Then we "tweak" things up for the true digital signals by modeling the <u>differences</u> between the discrete-time signals and the true digital signals.

    - This modeling is beyond the scope of ECE 2713 (and even ECE 3793!).
    - It is often taught in an introductory graduate-level DSP course.

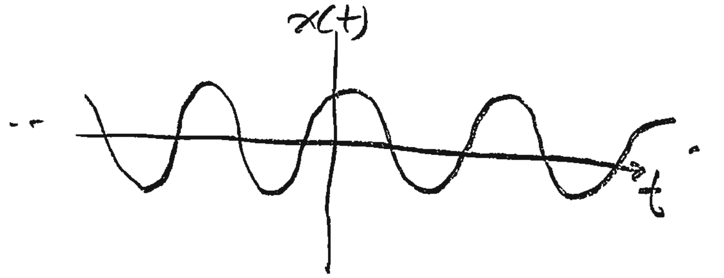
- So, in ECE 2713 we will work with discrete-time signals a lot, but we won't really deal with true digital signals.

Some Continuous-Time Signals

- In module 1, we worked with things like $\cos\theta$, $\sin\theta$, and $e^{j\theta}$ a lot. These were <u>numbers</u>.
- By replacing the number θ with a <u>function</u> of t, we get a number for every t...
  - → In other words, we get a signal.

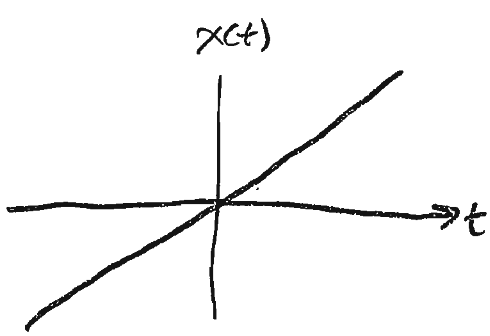
<u>EX</u>: $x(t) = \cos\left(\frac{\pi}{17}t\right)$

Domain: $\mathbb{R}$
Range: $\mathbb{R}$



- Here is another continuous-time signal:

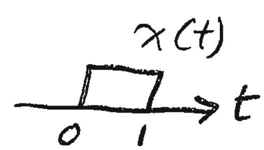
$x(t) = t$



And another:

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

=



"Boxcar"

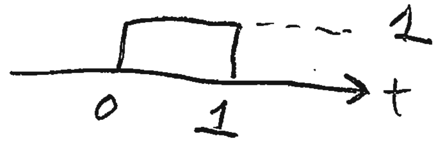
Time Shifting

- also known as "translation."
- Sometimes we will have the graph of a signal $x(t)$ and we will need to draw the graph of $x(t - t_0)$, where $t_0 \in \mathbb{R}$.
  - EX: $x(t-1)$ $\quad t_0 = 1$
    - what happens is that x is now gobbling up "t"s that have been reduced by 1.
      - So at $t=2$, $x(t-1)$ is doing what $x(t)$ did at 1.
      - At $t=3$, $x(t-1)$ is doing what $x(t)$ did at 2.
      - The effect is to slide the whole graph to the right by 1.
- More generally, to make the graph of $x(t-t_0)$, you take the graph of $x(t)$ and slide it to the right by t_0.

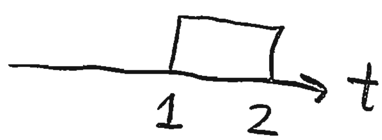
- This also works when t_0 is negative... for example, sliding right by $t_0 = -1$ will actually move the graph left by 1.

EX:

$x(t)$

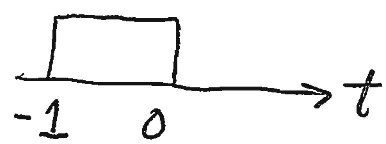


$x(t-1)$



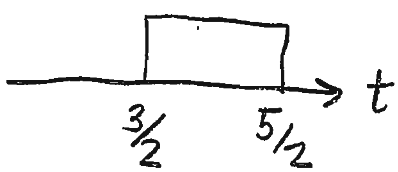
$t_0 = 1$

$x(t+1) = x(t - -1)$



$t_0 = -1$

$x(t - \frac{3}{2})$



$t_0 = \frac{3}{2}$

- You should always think of the shift as $x(t - t_0)$ and think of it as a <u>shift</u> <u>right</u> by t_0 ... even when t_0 is negative

→ Think of $x(t+2)$ as $x(t--2)$ with $t_0 = -2$.

Why?

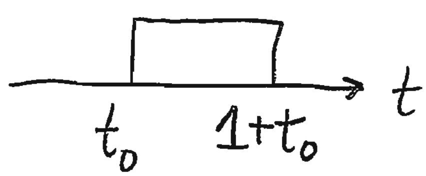
- Because sometimes we will have to make the graph of $x(t-t_0)$ when we <u>don't know</u> what t_0 is... when it could be positive <u>or</u> negative...

- We will need to draw the graph so that it is still good no matter what t_0 is.

<u>EX</u>:

$x(t)$

$x(t-t_0)$



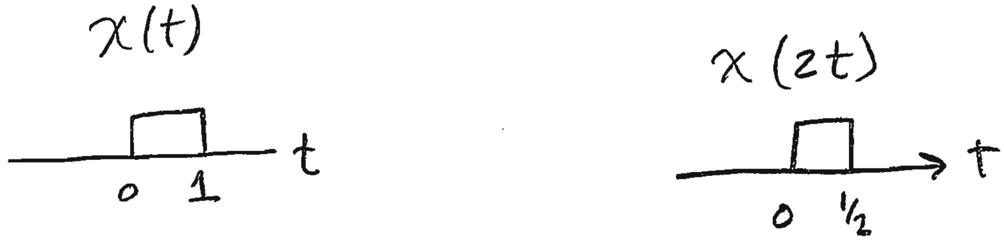
This graph is good no matter what t_0 is.

- So <u>always</u> think of it as $x(t-t_0)$ and <u>always</u> think of it as a shift right by t_0... even if t_0 is negative!

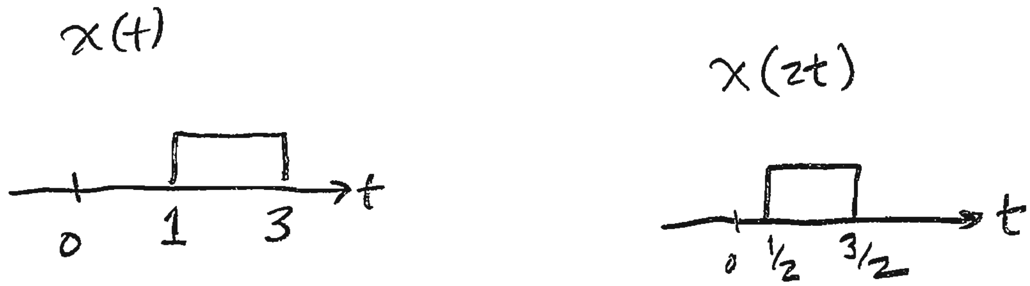
Time Scaling

- Sometimes we will have the graph of $x(t)$ and we will need to draw the graph of $x(at)$, where $a \in \mathbb{R}$ is a "scale factor."
- For example, we might need to draw the graph of $x(2t)$.
    - This makes the t's go by twice as fast as far as x can see.
    - So whatever $x(t)$ did from -2 to 2 will now happen in $x(2t)$ when t goes from -1 to 1.
  - The effect of this is to "squish" the graph by a factor of 2.

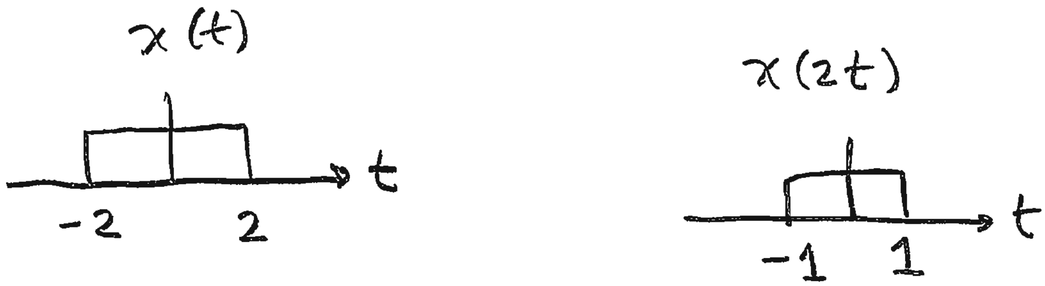
EX:


x(t)
0
1
t
x(2t)
0
1/2
t

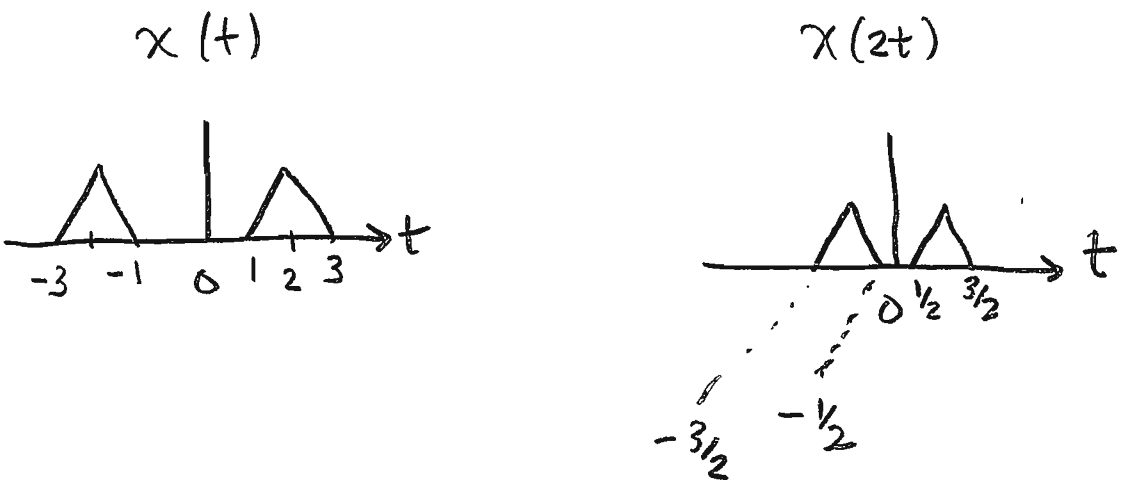
EX:


x(t)
0
1
3
t
x(2t)
0
1/2
3/2
t

EX:


x(t)
-2
2
t
x(2t)
-1
1
t

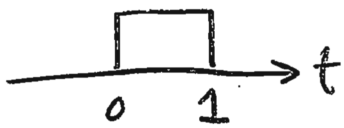
EX:


x(t)
-3
-1
0
1
2
3
t
x(2t)
0
1/2
3/2
-1/2
-3/2
t

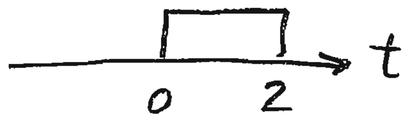
- If $0 < a < 1$, then the graph will get stretched instead squished.
- For example, if $a = \frac{1}{2}$ then we've got to draw the graph of $x(\frac{1}{2}t)$ given the graph of $x(t)$.
  - This makes the t's go by twice as slowly as far as x can see.
  - Whatever $x(t)$ did when t went from -1 to 1 will now be done by $x(\frac{1}{2}t)$ when t goes from -2 to 2... because t has to go from -2 to $+2$ in order for $x(\frac{1}{2}t)$ to "see" a -1 to $+1$.
  - So the graph gets <u>stretched</u>.

EX

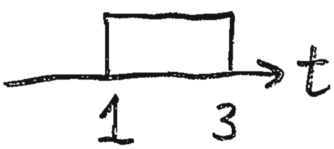
$x(t)$



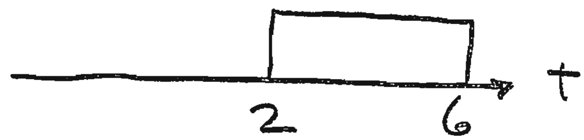
$x(\frac{1}{2}t)$



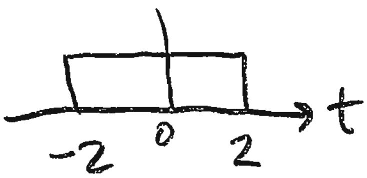
$x(t)$



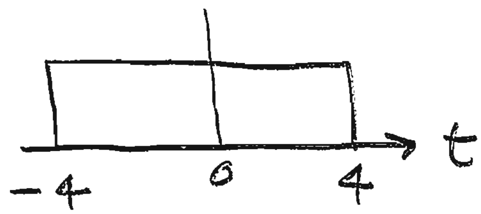
$x(\frac{1}{2}t)$



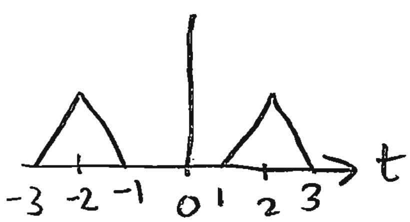
$x(t)$



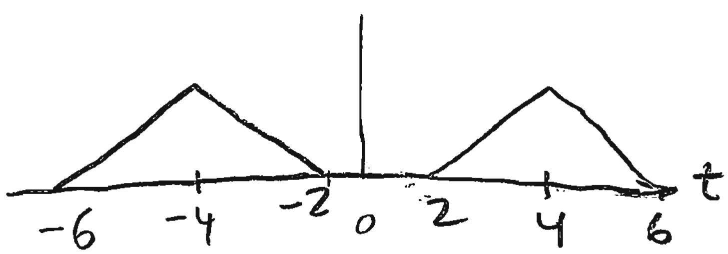
$x(\frac{1}{2}t)$



$x(t)$

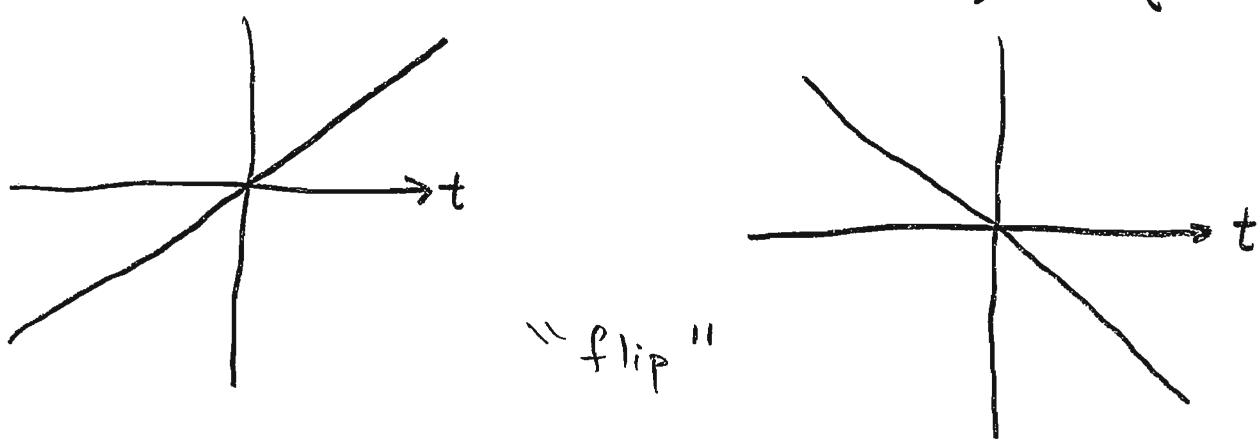


$x(\frac{1}{2}t)$



- If $a<0$, then the graph also gets flipped around the vertical axis.
  - For example, if $a=-1$, then all the buddy assignments that used to get made for t going from 0 to ∞ will now get made from 0 to $-\infty$... and vice versa.

EX $x(t) = t$ $\qquad$ $x(-t) = -t$

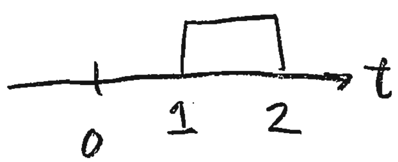


$x(t)$ $\qquad$ $x(-t)$

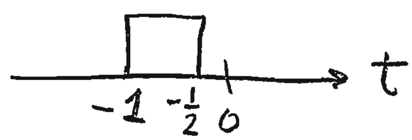
- More generally, when $a<0$ but a is not -1, there will be a flip <u>and</u> a stretch or squish.
  - if $a<0$ and $|a|>1$, it's a flip and a squish.
  - if $a<0$ and $|a|<1$, it's a flip and a stretch.

<u>Ex</u> :

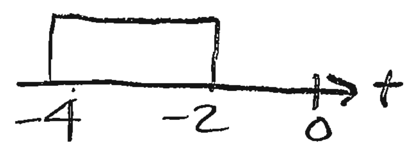
$x(t)$



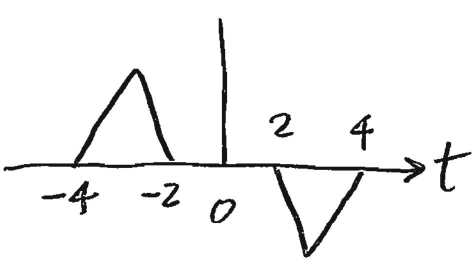
$x(-2t)$



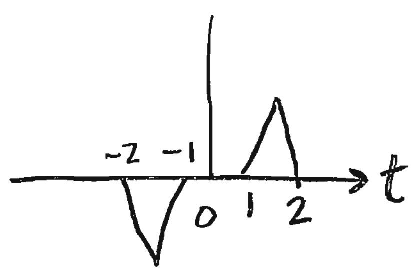
$x(-\frac{1}{2}t)$



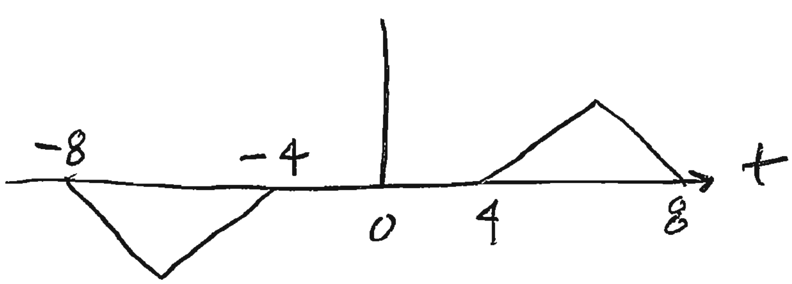
$x(t)$



$x(-2t)$



$x(-\frac{1}{2}t)$



Shift and Scale

- Sometimes we will have a shift and a scale at the same time.

- **NOTE:** $x(at - t_0)$ is <u>NOT</u> the same as $x(a(t-t_0))$!!!

★★★

→ To keep from making mistakes on a test, you should always handle this the <u>same way</u>.

→ You should think of it the first way: $x(at - t_0)$.

→ If you get $x(a(t - t_1))$, then multiply it out to make

$$x(at - \underbrace{at_1}_{t_0 = at_1})$$

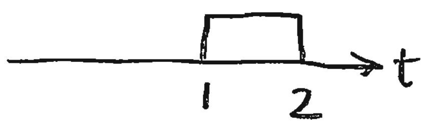
- If you have the graph of $x(t)$ and you need the graph of $x(at - t_0)$,

  - Do the shift first: shift the graph of $x(t)$ to the right by t_0.

  - Then scale the resulting graph by a.

EX:

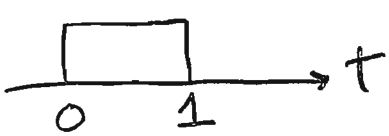
$x(t)$



$x(-t+1)$ $\quad a = -1$

$= x(-t - -1)$ $\quad t_0 = -1$

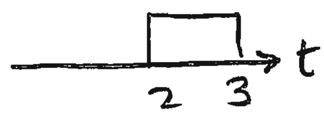
↓ shift first



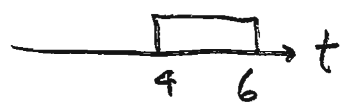
↓ Scale second

$x(\frac{1}{2}t - 1)$ $\quad a = \frac{1}{2}$, $t_0 = 1$

↓ Shift first



↓ Scale second



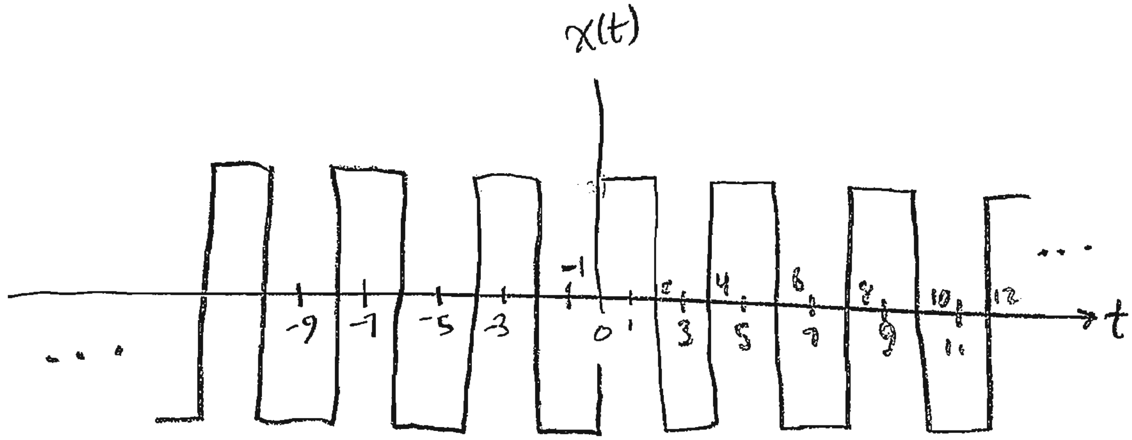
Periodic Signals

- if $x(t)$ is a signal
  - and there is a real number $T > 0$
    - such that for <u>any</u> t,
      - if you go ahead by T you get the same buddy, i.e. $x(t) = x(t+T)$,
    - Then we say that $x(t)$ is periodic with period T.

<u>NOTE</u>: it has to hold true for <u>all</u> of the t's, not just one or a few.

- It means that if you go ahead by T, you always get the same buddy.

- So if you go ahead by T once, you get the same buddy. If you go ahead by T more, you have to get the same buddy <u>again</u>.
  - So any $x(t)$ that is periodic with period T is also periodic with period $2T$ and $3T$ and $4T$ and kT for any $k \in \mathbb{N}$.



- This $x(t)$ is periodic with period $T=4$.
  - because if you go ahead by 4, you will always get the same thing.
  - It is also periodic with period $T=8$ and $T=12$ and $T=4k$ for any $k \in \mathbb{N}$.
- Here is how to write the definition of "periodic" using rigorous math:

  If $\exists T \in \mathbb{R}$, $T>0$, such that $x(t+T)=x(t)$ $\forall t \in \mathbb{R}$, then $x(t)$ is periodic with period T.

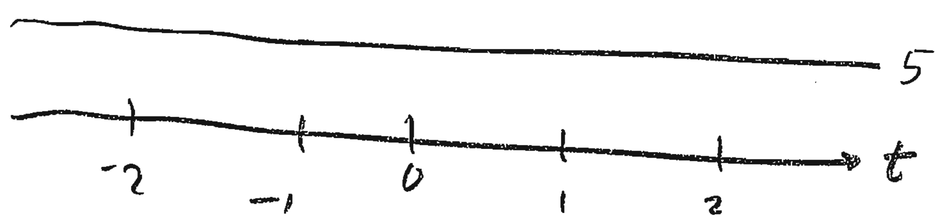
- In most cases, there will be a <u>smallest</u> positive number T that makes this all work.
  - That smallest one is called the <u>fundamental</u> <u>period</u> and is usually written T_0.
  - The fundamental period of the signal $x(t)$ shown above is $T_0 = 4$.

- Now consider this signal: $x(t) = 5$.

  → This does <u>not</u> mean that $x(t)$ is the <u>number</u> 5.

  → Rather, it means that $x(t)$ is the signal (function) that assigns to <u>every</u> t the buddy 5.

- Here is the graph of $x(t)$:



- This $x(t)$ is periodic with period $T=4$, because if you go ahead by 4 you get the same buddy (namely 5).
- This $x(t)$ is <u>also</u> periodic with period <u>any</u> $T>0$!!
  - Because no matter how you pick T, you will always get the same buddy if you go ahead by T... $x(t+T) = x(t) = 5$ no matter how you pick T.
  - So there is no <u>unique</u> smallest T that makes it work.

- So a constant signal like $x(t)=5$ is periodic with any positive period $T_0>0$ that you care to pick,

  - but it does not have a fundamental period.

  - This is a peculiarity of the continuous-time constant signals. As we will see later, this does not happen in discrete time.

Symmetry

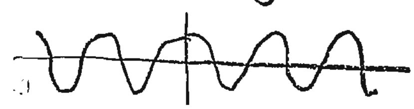
- if it is true that for every t, $x(t)$ assigns the same buddy to both t and $-t$, then $x(t)$ is called "even symmetric" or just "even."

  - It means that the graph of $x(t)$ on the negative t's is the reflection of the graph on the positive t's, reflected through the vertical axis.

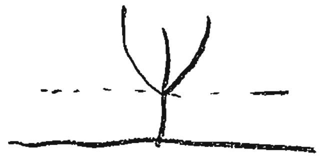
  - Here's how you write it in math:

    DEF: if $x(t)=x(-t)\ \forall\ t\in\mathbb{R}$, then $x(t)$ is even symmetric.

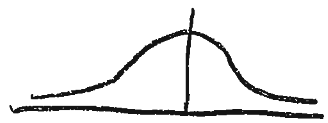
- Here are some examples of even signals:

$x(t) = \cos(\omega_0 t)$ 

$x(t) = t^2$ 

$x(t) = t^4 + 5$ 

$$x(t) = \frac{1}{\sqrt{2\sigma^2}} e^{-t^2/2\sigma^2}$$ (Gaussian)



- If the buddy assignments on the negative t's are the <u>negatives</u> of the assignments on the positive t's, then $x(t)$ is called "odd symmetric" or just "odd".

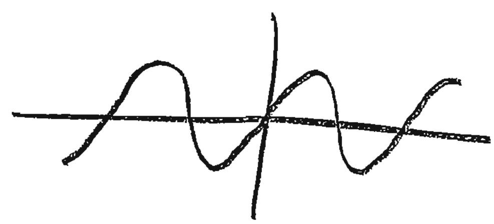
  - It means that the graph on the negative t's can be obtained by
    1. take the graph from the positive t's and reflect it through the vertical axis
    2. take that and reflect it through the horizontal axis (this "flips" it upside down).

- Here's how to write the definition of odd in math:

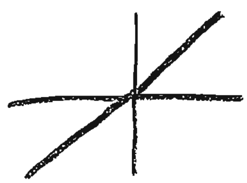
DEF: if $x(t) = -x(-t) \;\; \forall\, t \in \mathbb{R}$, then $x(t)$ is called odd symmetric.

- Here are some examples of odd signals:

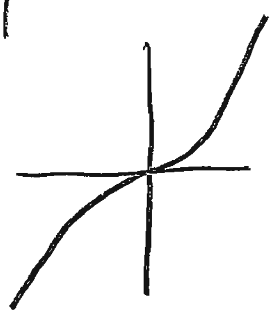
$x(t) = \sin(\omega_0 t)$



$x(t) = t$



$x(t) = t^3$



NOTE: if $x(t)$ is odd, then $x(0) = 0$.

why? To be odd, you must have

$$x(t) = -x(-t) \quad \forall\, t.$$

Plugging in $t=0$, we get

$$x(0) = -x(0)$$

→ Since zero is the only number that is it's own additive inverse, this means $x(0) = 0$.

<u>NOTE</u>: the constant signal $x(t) = 0$ ... i.e., the signal that assigns the buddy zero to every t, is <u>both</u> even <u>and</u> odd.

- There are a couple of more types of symmetry that are mostly useful for complex-valued signals.

<u>DEF</u>: if $x(t) = x^*(-t) \ \forall\, t \in \mathbb{R}$, then $x(t)$ is <u>conjugate</u> <u>symmetric</u>.

  - This means that, for every t, the buddy that $x(t)$ assigns to t is the complex conjugate of the buddy that $x(t)$ assigns to $-t$.
  - This means that the <u>real</u> part of $x(t)$ is <u>even</u>.
  - This means that the <u>imaginary</u> part of $x(t)$ is <u>odd</u>

- For real signals, conjugate symmetric is the same as even, because there is no imaginary part.

<u>DEF</u>: if $x(t) = -x^*(-t) \ \forall t \in \mathbb{R}$, then $x(t)$ is <u>conjugate</u> <u>antisymmetric</u>.

- It means that, for every t, the buddy that $x(t)$ assigns to t is the <u>negative</u> of the <u>conjugate</u> of the buddy that $x(t)$ assigns to $-t$.
- This means that the <u>real</u> part of $x(t)$ is <u>odd</u>.
- This means that the <u>imaginary</u> part of $x(t)$ is <u>even</u>.

- For real signals, conjugate antisymmetric is the same as odd, because there is no imaginary part.

Continuous-Time Sinusoidal Signals

- On page 1 of the course formula sheet, you will find the trig identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

- Let $A = \theta$ and $B = \frac{\pi}{2}$

Then $\cos(\theta - \frac{\pi}{2}) = \cos\theta \underbrace{\cos\frac{\pi}{2}}_{\text{zero}} + \sin\theta \underbrace{\sin\frac{\pi}{2}}_{\text{one}} = \sin\theta$

- Replacing the number θ with the function $\omega_0 t$, $\omega_0 \in \mathbb{R}$, we get

$$\sin(\omega_0 t) = \cos(\omega_0 t - \pi/2)$$

- Replacing θ with $\omega_0 t + \phi$, $\omega_0, \phi \in \mathbb{R}$, we get

$$\sin(\omega_0 t + \phi) = \cos(\omega_0 t + \phi - \pi/2)$$

$\Rightarrow$ So any "sine" signal can be written as a cosine signal.

- By convention, we usually write continuous-time sinusoidal signals in the form

$$x(t) = A\cos(\omega_0 t + \phi)$$

A: real : "amplitude" of $x(t)$

ω_0: real : "frequency" of $x(t)$

ϕ: real : "initial phase" of $x(t)$. Also sometimes called "phase offset."

NOTE : It is comman to write the signal as
$$x(t) = A\cos(\omega_0 t + \phi).$$

-But if we want to think of "ϕ" as a time shift, then we should think of this as $x(t) = A\cos(\omega_0 t - -\phi)$.

$\Rightarrow$ If you have the graph of $x_1(t) = A\cos(\omega_0 t)$

-and you need the graph of
$$x_2(t) = A\cos(\omega_0 t + \phi),$$

$\rightarrow$ Then what you've got for the difference between $x_1(t)$ and $x_2(t)$ is just a time shift, but <u>not</u> a scale,

-because they both have "$\omega_0 t$".

- So, to make the graph of $x_2(t)$ from the graph of $x_1(t)$, we are going to have to time shift (translate) the graph of $x_1(t)$.

- But by how much?
  - we need to write $x_2(t) = x_1(t-t_0)$, as a function of t minus some shift amount t_0.
  - Like this:

$$
\begin{aligned}
x_2(t) &= A\cos(\omega_0 t + \phi) \\
&= A\cos(\omega_0 t - -\phi) \\
&= A\cos\left(\omega_0 t - \omega_0\left(\frac{-\phi}{\omega_0}\right)\right) \\
&= A\cos\Big(\omega_0\big(t - \underbrace{\tfrac{-\phi}{\omega_0}}_{t_0 = \frac{-\phi}{\omega_0}}\big)\Big) \\
&= x_1(t-t_0)
\end{aligned}
$$

$\Rightarrow$ So to get the graph of $x_2(t)$, take the graph of $x_1(t)$ and shift it right by $\frac{-\phi}{\omega_0}$.

- Now keep $x_2(t)$ the same:

$$x_2(t) = A\cos(\omega_0 t + \phi).$$

- But suppose that, instead of the graph of $x_1(t)$, we have the graph of

$$x_0(t) = A\cos(t).$$

- So this time we need to start with the graph of $x_0(t)$ and make the graph of $x_2(t)$.
- Compared to $x_0(t)$, $x_2(t)$ has <u>both</u> a scale <u>and</u> a shift.
- So this time, we write

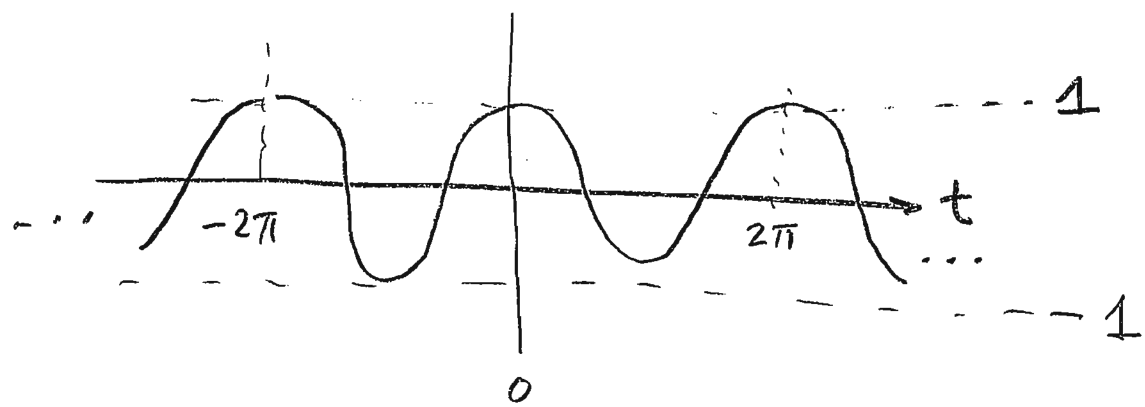
$$x_2(t) = A\cos(\omega_0 t + \phi)$$
$$= A\cos(\underset{\uparrow \text{ scale}}{\omega_0} t - \underbrace{-\phi}_{\text{shift}})$$
$$= x_0(\omega_0 t - (-\phi))$$

- So, as we said on page 2.21, to make the graph of $x_2(t)$ we:
  1. Shift the graph of $x_0(t)$ right by $-\phi$,
  2. Take the resulting graph and time scale by ω_0.

So now let's take a closer look at the continuous-time cosine signal.

- Suppose that time is measured in units of seconds.
- The most basic cosine signal is

$$x(t) = \cos(t)$$



- This signal is 1 at $t=0$, 0 at $t=\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and -1 at $t=\pi$.
- The amplitude is $A=1$.
- The fundamental period is $T_0 = 2\pi$
- The frequency in Hertz, where 1 Hz is one cycle per second, is given by

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi} \text{ Hz}$$

- with $\omega = 2\pi f$, we see that the radian frequency is $\omega_0 = \frac{2\pi}{T_0} = 1$ rad/sec.

- Using a different amplitude like $A=5$ will change the max and min values to ± 5, but it does not change the fundamental period or the frequency.

- Adding an initial phase ϕ will change the value of the signal at $t=0$, but it will not change the fundamental period or the frequency.

- These signals all have fundamental period $T_0 = 2\pi$ and frequency $\omega_0 = 1$ rad/sec:

$x(t) = \cos(t)$ $\quad$ $x(t) = 5\cos(t)$

$x(t) = \cos(t + \frac{\pi}{7})$ $\quad$ $x(t) = 5\cos\left(t - \frac{\pi}{9}\right)$

- Now let us consider the signal

$$x(t) = \cos(\omega_0 t), \quad \omega_0 \in \mathbb{R}$$

- what is the fundamental period?
  - → It is when $\omega_0 t$ goes through 2π rad
    - → which is when t goes through $\frac{2\pi}{|\omega_0|}$ sec.
    - → The fundamental period is

$$T_0 = \frac{2\pi}{|\omega_0|}.$$

    - → In terms of f_0, where $\omega_0 = 2\pi f_0$, we have

$$T_0 = \frac{1}{|f_0|}.$$

- Similarly, $\sin(\omega_0 t)$ and $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$ also have fundamental period

$$T_0 = \frac{2\pi}{|\omega_0|} = \frac{1}{|f_0|}.$$

- Adding an amplitude A and or an initial phase ϕ does not change the frequency and does not change the fundamental period.

So:

$$A\cos(\omega_0 t + \phi)$$

$$A\sin(\omega_0 t + \phi) = A\cos(\omega_0 t + \phi - \pi/2)$$

$$Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

- All have frequency ω_0 and fundamental period $T_0 = \frac{2\pi}{|\omega_0|} = \frac{1}{|f_0|}$.

- All have amplitude A.
- All have initial phase ϕ.

Phasors

BOOK ↓

Let $z(t) = Ae^{j(\omega_0 t + \phi)}$ (2.13)

$= A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$ (2.14)

↑ BOOK

NOTE: $e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$

is a complex sinusoid with unit amplitude ($A = 1$) and zero initial phase ($\phi = 0$).

- We can write $z(t)$ in terms of this signal ⟶

$$z(t) = Ae^{j(\omega_0 t + \phi)}$$

$$= \{Ae^{j\phi}\} e^{j\omega_0 t}$$

$$= \underbrace{\{Ae^{j\phi}\}}_{\uparrow \text{ phasor}} \{\cos(\omega_0 t) + j\sin(\omega_0 t)\}$$

- The quantity $Ae^{j\phi}$ is called a "phasor".
- It is also sometimes called the "complex amplitude" of $z(t)$, although this is an oxymoron.
- The phasor $X = Ae^{j\phi}$ is a complex number
  - The magnitude of the phasor

    $$A = |X|$$

    is the amplitude of $z(t)$.
  - The angle of the phasor

    $$\phi = \arg X$$

    is the initial phase of $z(t)$.
- So the phasor $X = Ae^{j\phi}$ is a complex constant that tells how $z(t)$ is different from a unit amplitude, zero phase complex sinusoid at the same frequency.

- How is this useful?

  FACT: a linear system can not change the frequency of a sinusoid. But it can change the amplitude and initial phase.

  - So it actually comes up that you have a circuit with lots of signals in it that are all sinusoids with the same frequency... they just have different amplitudes and initial phases.

  - This happens especially often in electric power systems where the generator(s) make a signal $120\cos(2\pi\cdot 60t)$.

    - To the extent that the electric power system is linear, all of the signals in the system will be of the form $A\cos(2\pi 60t+\phi)$.

    - They can all be represented by a phasor $X=Ae^{j\phi}$.

- So, a phasor is a complex number $X=Ae^{j\phi}$ that is used to represent the signal

$$z(t) = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi).$$

- The phasor $X = Ae^{j\phi}$ is also sometimes used to represent the signals

$$x(t) = A\cos(\omega_0 t + \phi) = \mathrm{Re}\{z(t)\}$$

and

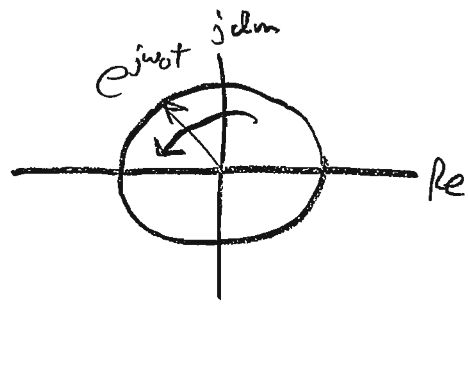
$$x(t) = A\sin(\omega_0 t + \phi) = \mathrm{Im}\{z(t)\}.$$

- This is mostly useful when <u>all</u> of the signals that are present have the same frequency ω_0.

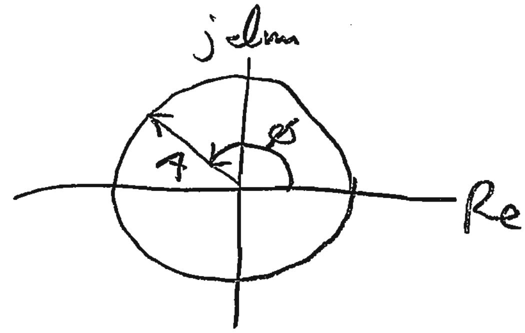
- For any $\theta \in \mathbb{R}$, $e^{j\theta} = \cos\theta + j\sin\theta$ is a complex number. It has unit magnitude: $|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta}$
$= 1$

  → So $e^{j\theta}$ is a point that lies on the unit circle of the complex plane.

  → So $e^{j\omega_0 t}$ describes a point that spins counterclockwise around the unit circle as t advances:



- The phasor $X = Ae^{j\phi}$ is a fixed point in the complex plane that lies on a circle of radius A at an angle of ϕ:



- The signal $z(t) = Ae^{j(\omega_0 t + \phi)}$

$$= A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

can be thought of as a point in the complex plane that lies on a circle of radius A and spins counterclockwise around this circle as t advances, starting at an initial angle of ϕ when $t = 0$.

- In other words,

$$z(t) = \overbrace{Ae^{j\phi}}^{\text{phasor } X} \underbrace{e^{j\omega_0 t}}_{\text{spins it around as } t \text{ advances}}$$

radius

starting angle (phase) when $t=0$

- In this sense, $z(t)$ is sometimes called a "rotating phasor."

$$z(t) = \underset{\substack{\uparrow \\ \text{fixed} \\ \text{number}}}{X} \underbrace{e^{j\omega_0 t}}_{\text{rotates}}$$

$$X = Ae^{j\phi}$$

NOTE: In some books you will see the phasor written this way: $X = A\angle\phi$.

- It means the same thing as $X = Ae^{j\phi}$... i.e., X is a complex number with magnitude A and angle (or argument) ϕ.

Another Way to Think About Real Signals

- Applying Euler's formula $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$, we get

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j(\omega_0 t + \phi)} + \frac{A}{2}e^{-j(\omega_0 t + \phi)}$$

$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$\longrightarrow$

- You can think of $\frac{A}{2}e^{j\phi}$ and $\frac{A}{2}e^{-j\phi}$ as conjugate phasors

  - As t advances,

    $\frac{A}{2}e^{j\phi}e^{j\omega_0 t}$ is a rotating phasor that rotates counterclockwise

    $\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$ is a rotating phasor that rotates clockwise.

  - Because these two rotating phasors are conjugates at every t, the imaginary parts cancel and the real parts add, leaving you with a real signal.

  - Each term contributes half the amplitude of the real signal.

Phasor Addition

- Especially in circuit analysis,

  it will sometimes happen that you have a bunch of cosine signals,

  - all with the same frequency w_0,
  - but each having a separate amplitude and initial phase,

  → And you will need to add them up.

- Phasors can help with this.

- Suppose the cosine signals are given by

$$x_1(t) = A_1 \cos(w_0 t + \phi_1)$$

$$x_2(t) = A_2 \cos(w_0 t + \phi_2)$$

$$x_3(t) = A_3 \cos(w_0 t + \phi_3)$$

$$\vdots$$

$$x_N(t) = A_N \cos(w_0 t + \phi_N).$$

- Problem: we need to compute the sum of these. Call the sum $y(t)$.

  Then $y(t) = \sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} A_k \cos(w_0 t + \phi_k)$

FACT: $y(t)$ is a cosine signal with frequency ω_0. In other words, for two real numbers A and ϕ (no subscripts) where $A \geq 0$, it is true that

$$y(t) = A\cos(\omega_0 t + \phi).$$

→ Moreover, each of the signals $x_k(t)$ can be described by a phasor $X_k = A_k e^{j\phi_k}$.

- This is because

$$\begin{aligned} x_k(t) &= A_k \cos(\omega_0 t + \phi_k) \\ &= \mathrm{Re}\{A_k e^{j(\omega_0 t + \phi_k)}\} \\ &= \mathrm{Re}\{\underbrace{A_k e^{j\phi_k}}_{X_k} e^{j\omega_0 t}\} \\ &= \mathrm{Re}\{X_k e^{j\omega_0 t}\} \end{aligned}$$

→ The sum signal $y(t)$ can also be described by a phasor $X = Ae^{j\phi}$ (no subscripts),

⇒ And $X = X_1 + X_2 + X_3 + \dots + X_N$

$$= \sum_{k=1}^{N} X_k$$

- In other words, the phasor for the <u>sum signal</u> $y(t)$ is given by the <u>sum</u> of the phasors for the individual cosine signals $x_k(t)$.

- Here is a proof of these things. There is also a proof in the book on page 30.

$$y(t) = \sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= \sum_{k=1}^{N} Re\{A_k \cos(\omega_0 t + \phi_k) + j A_k \sin(\omega_0 t + \phi_k)\}$$

$$= Re\left\{\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) + j A_k \sin(\omega_0 t + \phi_k)\right\}$$

$$= Re\left\{\sum_{k=1}^{N} A_k e^{j(\omega_0 t + \phi_k)}\right\}$$

$$= Re\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$

→ but $e^{j\omega_0 t}$ is the same for every term of the sum

→ so it can be pulled out of the sum

$$= Re\left\{e^{j\omega_0 t} \sum_{k=1}^{N} A_k e^{j\phi_k}\right\}$$

→

- Now, the sum in the last line on page 2.46 is just the sum of the phasors for all the cosine signals $x_1(t)$, $x_2(t)$, ..., $x_N(t)$.
  - → It is a sum of N complex numbers.
  - → So it is itself just a complex number.
  - → Call it "X" and write it in polar form as $X = Ae^{j\phi}$.
- Then the last line on page 2.46 becomes

$$y(t) = \mathrm{Re}\left\{e^{j\omega_0 t} \underbrace{\sum_{k=1}^{N} A_k e^{j\phi_k}}_{X = Ae^{j\phi}}\right\}$$

$$= \mathrm{Re}\{Xe^{j\omega_0 t}\} = \mathrm{Re}\{Ae^{j\phi}e^{j\omega_0 t}\}$$

$$= \mathrm{Re}\{Ae^{j(\omega_0 t + \phi)}\}$$

$$= \mathrm{Re}\{A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)\}$$

$$= A\cos(\omega_0 t + \phi).$$

//

Summary:

- If you have to add up several cosine signals that all have the same frequency,
  - Then the sum is also a cosine signal with the same frequency.
  - The phasor for the sum is given by the sum of the phasors for the individual signals that you have to add up.

- So you can compute the sum as follows:

  (1) Make a phasor for each input signal by reading off the amplitude A_K and phase ϕ_K to get the phasor $X_K = A_K e^{j\phi_K}$.

  (2) Convert the phasors X_K to rectangular form and add them up to get

$$X = X_1 + X_2 + \cdots + X_N$$

$\longrightarrow$

(3) Convert X from rectangular form to polar form to get $X = Ae^{j\phi}$.

(4) The sum is given by

$$y(t) = A\cos(\omega_0 t + \phi).$$

For example, if $\omega_0 \in \mathbb{R}$,

and $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$,

and $A_1, A_2, A_3 \in \mathbb{R}$ such that $A_1 \geq 0$, $A_2 \geq 0$, and $A_3 \geq 0$,

and

$$x_1(t) = A_1\cos(\omega_0 t + \phi_1)$$

$$x_2(t) = A_2\cos(\omega_0 t + \phi_2)$$

$$x_3(t) = A_3\cos(\omega_0 t + \phi_3)$$

$\Rightarrow$ And you have to compute

$$y(t) = x_1(t) + x_2(t) + x_3(t),$$

$\longrightarrow$

-Then you do it like this;

Phasors: $X_1 = A_1 e^{j\phi_1}$

$X_2 = A_2 e^{j\phi_2}$ (step ①)

$X_3 = A_3 e^{j\phi_3}$

Step ②: convert to rectangular and add them up:

$X_1 = A_1 \cos\phi_1 + jA_1 \sin\phi_1$

$X_2 = A_2 \cos\phi_2 + jA_2 \sin\phi_2$

$X_3 = A_3 \cos\phi_3 + jA_3 \sin\phi_3$

$$X = X_1 + X_2 + X_3$$

$$= [A_1\cos\phi_1 + A_2\cos\phi_2 + A_3\cos\phi_3]$$

$$+ j[A_1\sin\phi_1 + A_2\sin\phi_2 + A_3\sin\phi_3]$$

Step ③; convert X to polar form to get $X = Ae^{j\phi}$

Step ④ $y(t) = A\cos(\omega_0 t + \phi)$.

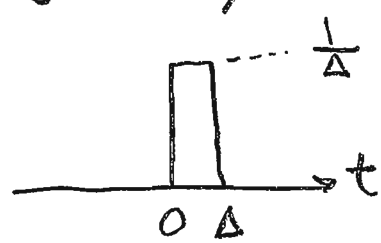
Continuous-Time Unit Impulse

- There are a couple of continuous-time signals that we <u>cannot</u> model with functions.

- One of them is the "unit impulse" signal, usually written $\delta(t)$.
    - Intuitively, this signal is a <u>very</u> skinny, <u>very</u> tall pulse that has unit area.
    - It is also called the "Dirac delta" in honor of P.A.M. Dirac, an English physicist who lived from 1902 to 1984.
    - It is also sometimes called the "delta function" or the "Dirac delta function", but these names are oxymorons.
        - Because it is <u>incorrect</u> to think of $\delta(t)$ as a <u>function</u>.
- 20th century math is required to model $\delta(t)$ correctly.
- This math is called "distribution theory" or "the theory of generalized functions."
    → It is a <u>different</u> use of the word "distribution" than in probability theory. Generalized functions have nothing to do with probability distributions.
    → This math was developed rigorously in the 1940's, most notably by French mathematician Laurent Schwartz who lived from 1915 to 2002.

- The details of distribution theory are beyond the scope of ECE 2713.
    - Some instructors may teach the rigorous math details in ECE 3793
    - The details are also taught in ECE 4213.
    - There is a decent Wikipedia article on distribution theory if you want to get the "flavor" of it.
- Unfortunately, many textbooks contain explanations of $\delta(t)$ that are mathematically <u>incorrect</u> or even complete <u>nonsense</u>.
- Our goal in ECE 2713 will be to
    1. learn how to work with $\delta(t)$ in a way that gives correct answers, and
    2. <u>not</u> learn anything that is <u>wrong</u>.
- To make sense of all this in a way that is correct, we need to start by thinking of an ordinary continuous-time signal $d_1(t)$ that is given by

$$d_1(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases} =$$



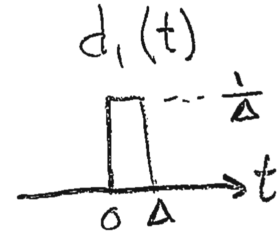
where $\Delta \in \mathbb{R}$.

Note: $d_1(t)$ is a plain old continuous-time signal.

$\Rightarrow$ We can model $d_1(t)$ with a function.

- The area under the curve is the area of a rectangle

$$= \text{base} \times \text{height}$$
$$= \Delta \cdot \frac{1}{\Delta} = 1.$$



- So $d_1(t)$ has unit area, and we see that

$$\int_{-\infty}^{\infty} d_1(t)\,dt = 1.$$

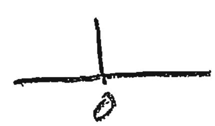
- The trick here is that we need to think of Δ as being a really really tiny number, like $\Delta = 10^{-40}$ seconds!

- So $d_1(t)$ is really skinny and really tall.

- In fact, $d_1(t)$ is so skinny that we cannot see it or even measure it.

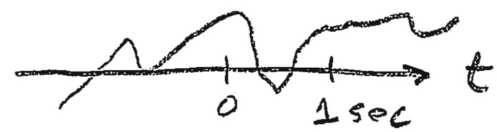
- If we try to look at $d_1(t)$ on an oscilloscope, we won't be able to see any of the details of the graph.

$\longrightarrow$ all we will be able to see is this:



- But for now, pretend like we have a super-duper microscope that allows us to look at $d_1(t)$.

- Let $x(t)$ be an ordinary continuous-time signal like we might find in a circuit in Lab I.
  - To our eyeballs, $x(t)$ might appear to have a lot of variation ... to be "busy":

$x(t) =$ 

  - But on a super microscopic time scale like $\Delta = 10^{-40}$ sec, $x(t)$ is <u>virtually constant</u>.

- Now, what we need to do is integrate the product $d_1(t)\,x(t)$ ... the product of the super skinny guy $d_1(t)$ times the "normal" guy $x(t)$.
  - In other words, we need to compute

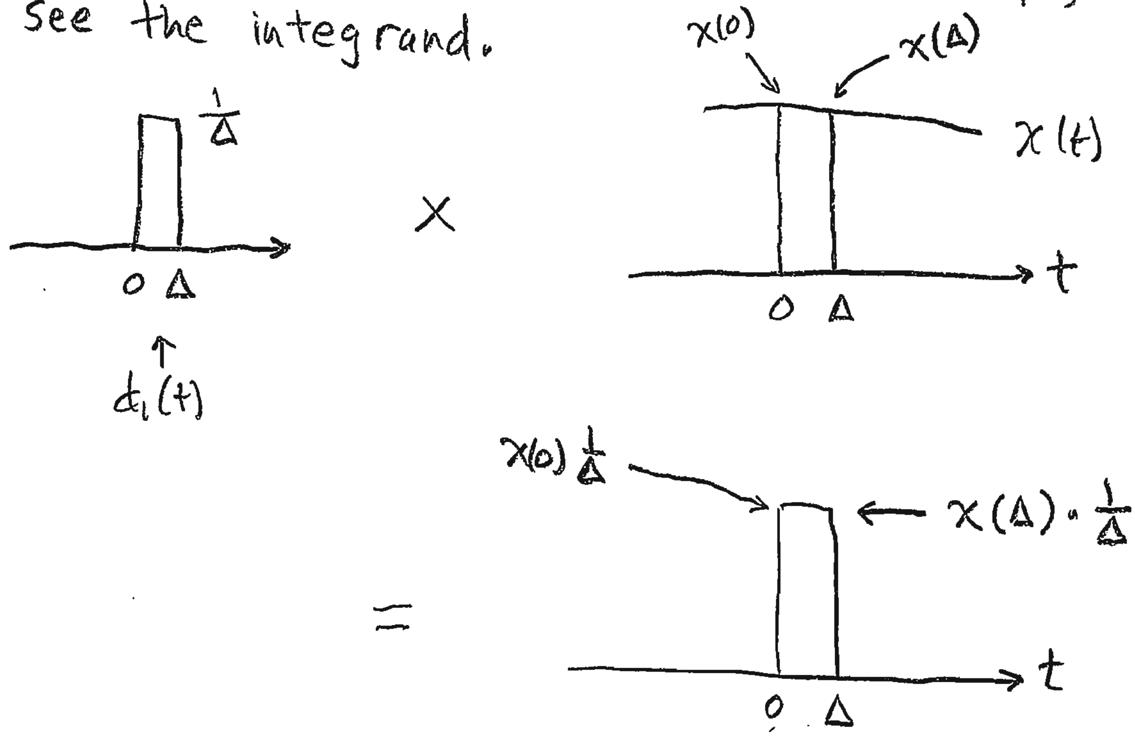
$$\int_{-\infty}^{\infty} d_1(t)\,x(t)\,dt$$

→ This is a <u>number</u>.

→ It is the <u>area</u> under the curve $d_1(t) \times x(t)$.

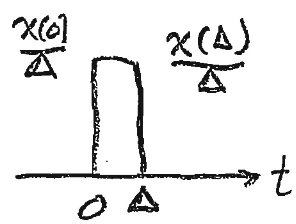
- Looking through our super duper pretend microscope, we can see the integrand.

- It is:



- This integrand $d_1(t)\,x(t)$ is a <u>function</u>.
- So we <u>can</u> integrate it.

$$\int_{-\infty}^{\infty} d_1(t)\,x(t)\,dt = \text{Area under the curve}$$



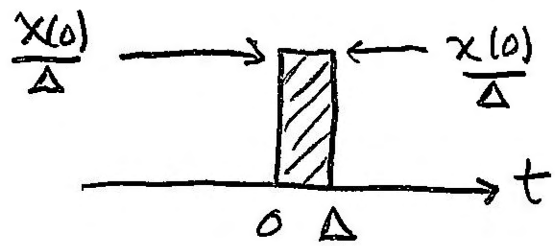
- But remember that Δ is <u>tiny</u>.

→ Δ is <u>so</u> <u>tiny</u> that we <u>can</u> <u>not</u> see or measure the difference between the number $\frac{x(0)}{\Delta}$ and the number $\frac{x(\Delta)}{\Delta}$)

→ Because $x(t)$ can <u>not</u> make any measurable change on a time scale of only Δ seconds.

- So our integral is equal to the area under the curve of the integrand,

→ which is not <u>measurably</u> different from the area under this rectangle:



- The area under this rectangle is:

$$\text{Area} = \text{base} \times \text{height}$$

$$= \frac{x(0)}{\Delta} \cdot \Delta = x(0)\text{, a } \underline{\text{number}}.$$

☆☆ Therefore, as far as we can measure it,

- as far as we can tell,
- our integral is given by

$$\int_{-\infty}^{\infty} d_1(t)\, x(t)\, dt = x(0).$$

- Now, in truth this is an approximation.
  - But it is an approximation that is good to at least 40 decimal places!
  - The error is <u>too</u> <u>small</u> to measure!

- So up to now, we had $d_1(t)$ and $x(t)$ and we could model them both with functions.

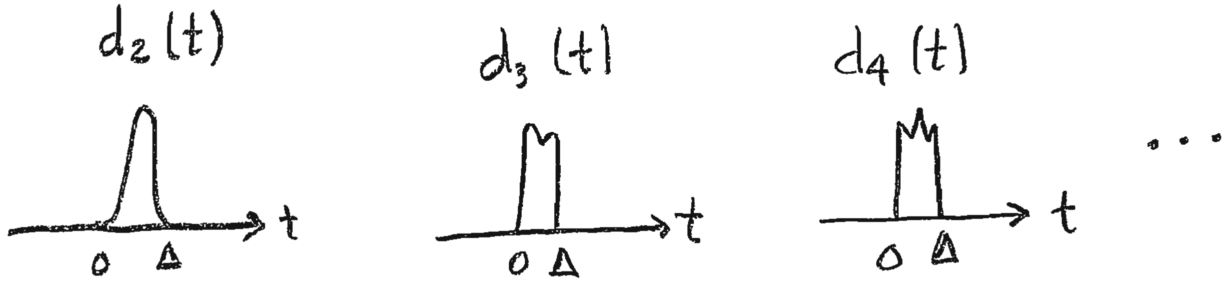
- And we had $\int_{-\infty}^{\infty} d_1(t)\, x(t)\, dt$

  → The integrand can also be modeled just fine as a function.

  → So if we had to, we could work it as a Riemann integral and we would get an answer <u>not</u> <u>measurably</u> <u>different</u> from the number $x(0)$:

$$\int_{-\infty}^{\infty} d_1(t)\, x(t)\, dt = x(0)$$

- But the trouble is that there are also <u>other</u> super skinny guys



that <u>all</u> <u>have</u> <u>unit</u> <u>area</u> ...

- And all have the property that

$$\int_{-\infty}^{\infty} d_2(t)\,x(t)\,dt = \int_{-\infty}^{\infty} d_3(t)\,x(t)\,dt$$

$$= \int_{-\infty}^{\infty} d_4(t)\,x(t)\,dt$$

$$= \int_{-\infty}^{\infty} d_1(t)\,x(t)\,dt = x(0).$$

- And they are all <u>so</u> <u>skinny</u> that we can't see or measure the differences between them!

→ Because in the <u>real</u> <u>world</u> we don't get to use our super duper <u>pretend</u> microscope.

- So the truth is, when one of these guys shows up in a circuit, all we can see on the oscilloscope is:  .

⇒ The truth is that we can't even tell which one we've got!!!

- But no matter which one it is,
  - whether it's $d_1(t)$ or $d_2(t)$ or $d_3(t)$ or $d_4(t)$ ...
    or even some other similar guy,

$\Rightarrow$ we know that

$$\int_{-\infty}^{\infty} [\text{our guy}]\, x(t)\, dt = x(0).$$

- But we <u>can't</u> work this as a Riemann integral.
- To work it as a Riemann integral, we have to know the integrand as a <u>function</u>.
- we have to know the buddy assignments for all the t's on a microscopic level...
- But we can't know that because we can't even tell which skinny unit area guy we've actually got... is it $d_1(t)$? $d_2(t)$? $\rightarrow$ No way to know!

☆☆☆ $\delta(t)$ is <u>one</u> math object that models <u>all</u> of these super skinny guys $d_1(t)$, $d_2(t)$, $d_3(t)$ ... <u>at the same time</u>.

$\Rightarrow$ $\delta(t)$ stands for <u>whichever</u> <u>one</u> we've got.

- Because of this, $\delta(t)$ <u>can not</u> be a function!

    - To be a function, we would have to know which one we've actually got...
    - And we can't know that...
    - So we can't know the buddy assignments for all the t's.

→ $\delta(t)$ stands for <u>any one</u> of the super skinny unit area pulses $d_1(t)$, $d_2(t)$, $d_3(t)$, ...

- And whichever one it is, we know that

$$\int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(0) \qquad (*)$$

    - But this can't be a Riemann integral, because there is no way to know what the integrand is as a function.
    - So the truth is that it is a <u>fake</u> integral.
        - We know what the answer is!
            → It is the number $x(0)$.

    ⇒ But there is <u>no</u> <u>calculus</u> in equation (*).

    ⇒ There is <u>no</u> <u>integration</u>.

⇒ We know what the answer has to be, so we just write it down!!

- So why even write a fake integral sign ??
  - Because our pal Laurent Schwartz set up the theory of distributions so that...
    - If we need to do a <u>change of variable</u>,
    - We can use the fake integral sign as a <u>crutch</u> ...
    - And do the change of variable as though it was a <u>real integral</u>...
      - And we will get the right answer every time!!
- Here is an example:
  - We know that $\int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(0)$.

But what about:

$$\int_{-\infty}^{\infty} \delta(t-5)\, x(t)\, dt \quad ???$$

- The theory of distributions says: do a change of variable on the <u>fake</u> integral as though it was a <u>real</u> integral:

$$\int_{-\infty}^{\infty} \delta(t-5)\, x(t)\, dt$$

Let $u = t-5$ $\quad t = u+5$

$du = dt \quad dt = du$

when $t \to \infty$, $u \to \infty$

when $t \to -\infty$, $u \to -\infty$

$$\int_{-\infty}^{\infty} \delta(t-5)\, x(t)\, dt = \int_{-\infty}^{\infty} \delta(u)\, x(u+5)\, du$$

$$= \int_{-\infty}^{\infty} \delta(u)\, x_2(u)\, du = x_2(0)$$

where $x_2(u) = x(u+5)$

- And so:

$$\int_{-\infty}^{\infty} \delta(t-5)\, x(t)\, dt = x_2(0)$$

$$= x(0+5)$$

$$= \underline{\underline{x(5)}} \quad \text{(a number)}$$

- Doing the same kind of change of variable to the fake integral, we get more generally that:

$u = t - t_0 \qquad t = u + t_0 \qquad du = dt$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,x(t)\,dt = \int_{-\infty}^{\infty} \delta(u)\,x(u+t_0)\,dt$$

$$= x(0+t_0) = \underline{\underline{x(t_0)}}$$

- This is called the "sifting property" of $\delta(t)$.
- There is a "rule of thumb" that makes it easier to think about:
  - Think of $\delta(t)$ as being "turned on" at $t=0$.
  - So $\delta(t-t_0)$ is "turned on" when $t-t_0=0$
  - In other words, $\delta(t-t_0)$ is "turned on" when $t=t_0$.

<u>RULE</u>: $\int_{-\infty}^{\infty} \delta(t-t_0)\,x(t)\,dt$ = the value of $x(t)$ where $\delta(t-t_0)$ is turned on

$= x(t_0)$.

EXAMPLES :

$$\int_{-\infty}^{\infty} \delta(t) \overbrace{\left[\tfrac{1}{4} e^{-t} \right]}^{x(t)} dt = \tfrac{1}{4} e^{-t} \Big|_{t=0} = \tfrac{1}{4}$$

$$\int_{-\infty}^{\infty} x(t)\, \delta(t+5)\, dt = x(-5)$$

(because $\delta(t+5)$ is turned on at $t=-5$)

$$\int_{-\infty}^{\infty} \delta(t-3)\, x(t+2)\, dt = x(t+2) \Big|_{t=3} = x(5)$$

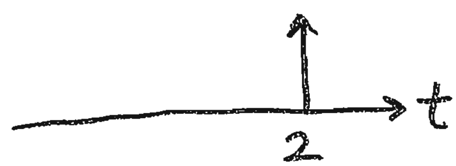
$$\int_{-\infty}^{\infty} \delta(t+2) \underbrace{\left[t^2 \right]}_{x(t)} dt = t^2 \Big|_{t=-2} = 4$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, x(2t-5)\, dt = x(2t-5) \Big|_{t=t_0}$$

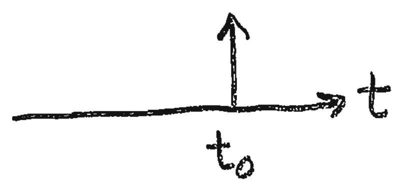
$$= x(2t_0 - 5)$$

- Here is how we draw the graph of $\delta(t)$:

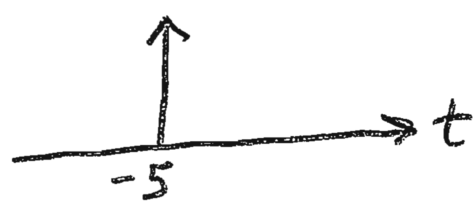
- Here is the graph of $\delta(t-2)$:



- Here is the graph of $\delta(t-t_0)$:



- Here is the graph of $\delta(t+5) = \delta(t - -5)$:



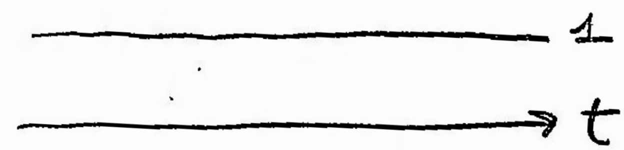
- Now here is a trick that shows how to prove that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

using distribution theory:

- Let $x(t) = 1$ ... the signal that assigns the buddy 1 to every t.

- Graph of $x(t)$:



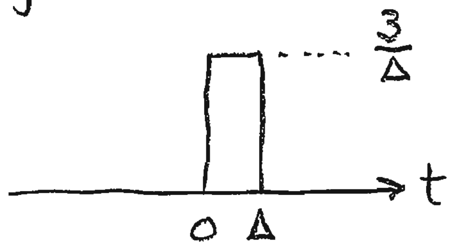
$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{-\infty}^{\infty} \delta(t) \cdot 1\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt$$

$$= x(t)\Big|_{t=0} = 1$$

- If you have $3\delta(t)$, then you have to think of it as the number 3 times one of the super skinny guys like $d_1(t)$ or $d_2(t)$ or $d_3(t)$ ...

- So, using $d_1(t)$, you get

$3d_1(t) =$ 

- Notice that this changes the <u>area</u>.

$$\text{Area} = \text{base} \times \text{height}$$
$$= \Delta \cdot \frac{3}{\Delta} = \underline{\underline{3}}.$$

- So $\int_{-\infty}^{\infty} 3\delta(t)\,dt = 3$

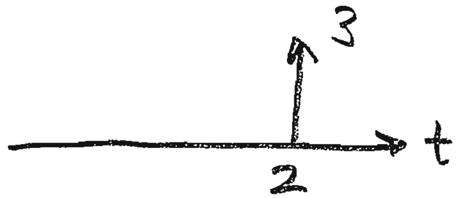
- By manipulating the <u>fake</u> integral as though it was a real integral (thank you Laurent Schwartz!), you can also think of this as:

$$\int_{-\infty}^{\infty} 3\delta(t)\,dt = 3\int_{-\infty}^{\infty} \delta(t)\,dt = 3 \cdot 1 = 3$$

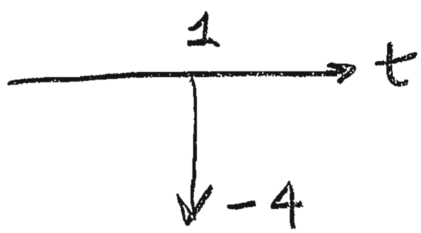
→ The guy $3\delta(t)$ is called a "weighted impulse."

- Graph of $3\delta(t)$:

- Graph of $3\delta(t-2)$:



- Graph of $-4\delta(t-1)$;



<u>NOTE</u> ; $\int_{-\infty}^{\infty} K\delta(t-t_0)\, x(t)\, dt = Kx(t_0)$

<u>EX</u> : $\int_{-\infty}^{\infty} 3\delta(t-2)\, x(t)\, dt = 3x(2)$

- if $x(t) = t^2$, then

$$\int_{-\infty}^{\infty} 4\delta(t-2)[t^2]\, dt = 4[t^2]_{t=2} = 4 \cdot 4 = 16$$

Continuous-Time Unit Step

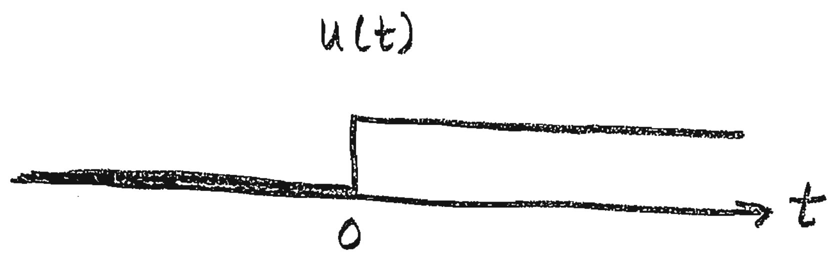
- The unit step "function" is written $u(t)$.
- The idea is that $u(t) = 1$ when $t > 0$
  $u(t) = 0$ when $t < 0$

  - But what is $u(t)$ when $t = 0$??

- Most of the time in ECE 2713, it will be fine to say that $u(0) = 1$.

- This makes $u(t)$ a perfectly good function:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

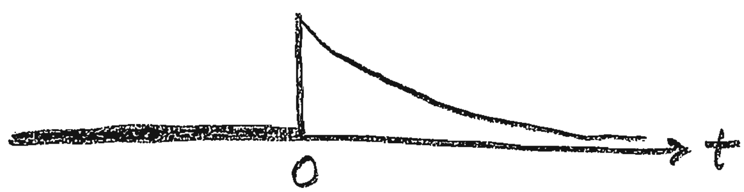


- This is usefull for "cutting off" signals that would otherwise blow up.

- For example, $x(t) = e^{-2t}$ blows up as $t \to -\infty$.

- But if you multiply e^{-2t} by $u(t)$, it chops off the part where e^{-2t} blows up:

$$e^{-2t}u(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

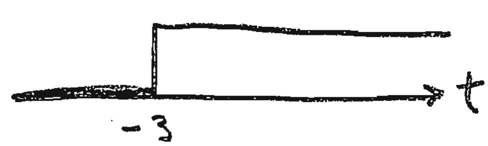


- The signal $u(t-t_0)$ turns on at $t=t_0$ instead of at $t=0$:

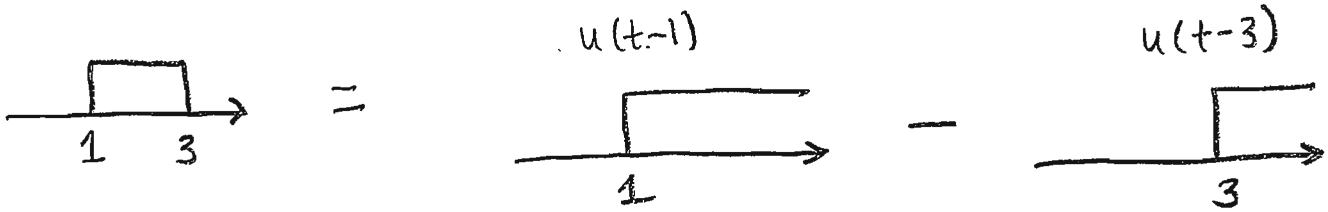
$u(t-t_0)$

EX: $u(t-5)$

EX: $u(t+3) = u(t - -3)$ (shift right by $t_0 = -3$)



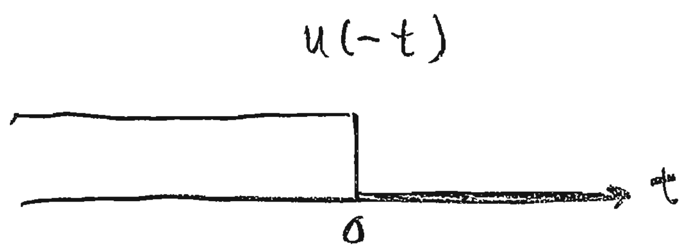
- You can think of a boxcar as being the difference of two step functions:



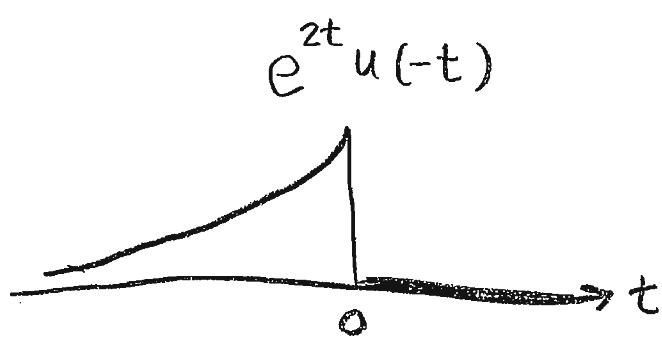
This guy turns it on at $t=1$

But this guy kills it starting at $t=3$.

- The signal $u(-t)$ turns on at $t=0$, but goes to the left... he is turned on for the negative t's:



EX:



- For $u(-t-t_0)$, you can use our rule from page 2.21:

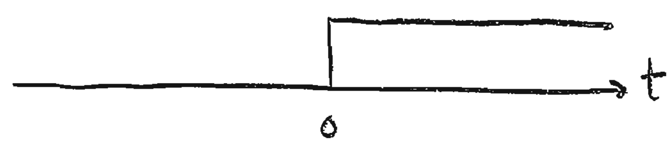
  - You have a scale by $a=-1$
  - And a shift right by t_0.

    → <u>NOTE</u>: we don't know if t_0 is positive or negative, but we should always think of it as a shift right by t_0.

    → Review page 2.13 if you have forgotten about this...

  - The rule on page 2.21 says do the shift first, then the scale.

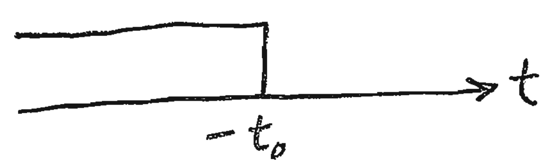
$u(t)$



Shift:

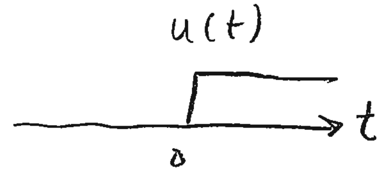
$u(-t-t_0)$

Scale (a flip in this case)

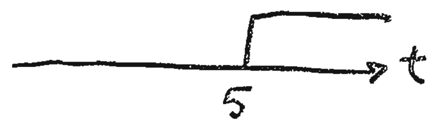


→ These graphs are good no matter what t_0 is!

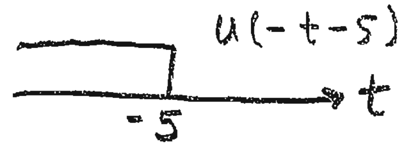
EX: $u(-t-5)$:



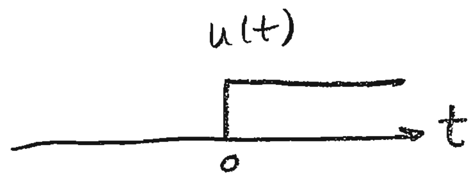
Shift:



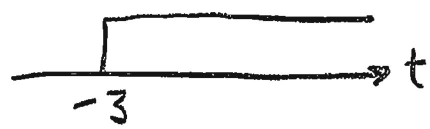
Scale (flip):



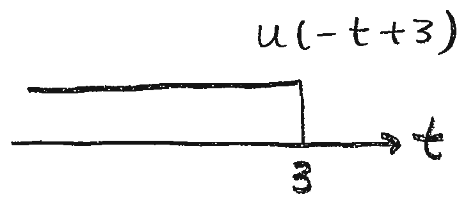
EX: $u(-t+3) = u(-t - -3)$:



Shift:



Scale (Flip):



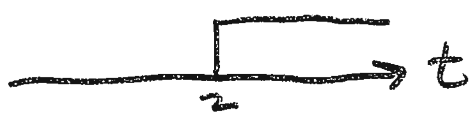
- For $u(t)$ <u>only</u>, there is an easier way to get the graph.

→ <u>DO NOT TRY TO APPLY THIS TRICK TO OTHER SIGNALS</u>!

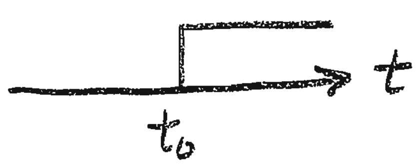
→ <u>IT ONLY WORKS FOR u(t)</u>!

- The unit step function turns on when he eats a zero.
- If there is no "minus" on t, then he goes to the right.
- If there is a "minus" on t, then he goes to the left.

<u>EX</u>: $u(t-2)$: turns on at 2, goes right:



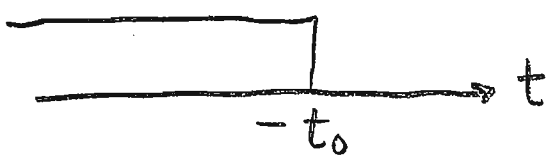
<u>EX</u>: $u(t-t_0)$: turns on at t_0, goes right:



EX : $u(-t-5)$: turns on at -5, goes left:

EX : $u(-t+3)$: turns on at 3, goes left:

EX : $u(-t-t_0)$: turns on at $-t_0$, goes left:



Unit Step as a Distribution

- Sometimes we have to be "flexible" about the value of $u(t)$ at $t=0$.
  - Usually it's just fine to have $u(0)=1$.
  - But sometimes we need $u(0)=0$. I don't think this will come up in ECE 2713.
  - There is also a special version called the "Heaviside Step Function" where $u(0)=\frac{1}{2}$.
    - It is then given by

$$u(t) = \begin{cases} 1, & t>0 \\ 1/2, & t=0 \\ 0, & t<0 \end{cases}$$

      - I think that this also won't come up in ECE 2713, although it is defined in Matlab.

- Yet, all of the above are still <u>functions</u>, because they make $u(t)$ assign one and only one buddy to every t.

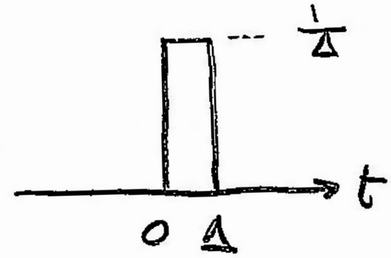
- There are other cases where it is impossible to know what the values of $u(t)$ are in a tiny neighborhood around $t=0$.

  → I am talking <u>tiny</u>, like 10^{-40} seconds.

- How can this happen?

- Recall our super skinny unit area signal $d_1(t)$ from page 2.52:

$$d_1(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases} =$$



  where $\Delta \in \mathbb{R}$

  and we are thinking of Δ as super tiny,

  - like $\Delta = 10^{-40}$ sec.

  - Recall that $d_1(t)$ is a perfectly good function, because he assigns one and only one buddy to every t.

- Now let $m_1(t) = \int_{-\infty}^{t} d_1(\alpha)\, d\alpha$

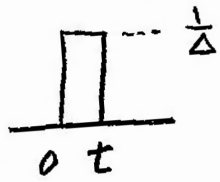
  - when $t<0$, we don't get any area.

    → So $m_1(t) = 0$ when $t<0$.

- when $t > \Delta$, we get all of the area,

$\rightarrow$ so $m_1(t) = 1$ when $t > \Delta$.

- when $0 \le t \le \Delta$, we pick up the area from 0 to t:



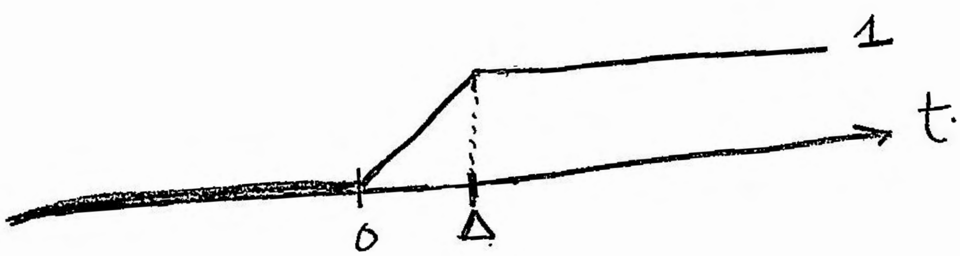
$= \text{base} \times \text{height}$

$= t \cdot \frac{1}{\Delta} = \frac{t}{\Delta}$

- This area increases linearly from zero at $t=0$ to one at $t=\Delta$.

- So the graph of $m_1(t) = \int_{-\infty}^{t} d_1(\alpha)\, d\alpha$

looks like this:



- But remember that Δ is tiny... like $\Delta = 10^{-40}$ sec.

- So on the oscilloscope, all we can see is this:

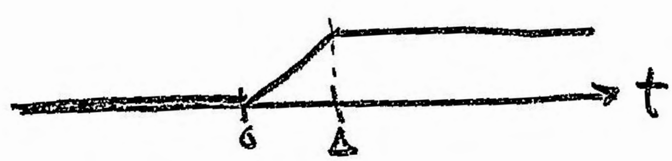
- And in the real world where we do not have our super duper pretend microscope,

→ how are we going to know if we've got the integral of $d_1(t)$ or of $d_2(t)$ or of $d_3(t)$ or of ... one of the other super skinny guys ??

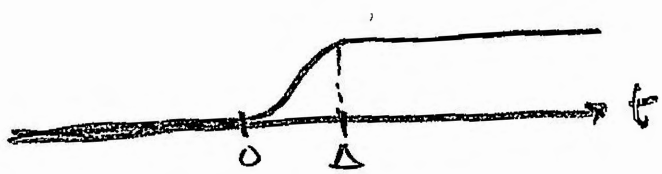
→ we <u>can't</u> know. Because we can't see or measure the differences between any of them.

- But they all give a slightly different behavior for the integral in a tiny neighborhood around $t=0$:

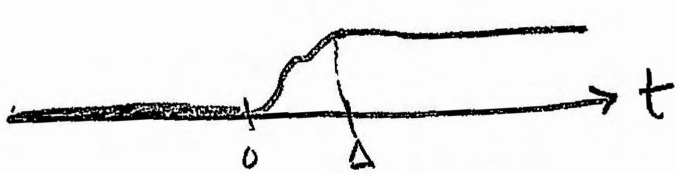
$$m_1(t) = \int_{-\infty}^{t} d_1(\alpha)\,d\alpha \;:$$



$$m_2(t) = \int_{-\infty}^{t} d_2(\alpha)\,d\alpha \;:$$



$$m_3(t) = \int_{-\infty}^{t} d_3(\alpha)\,d\alpha \;:$$



- Since we can't tell the difference between
  $d_1(t)$, $d_2(t)$, $d_3(t)$ ...
  → There's no way to know whether we've actually got $m_1(t)$, $m_2(t)$, $m_3(t)$, ... ??

- In this situation, we have to use distribution theory.
  - we let $\delta(t)$ stand for <u>all</u> of the skinny guys $d_1(t)$, $d_2(t)$, $d_3(t)$ ... at <u>once</u>.
  - we let it model them all simultaneously, so it can stand for whichever one we've actually got.
  - Remember that the neighborhood in question is <u>tiny</u>... $\Delta = 10^{-40}$ ... we can't even see it or measure it.
- Then we let $u(t)$ be a distribution that models $m_1(t)$, $m_2(t)$, $m_3(t)$ --- all at the same time... it stands for whichever one we've got.
- And we write the distributional integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau .$$

<u>NOTE</u>: this is <u>not</u> a fake integral this time. But it is also <u>not</u> a Riemann integral.

- You cannot work this integral using the Riemann Calculus you learned in freshman calc.

  - If you could somehow <u>know</u> that it was $d_1(t)$ or $d_2(t)$ ... then it would be a plain old Riemann integral.

  - But when it is $\delta(t)$, there is no way to know which one you've got. So it takes a more powerful kind of integration developed in the 20th century.

  - The rigorous details of the math are beyond the scope of ECE 2713.

- But here's what it means from a practical standpoint:

  - Most of the time it is just fine to let $u(0) = 1$ so that

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

    is a perfectly good function.

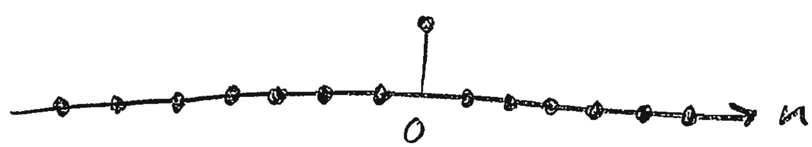
  - But sometimes $u(t)$ has to be treated as a distribution. In those cases, you just have to be a little bit flexible about what $u(0)$ is. (Remember: Δ is <u>tiny</u>)

SOME BASIC DISCRETE-TIME SIGNALS

- Discrete-time unit impulse:

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$

- Normally, we graph discrete-time signals using a circle with a vertical bar that connects the circle to the horizontal axis, like this: .
    - These are called "stems".
- Here is the graph of $\delta[n]$:



- The signal $\delta[n]$ is also sometimes called:
    - The unit sample function
    - Discrete-Time unit impulse
    - Kronecker delta function
        - In honor of German mathematician Leopold Kronecker who lived from 1823 to 1891.
        - There is a decent wikipedia article on the Kronecker delta.

- Unlike the Dirac delta $\delta(t)$, the Kronecker delta $\delta[n]$ is a perfectly good function.

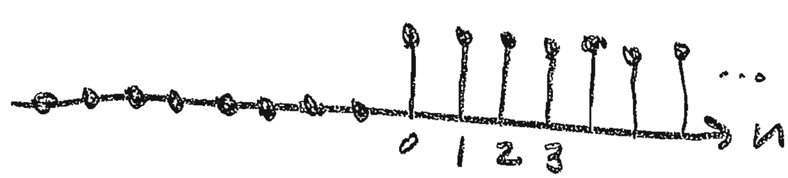
  - It does not suffer from any of the problems of $\delta(t)$.

  - No fancy 20th century math is needed to handle $\delta[n]$. He is just a plain old signal / function / guy.

- Discrete-time unit step function:

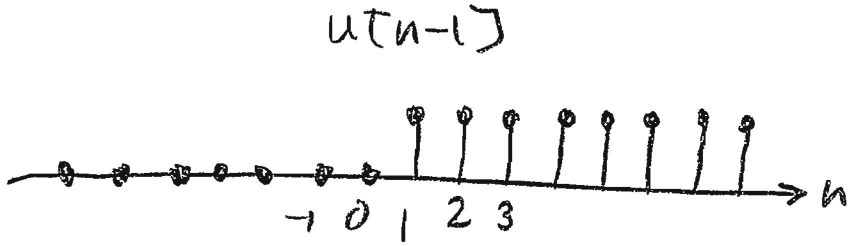
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

  - graph of $u[n]$:

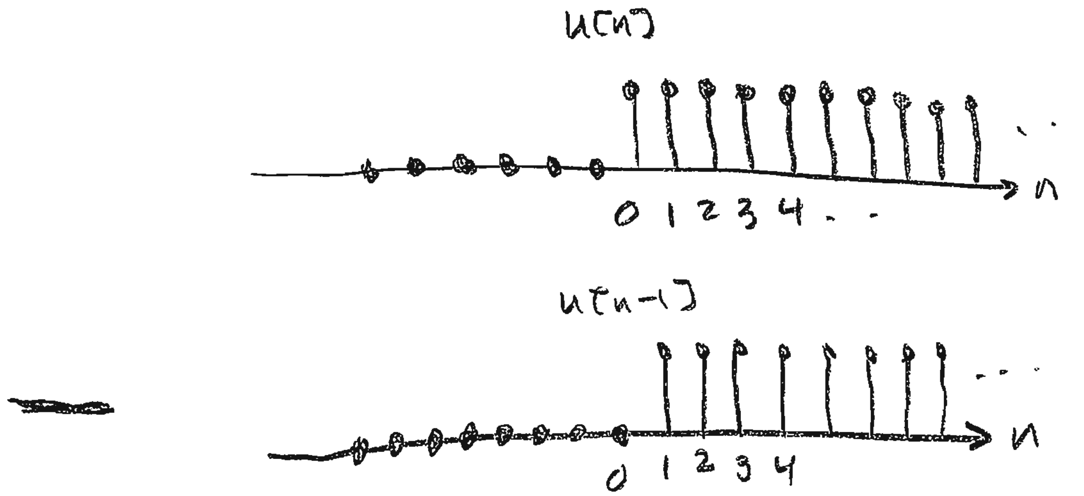


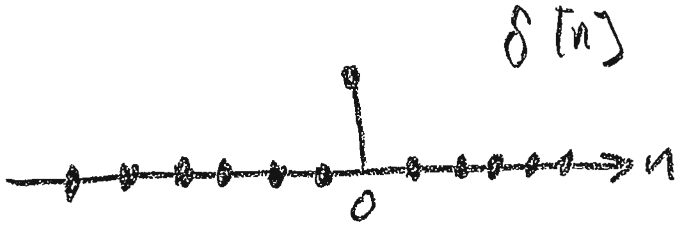
  - Like $\delta[n]$, $u[n]$ is a perfectly good function. Distribution theory is not need for $u[n]$.

- The graph of $u[n-1]$ is obtained by shifting the graph of $u[n]$ to the right by 1:



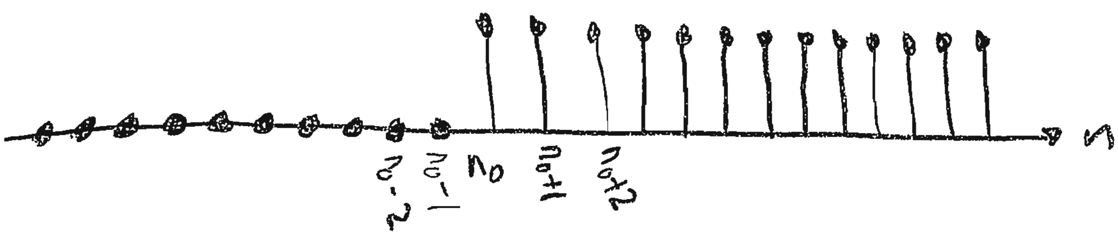
- From this, you can see that $u[n] - u[n-1] = \delta[n]$:



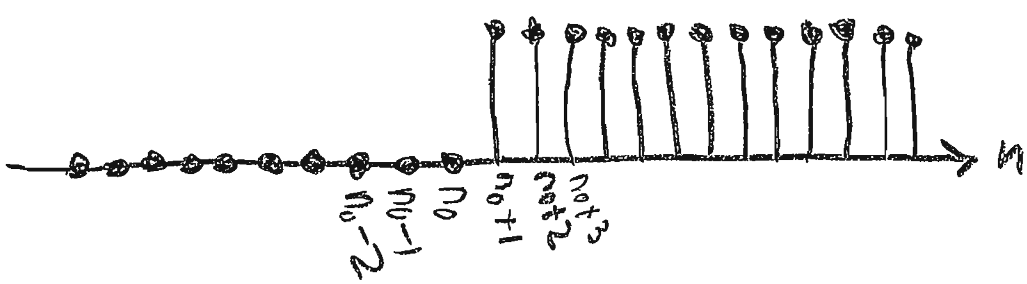


- More generally,
  - The graph of $u[n-n_0]$ starts at n_0 and "goes right"
  - The graph of $u[n-(n_0+1)] = u[n-n_0-1]$ starts at n_0+1 and goes right
  - So $u[n-n_0] - u[n-n_0-1]$ will just leave the one stem at $n=n_0$ ... in other words, it's $\delta[n-n_0]$:

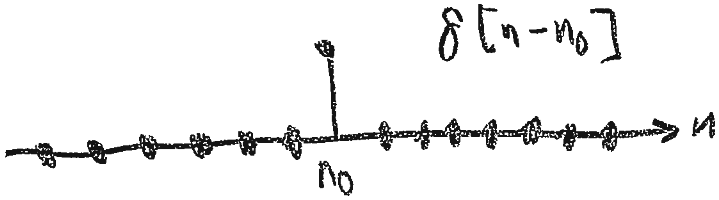
$u[n-n_0]$



$u[n-n_0-1]$

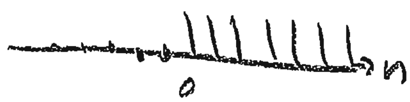


$\delta[n-n_0]$



- So we see that $u[n-n_0] - u[n-n_0-1] = \delta[n-n_0]$.

- The graph of $\delta[n]$ is "turned on" at $n=0$.
- The graph of $\delta[n-7]$ is "turned on" at $n=7$
- The graph of $\delta[n+3] = \delta[n - -3]$ is "turned on" at $n=-3$.
- The graph of $\delta[n-n_0]$ is "turned on" at $n=n_0$ for any $n_0 \in \mathbb{Z}$.

- So $u[n] =$  can be thought of as
  - The guy turned on at $n=0$
  - plus the guy turned on at $n=1$
  - plus the guy turned on at $n=2$
  - plus ...

- or: $u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \cdots$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

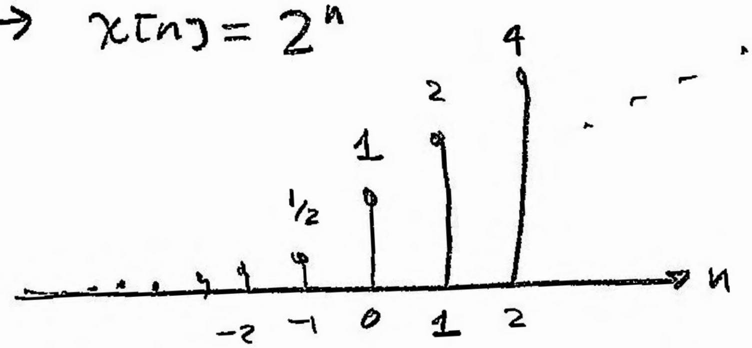
Discrete-Time Exponential Signals

- The discrete-time exponential signal is given by $x[n] = a^n$ where $a \in \mathbb{R}$ is a constant.

- There are four possible behaviors based on the value of a.

- If $a > 1$, then $x[n] = a^n$ is a growing exponential. The signal increases without bound (blows up) as $n \to \infty$.

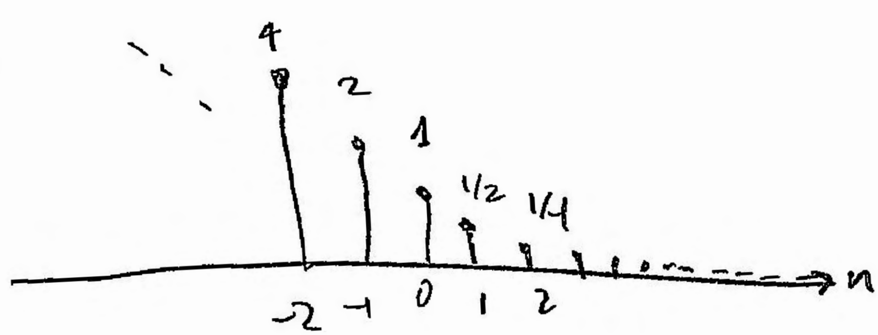
EX: $a = 2 \longmapsto x[n] = 2^n$



- If $0 < a < 1$, then $x[n] = a^n$ is a decaying exponential. The signal asymptotically approaches zero as $n \to \infty$ and it blows up as $n \to -\infty$.

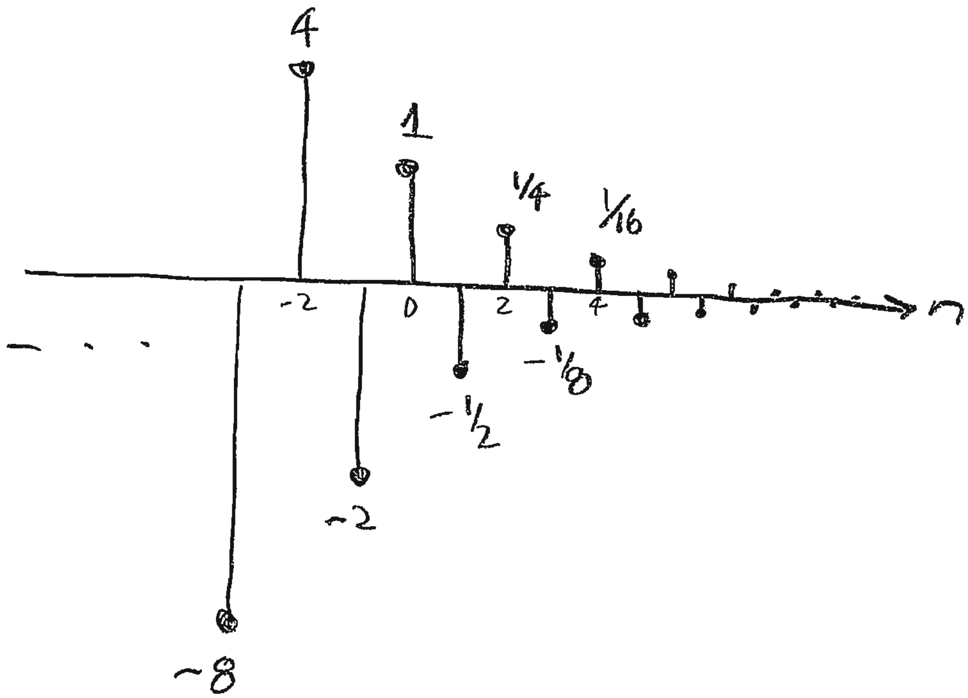
EX: $a = \frac{1}{2}$

$x[n] = \left(\frac{1}{2}\right)^n$



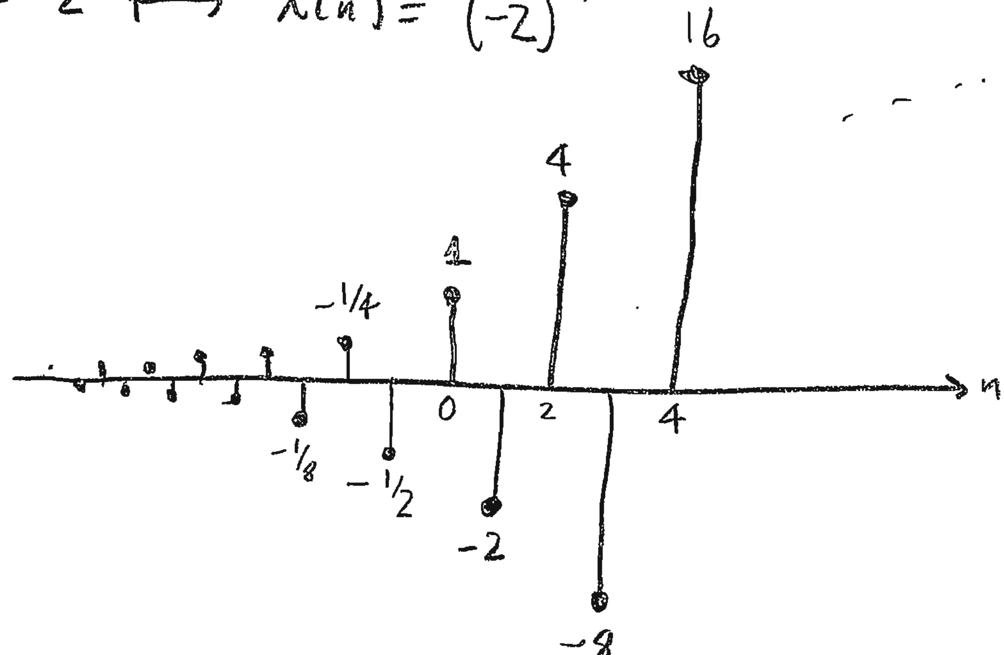
- If $-1 < a < 0$, then $x[n] = a^n$ is decaying... it goes to zero as $n \to \infty$ and it blows up as $n \to -\infty$... but because $a < 0$, the sequence also <u>alternates</u>:

<u>EX</u> $a = -\frac{1}{2} \longmapsto x[n] = \left(-\frac{1}{2}\right)^n$



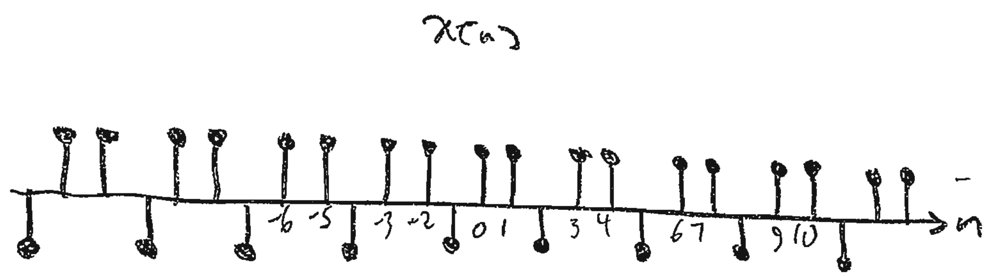
- If $a < -1$, then $x[n] = a^n$ is growing and alternating.

<u>EX</u>: $a = -2 \longmapsto x[n] = (-2)^n$



Discrete-Time Periodic Signals

- The notion of "periodic" for discrete-time signals is the same as for continuous-time signals.
- If you can find a positive integer N such that, no matter what n is, $x[n+N] = x[n]$, then we say that $x[n]$ is periodic with period N.
  - It has to be true for <u>every</u> n, not just one or a few of them.
  - It means that if you go ahead by N, you get the same value for the signal.
  - So if you go ahead by N, and then do it again, you still have to get the same value.
  - So a signal that is periodic with period N is also periodic with period $2N$ and $3N$ and kN for any integer $k > 0$.
- Here is a signal that is periodic with period $N=3$ and also $N=6$, $N=9$, $N=12$, etc.:

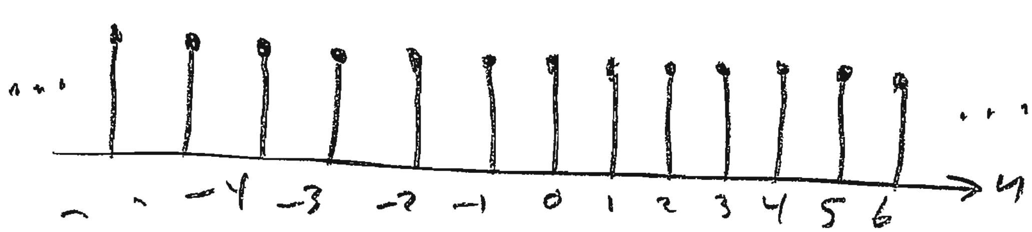


- Here is the mathematical definition of "periodic" for discrete-time signals:

<u>DEF</u>: if $\exists N \in \mathbb{N}$ such that $x[n+N] = x[n] \ \forall n \in \mathbb{Z}$, then $x[n]$ is <u>periodic with period N</u>.

- Every periodic discrete-time signal has a fundamental period. It is the smallest N that satisfies the definition above.
    - This is different from continuous time, where there are some periodic signals that don't have a fundamental period (see page 2.24).
    - But that problem <u>can't happen</u> in discrete time, because a discrete-time signal doesn't have values between integers on the n-axis... it's just not defined between integers.
    - So a discrete-time constant signal like $x[n] = 5$ <u>will</u> have a fundamental period of $N_0 = 1$:

$$x[n] = 5$$



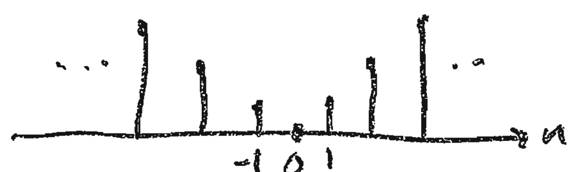
- There is a unique smallest positive integer N_0 such that $x[n+N_0] = x[n]$ $\forall n \in \mathbb{Z}$, and that value is $N_0 = 1$.

Symmetry for Discrete-Time Signals

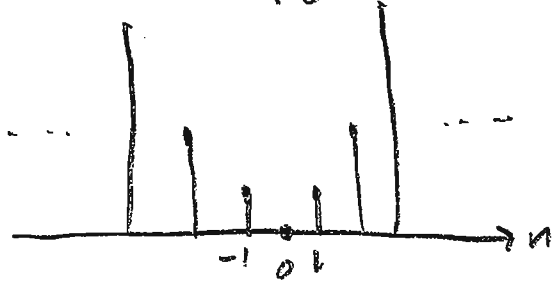
- Symmetry for the discrete-time signals works pretty much the same as for the continuous-time signals.
- If $x[n] = x[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called <u>even symmetric</u> (or just <u>even</u>).
    - It means that the graph of $x[n]$ on the negative n's can be obtained by taking the graph on the positive n's and reflecting it through the vertical axis.

<u>Examples</u>:

$x[n] = |n|$

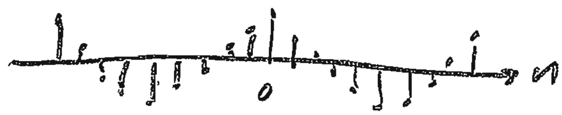


$x[n] = n^2$

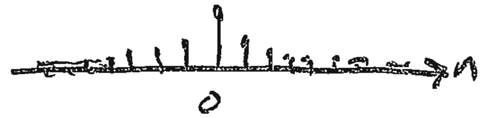


Even Examples...

$x[n] = \cos(\omega_0 n)$



$x[n] = \left(\frac{1}{2}\right)^{|n|}$

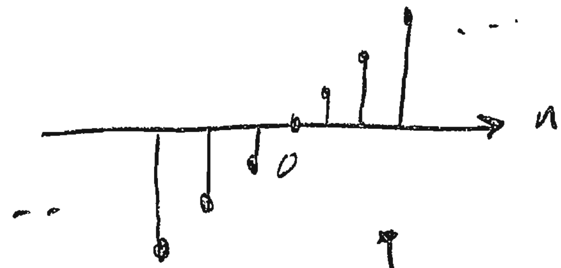


- If $x[n] = -x[-n]$ $\forall\, n \in \mathbb{Z}$, then $x[n]$ is called odd symmetric (or just odd).

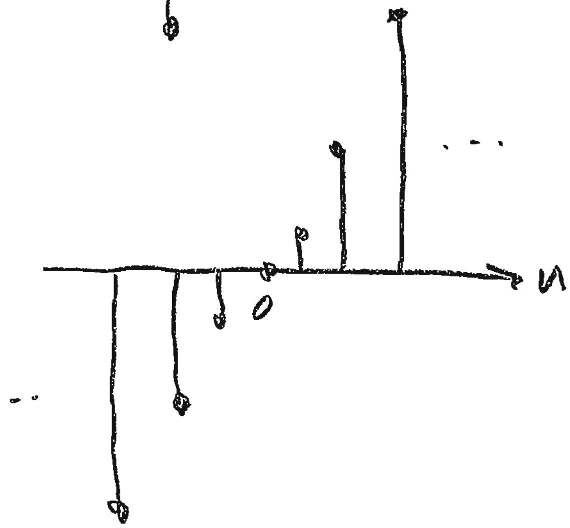
  - It means that the values of $x[n]$ on the negative n's are the negatives (additive inverses) of the values on the positive n's.
  - It means that the graph of $x[n]$ on the negative n's can be obtained by taking the graph on the positive n's, reflecting it through the vertical axis, and then reflecting that through the horizontal axis to "flip it upside down."

Examples of odd signals:

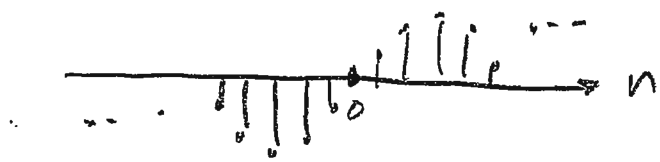
$x[n] = n$



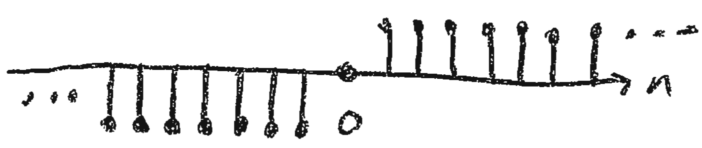
$x[n] = n^3$



$x[n] = \sin(\omega_0 n)$



$x[n] = u[n] - u[-n]$



- If $x[n]$ is odd, then $x[n] = -x[-n]$.
  - plugging in $n=0$, we get

$$x[0] = -x[0]$$

$\Rightarrow$ This means that $x[0] = 0$ for any odd discrete-time signal.

- If $x[n] = x^*[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called <u>conjugate</u> <u>symmetric</u>.
  - This means that the real part of $x[n]$ has to be even
  - And the imaginary part of $x[n]$ has to be odd.
  - For real signals, conjugate symmetric is the same as even, because a real signal has no imaginary part.

- If $x[n] = -x^*[-n]$ $\forall n \in \mathbb{Z}$, then $x[n]$ is called conjugate antisymmetric.

  - It means that the real part of $x[n]$ has to be odd.

  - And the imaginary part of $x[n]$ has to be even.

  - For real signals, conjugate antisymmetric is the same as odd... because a real signal has no imaginary part.

Shifting and Scaling for Discrete-Time Signals

- Time shifting for discrete-time signals works the same as for continuous-time signals, except that the shift amount must be an integer.
- So in general, we will know the graph of a signal $x[n]$ and we will need to be able to write down the graph of $x[n-n_0]$, where $n_0 \in \mathbb{Z}$.
- As with the continuous-time signals, this will shift the graph of $x[n]$ to the right by n_0.

- As with the continuous-time signals, we will sometimes need to do this when we <u>don't know</u> if n_0 (the shift amount) is positive or negative.

- So you should always think of the shifted signal as $x[n-n_0]$ and you should always think of it as a shift right by n_0, even if n_0 is negative.

$\rightarrow x[n-2] \; : \quad n_0 = 2 \mapsto$ shift right by 2.

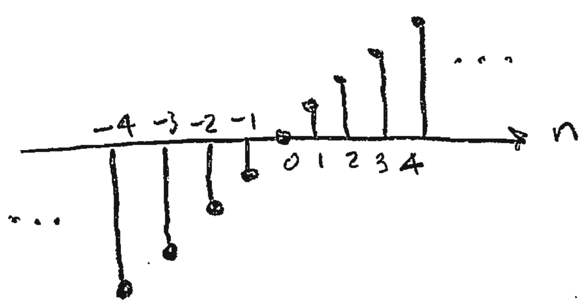
$\rightarrow x[n+2] = x[n - -2] \; : \quad n_0 = -2$

$\mapsto$ shift right by $\underline{\underline{-2}}$

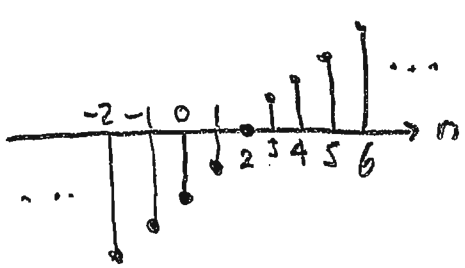
(which is the same as a shift <u>left</u> by +2)

<u>Examples</u>

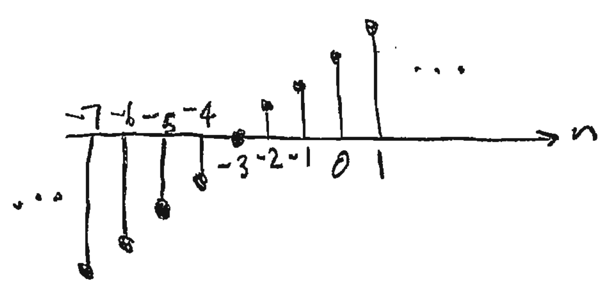
$x[n] = n$



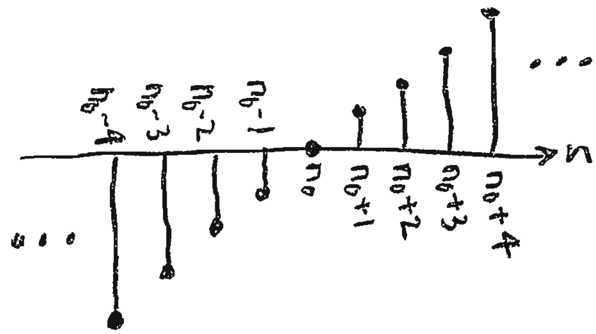
$x[n-2] = n-2$



$x[n+3] = x[n - -3] = n+3$



$$x[n-n_0] = n - n_0$$



- Note that this graph is good for <u>any</u> integer shift amount n_0 ... positive <u>or</u> negative.

- Recall from page 2.14 that time scaling for the continuous-time signals had the form $x(at)$ where $a \in \mathbb{R}$.
  - $\rightarrow$ $x(2t)$: squish by 2
  - $\rightarrow$ $x(\frac{1}{2}t)$: stretch by 2
  - $\rightarrow$ $x(-2t)$: squish by 2 plus a flip
  - $\rightarrow$ $x(-\frac{1}{2}t)$: stretch by 2 plus a flip

- Time scaling for the discrete-time signals is <u>much</u> <u>more</u> <u>restricted</u>.

- For ECE 2713, the only time scale that will be allowed is $a = -1$.
  - So we will need to be able to take the graph of $x[n]$ and write down the graph of $x[-n]$.

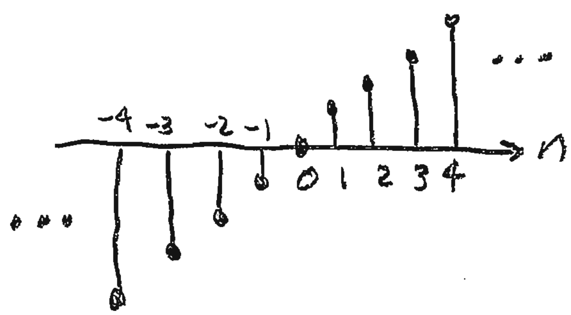
→ Other scale factors like $x[2n]$, $x[\frac{1}{2}n]$, $x[-2n]$ and $x[-\frac{1}{2}n]$ will <u>not</u> <u>be</u> <u>allowed</u>.

→ The problem with a scale factor like $x[\frac{1}{2}n]$ is: when $n=1$, we would need $x[\frac{1}{2}]$, which is <u>undefined</u> ... because the signal $x[n]$ only takes values at the <u>integers</u>.

→ The problem with a scale factor like $x[2n]$ is that half the signal values would be thrown away.

→ To understand this, consider the discrete-time signal

$$x[n] = n$$



→ If we try to make $y[n] = x[2n]$, we get

$\vdots$

$y[-1] = x[-2] = -2$

$y[0] = x[0] = 0$

$y[1] = x[2] = 2$

$\vdots$

But the values $x[1]$ and $x[-1]$ get thrown away... they never get used in $y[n]$.

NOTE: It IS possible to define operations like $x[2n]$ and $x[\frac{1}{2}n]$ for discrete-time signals. But this is an advanced topic called "multi-rate signal processing."

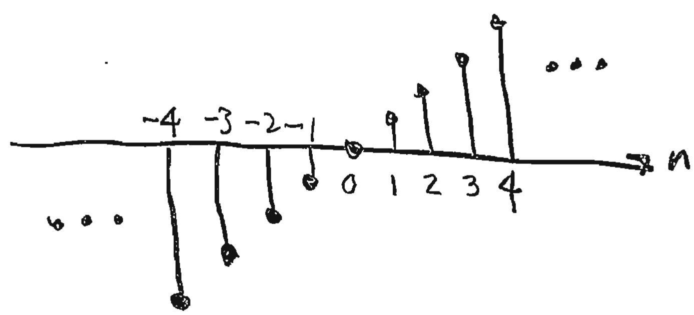
→ You may get introduced to it in ECE 3793.

→ It is taught in ECE 4213.

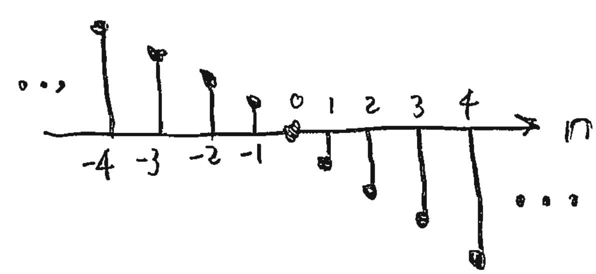
- But for ECE 2713, the only allowable scale factor for discrete-time signals will be $a = -1$, which simply reflects or "flips" the graph with respect to the vertical axis.

EX

$$x[n] = n$$



$$x[-n] = -n$$



Discrete-Time Signals: Shift plus Scale

- As we have just said, when we get a discrete-time signal with both a shift and a scale in ECE 2713,
  → The shift n_0 will always be integer.
  → The scale will always be $a = -1$.

- How this will come up: we will know the graph of $x[n]$ and we will need to write down the graph of $x[-n-n_0]$.

☆ The rule from page 2.21 still applies: do the shift first, then the scale.
  → The scale will always be a "flip" or reflection through the vertical axis.

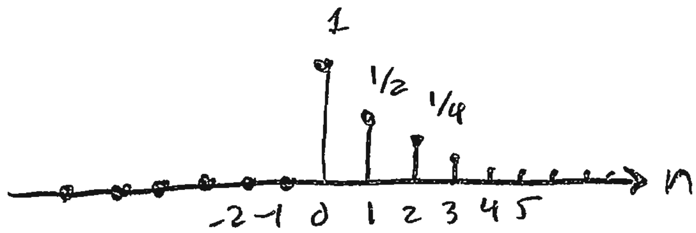
- So you will know or will be given the graph of $x[n]$.
  - To make the graph of $x[-n-n_0]$,
    1. Shift the graph of $x[n]$ right by n_0
    2. Flip the resulting graph around the vertical axis.

- You should always think of the new signal as $x[-n-n_0]$, even when n_0 is negative.

- If you are given the graph of $x[n]$ and asked to make the graph of $x[-n+3]$, think of it as $x[-n - -3]$...
  Scale: $\underline{a = -1}$
  Shift: $n_0 = -3$

Examples :

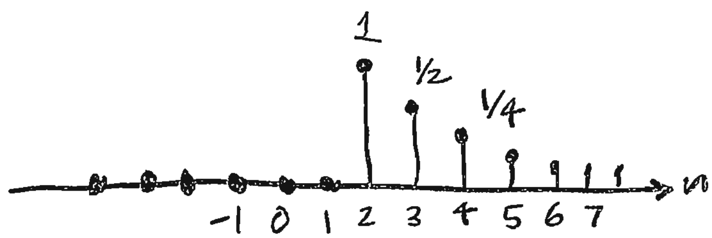
$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$



EX 1 : $x[-n-2] = \left(\tfrac{1}{2}\right)^{-n-2} u[-n-2]$

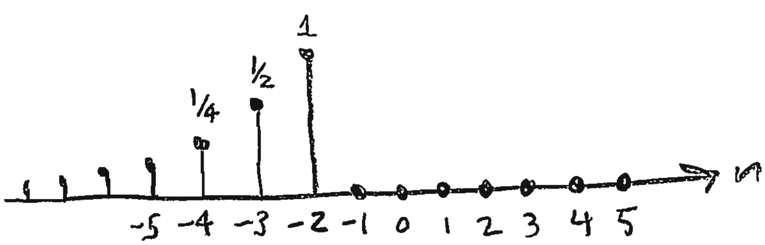
scale: $a = -1$ shift: $n_0 = 2$

① Shift the graph of $x[n]$ right by $n_0 = 2$



② Flip this graph left-to-right (scale):

$x[-n-2]$

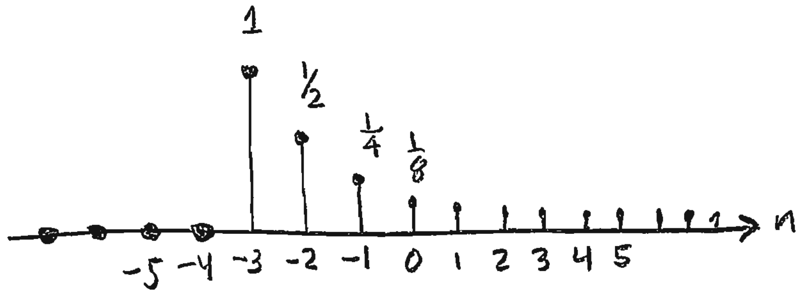


EX 2: $x[-n+3] = x[-n - -3]$

$$= \left(\tfrac{1}{2}\right)^{-n+3} u[-n+3]$$

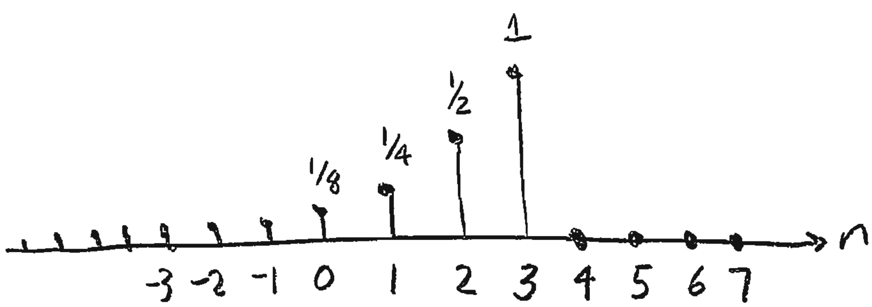
Scale: $a = -1$ Shift: $n_0 = -3$

(1) Shift the graph of $x[n]$ right by $n_0 = -3$. This means that whatever used to happen at $n=0$ now happens at $n=-3$



(2) Flip this graph left-to-right (scale):

$x[-n+3]$

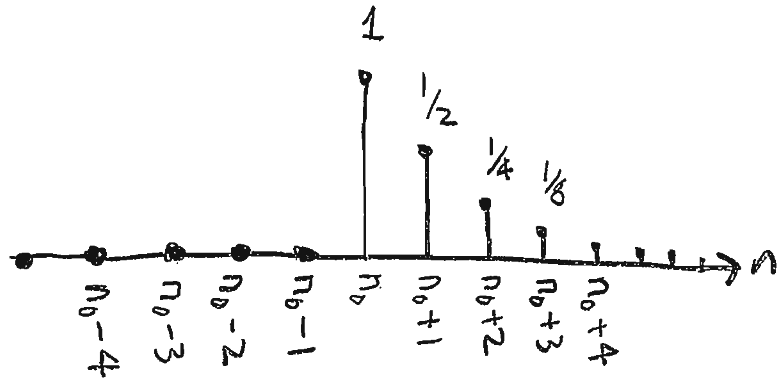


EX 3: $x[-n-n_0] = \left(\frac{1}{2}\right)^{-n-n_0} u[-n-n_0]$

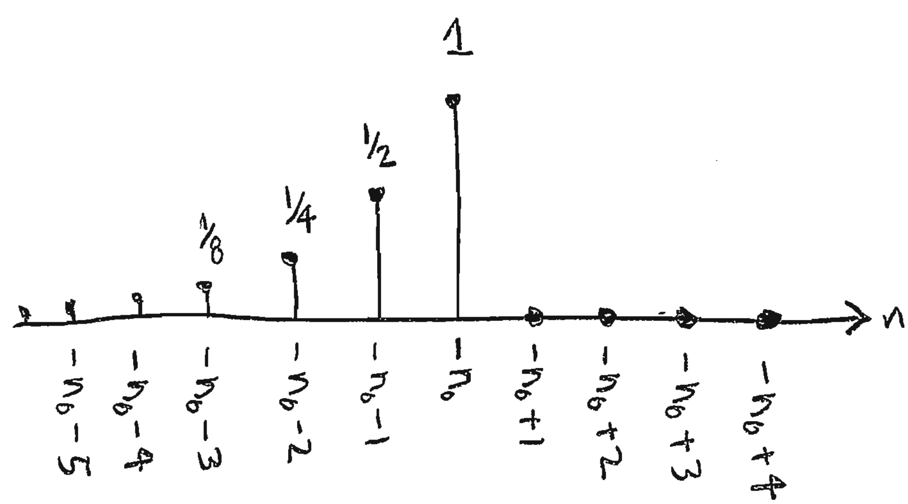
Scale: $a = -1$ Shift: n_0

$\Rightarrow$ But here, we don't know what n_0 is. It could be positive or negative (or even zero).

① Shift the graph of $x[n]$ right by n_0. whatever used to happen at $n=0$ now happens at $n=n_0$.



② Flip this graph left-to-right (scale):

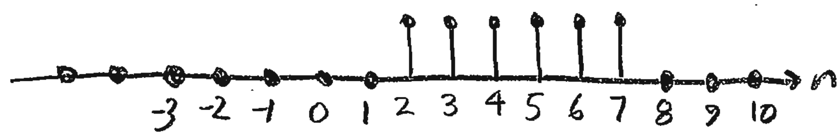


$\rightarrow$ This graph is good for <u>any</u> integer shift amount n_0.

Discrete-Time Boxcar Signals

- Here is an example of a discrete-time boxcar:

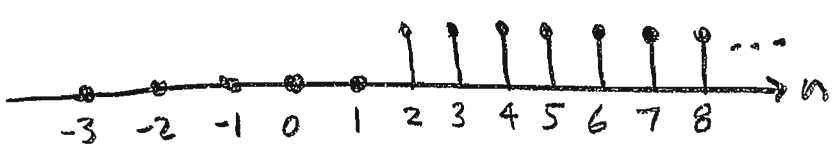
$$x[n] = \begin{cases} 1, & 2 \leq n \leq 7 \\ 0, & \text{otherwise} \end{cases}$$



- A discrete-time boxcar can be written as the difference of two unit step functions.

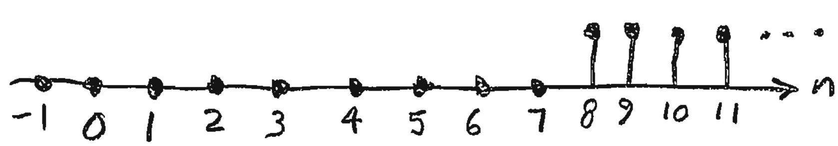
- For the example above, we need one step function to "turn it on" at $n=2$:

$$u[n-2]$$



- Then we need to subtract off a second unit step function to shut it off <u>starting</u> <u>at</u> <u>$n=8$</u>:

$$u[n-8]$$



- So we see that

$$x[n] = \begin{cases} 1, & 2 \le n \le 7 \\ 0, & \text{otherwise} \end{cases} = u[n-2] - u[n-8]$$

$=$ 

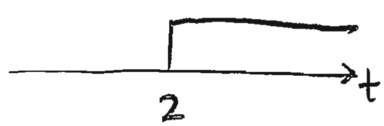
→ The first step function turns it on starting where we want the boxcar to *start*. ($n=2$ in this case)

→ The second step function shuts it off starting **one past** the last place where the boxcar is turned on. ($n=8$ in this case).

⇒ This is *different* from how it works in continuous time. For a continuous-time boxcar, the first step function turns it on where the boxcar starts and the second step function turns it off *where the boxcar ends*.

- *EX*: $$x(t) = \begin{cases} 1, & 2 \le t \le 7 \\ 0, & \text{otherwise} \end{cases} = u(t-2) - u(t-\underline{7})$$

$u(t-2)$ $\qquad$ $u(t-7)$



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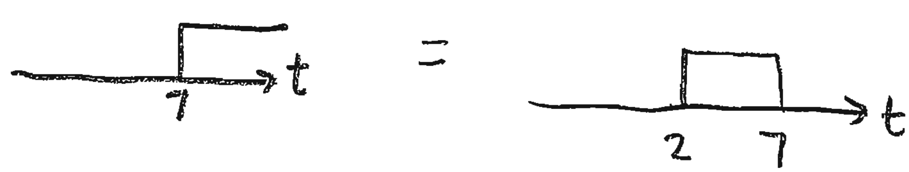


- Here's another example:

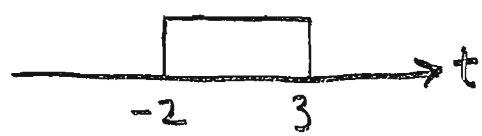
  - For a discrete boxcar that goes from $n = -2$ to $n=3$, we need to turn it on at $n=-2$ and turn it off starting at $n=4$. So

$$x[n] = \begin{cases} 1, & -2 \le n \le 3 \\ 0, & \text{otherwise} \end{cases} =$$

$$= u[n+2] - u[n-4]$$

  - But for a continuous-time boxcar that goes from $t=-2$ to $t=3$, we need to turn it on at $t=-2$ and turn it off at $t=3$. So

$$x(t) = \begin{cases} 1, & -2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases} =$$



$$= u(t+2) - u(t-3)$$

Discrete-Time Sinusoidal Signals

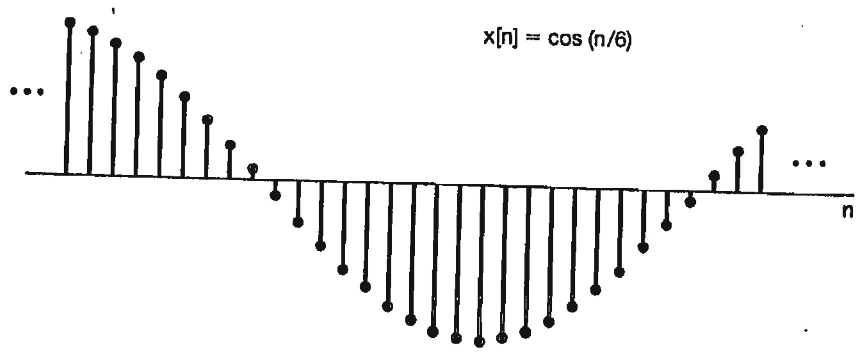
- The basic discrete-time sinusoidal signals are

$$x[n] = \cos(\omega_0 n)$$

$$x[n] = \sin(\omega_0 n) \qquad \omega_0 \in \mathbb{R}$$

$$x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

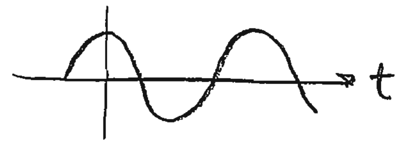
- often, but not always, the graph of the discrete-time sinusoid will look like a discrete version of the graph of the continuous-time sinusoid.



- But there are <u>important</u> differences between the discrete-time and continuous-time sinusoids.

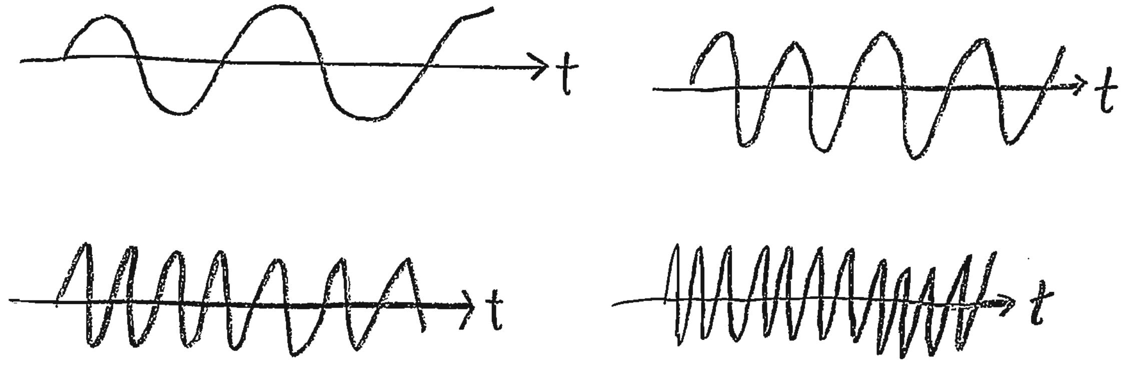
- To understand this, let's start by focussing our attention on just the discrete cosine signal $\cos(\omega_0 n)$.

- The $\cos(\cdot)$ function itself is like a continuous-time signal generator in a sense... if we evaluate $\cos(\omega_0 t)$ at all the t's, then we get a continuous-time cosine signal



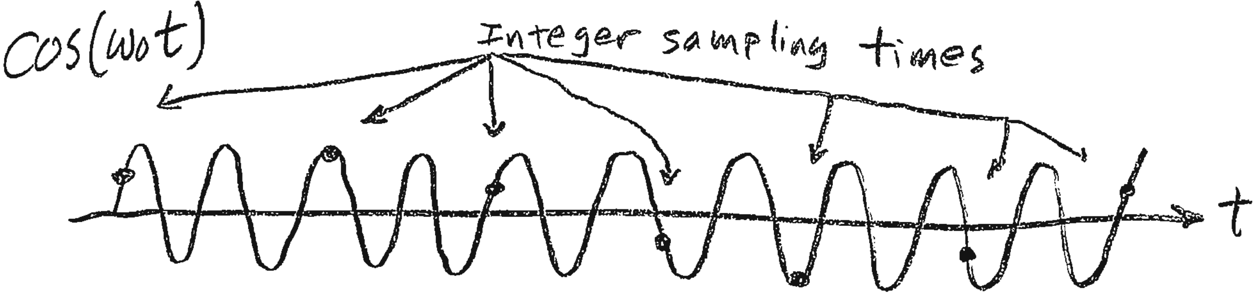
- If we only evaluate it at the integer times, then we get $\cos(\omega_0 n)$... which is a discrete-time signal containing <u>samples</u> of the continuous waveform... evaluated at just the integer times

- Now imagine that we have a knob that we can turn to control ω_0.
  - We start with ω_0 small... like $\omega_0 = \frac{2\pi}{1{,}000}$.
    → The graph of the discrete-time signal $\cos(\omega_0 n)$ looks a whole lot like the graph of the continuous-time signal $\cos(\omega_0 t)$.
  - Now suppose that we use the knob to start turning up ω_0.
    → As ω_0 gets bigger, the graph of $\cos(\omega_0 t)$ goes <u>faster</u> and <u>faster</u>

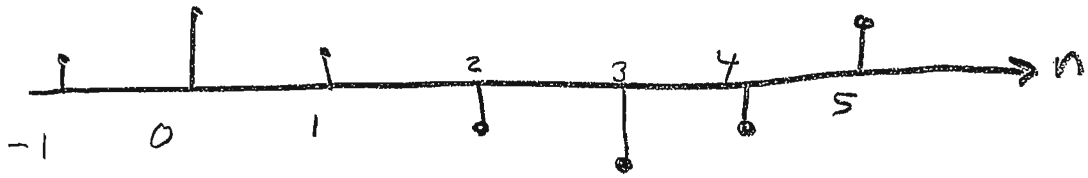


faster and faster...

- The graph of the discrete-time signal $\cos(\omega_0 n)$ also goes faster and faster... but only up to a point.
  - → As we continue to increase ω_0, the discrete-time signal $\cos(\omega_0 n)$ goes faster and faster up to a point, but then it starts to go slower.
- What??
- How can this happen?
  - → $\cos(\omega_0 n)$ samples the continuous waveform at the integers only.
  - → As ω_0 gets bigger and bigger, we reach a point where the continuous curve may go around one whole time or even more than once between integer times.
- → when this happens, the graph of $\cos(\omega_0 n)$ no longer looks like the graph of $\cos(\omega_0 t)$.



$\cos(\omega_0 n)$ looks like it goes __slow__.



☆☆ This is called __aliasing__. We will talk about it in more detail later and describe it mathematically

- For now, we just want to get the big idea.

- When ω_0 gets too big, aliasing makes the graph of $\cos(\omega_0 n)$ look like a cosine with a different frequency that is __smaller__.
  - The same thing happens with $\sin(\omega_0 n)$ and $e^{j\omega_0 n}$.

- In fact, if you add any integer multiple of 2π to the frequency, it does not change the graph of $\cos(\omega_0 n)$. It also does not change the graph of $\sin(\omega_0 n)$ and it does not change the graph of $e^{j\omega_0 n}$.

- To see this, let $k \in \mathbb{Z}$ be an integer and let $\omega_1 = \omega_0 + 2\pi k$.

- Then, for every $n \in \mathbb{Z}$, we get

$$\cos(\omega_1 n) = \cos\left([\omega_0 + 2\pi k] n\right)$$
$$= \cos\left(\omega_0 n + 2\pi k n\right) = \cos(\omega_0 n).$$

- So $\cos(\omega_1 n)$ and $\cos(\omega_0 n)$ have the same graph. They are just two different ways of writing the same signal.

- The same thing is true for $\sin(\omega_0 n)$ and $e^{j\omega_0 n}$...
  if $k \in \mathbb{Z}$ and $\omega_1 = \omega_0 + 2\pi k$, then for every n,

$$\sin(\omega_1 n) = \sin(\omega_0 n + 2\pi k n) = \sin(\omega_0 n)$$

$$e^{j\omega_1 n} = \cos(\omega_1 n) + j\sin(\omega_1 n) = \cos(\omega_0 n) + j\sin(\omega_0 n)$$
$$= e^{j\omega_0 n}$$

- What this means is that all possible graphs of the signal $\cos(\omega_0 n)$ can be made with frequencies ω_0 that are between 0 and π... such that $0 \le \omega_0 \le \pi$.
  - If $-\pi \le \omega_1 < 0$, then $\cos(\omega_1 n)$ has the same graph as $\cos(\omega_0 n)$ where $\omega_0 = -\omega_1$... So $0 \le \omega_0 \le \pi$. This is because cosine is even.
  - If $\omega_1 > \pi$ or $\omega_1 < -\pi$, then the graph of $\cos(\omega_1 n)$ is the same as the graph of $\cos(|\omega_0| n)$ where $\omega_0 = \omega_1 \pm 2\pi k$ for some $k \in \mathbb{Z}$ and $-\pi < \omega_0 \le \pi$. So $0 \le |\omega_0| \le \pi$.
- Since sine is odd, the graph of $\sin(-\omega_0 n)$ is not the same as $\sin(\omega_0 n)$.
  - The graph of $\sin(-\omega_0 n) = -\sin(\omega_0 n)$ is the negative of the graph of $\sin(\omega_0 n)$.
  - Because of this, to generate all possible graphs of $\sin(\omega_0 n)$, we need frequencies ω_0 going from $-\pi$ to π... such that $-\pi \le \omega_0 < \pi$.

- Similarly, because $e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$, we need frequencies ω_0 going from $-\pi$ to π in order to make all the possible graphs of the signal $e^{j\omega_0 n}$.

  - In other words, we only need "2π worth" of frequencies to make all the discrete-time sinusoidal signals.

    - If you have $\cos(\omega_1 n)$, $\sin(\omega_1 n)$, or $e^{j\omega_1 n}$,
      - and ω_1 is outside the range $-\pi$ to π,
      - Then you can add or subtract an integer multiple of 2π to get a new frequency $\omega_0 = \omega_1 \pm 2\pi k$,
        - such that $-\pi \leq \omega_0 \leq \pi$,

and

$$\cos(\omega_0 n) = \cos(\omega_1 n)$$
$$\sin(\omega_0 n) = \sin(\omega_1 n)$$
$$e^{j\omega_0 n} = e^{j\omega_1 n}$$

$$\forall\, n \in \mathbb{Z}.$$

- So, for example,

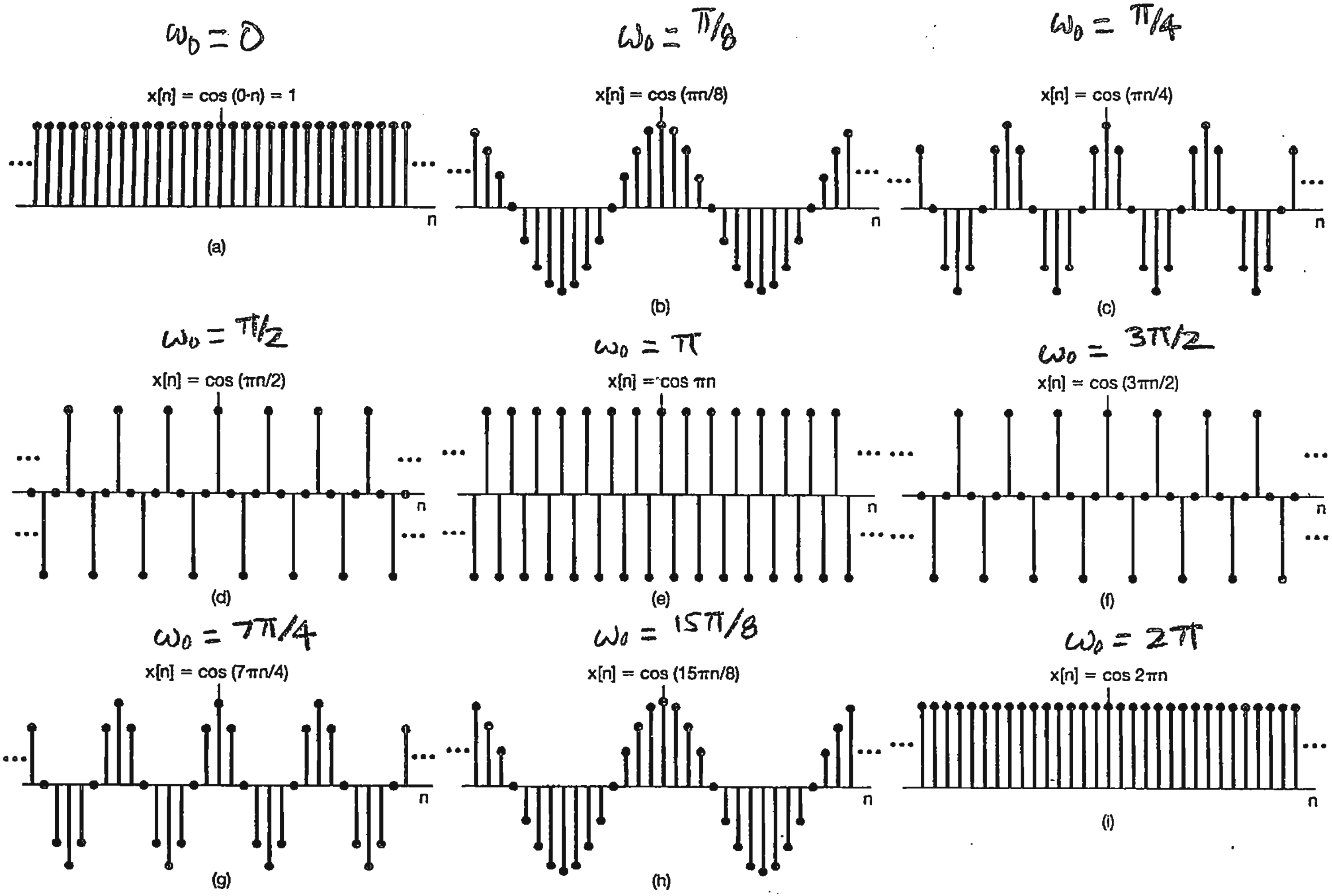
if $\omega_1 = 2\pi$, then the graphs are the same as with $\omega_0 = \omega_1 - 2\pi = \underline{0}$.

if $\omega_1 = \frac{15\pi}{8}$, then the graphs are the same as with $\omega_0 = \frac{15\pi}{8} - 2\pi = -\frac{\pi}{8}$.

$\rightarrow$ Since cosine is even, the graph of $\cos\left(\frac{15\pi}{8} n\right)$ is also the same as the graph of $\cos\left(\frac{\pi}{8} n\right)$.

etc...

$\rightarrow$ This is illustrated by the graphs of $\cos(\omega_0 n)$ shown on the next page.



- Here is another important difference between the discrete-time and continuous-time sinusoids:
  - → $\cos(\omega_0 t)$, $\sin(\omega_0 t)$, and $e^{j\omega_0 t}$ are periodic for any real choice of ω_0.
  - → $\cos(\omega_0 n)$, $\sin(\omega_0 n)$, and $e^{j\omega_0 n}$ are periodic if and <u>only</u> if $\frac{\omega_0}{2\pi} \in \mathbb{Q}$.
    - → In other words, if and <u>only</u> if $\frac{\omega_0}{2\pi}$ is a ratio of two integers.

- Notice that this means ω_0 must contain a factor of π in order for the discrete-time sinusoids to be periodic.

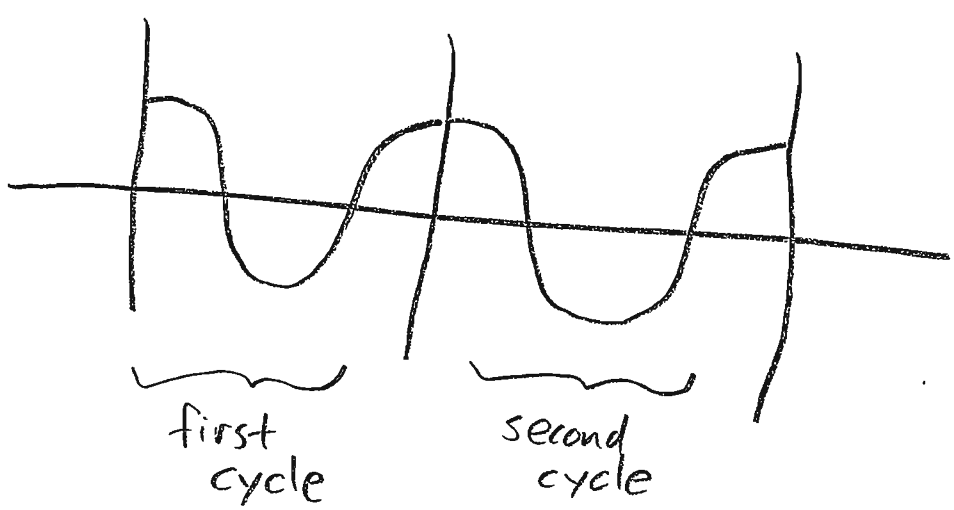
⇒ So let's try to understand how this seemingly counterintuitive fact can be true.

- Recall that the function $\cos(\cdot)$ is like a continuous-time signal generator. You can plug in $\omega_0 t$ and evaluate it at all the t's to get a continuous-time signal.

- The discrete-time signal $\cos(\omega_0 n)$ only evaluates the cosine at the integer times.

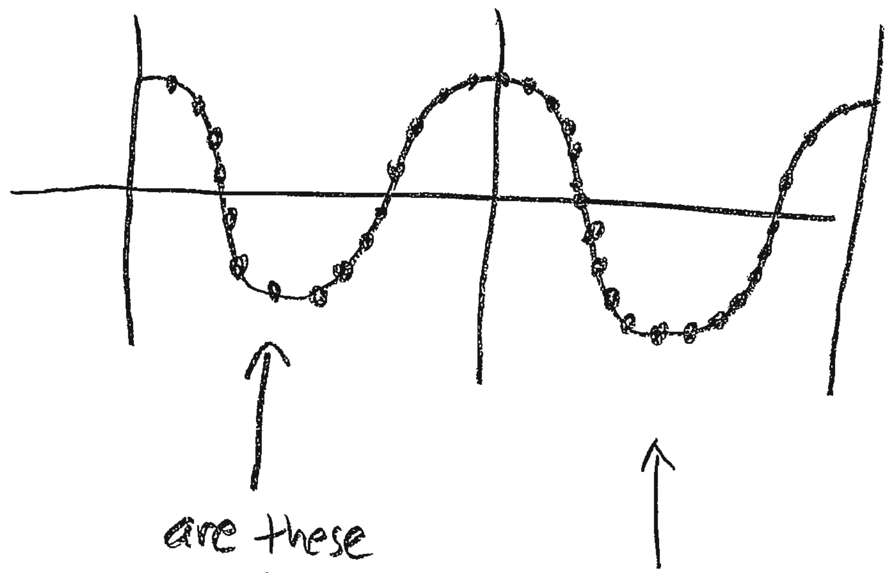
  → We can think of this as sampling the continuous waveform.

- Now imagine two cycles of $\cos(\omega_0 t)$:



→ $\cos(\omega_0 n)$ takes samples of this curve.

- Do you expect the samples to fall at the same places in the second cycle as they did in the first cycle?



are these samples in the same places ...

as these samples??

- In general they are not!!
- They may not repeat!!

⇒ This is why a discrete-time sinusoidal signal might not be periodic.

- So here's how you tell if a discrete-time sinusoid is periodic:

  - Take the frequency ω_0 and divide it by 2π.
  - If you get an irrational number... i.e., if you <u>don't</u> get a ratio of two integers...
    - → Then the discrete-time sinusoid is <u>not</u> periodic.
    - → The samples will <u>never</u> fall at the same places in any two cycles of the continuous waveform.

  - If you get a <u>rational</u> number, a ratio of two integers, then the discrete-time sinusoid <u>is</u> <u>periodic</u>.
    - remove any common factors between the numerator and denominator so that the fraction $\frac{\omega_0}{2\pi}$ is in <u>reduced</u> <u>form</u>.
    - Write it as

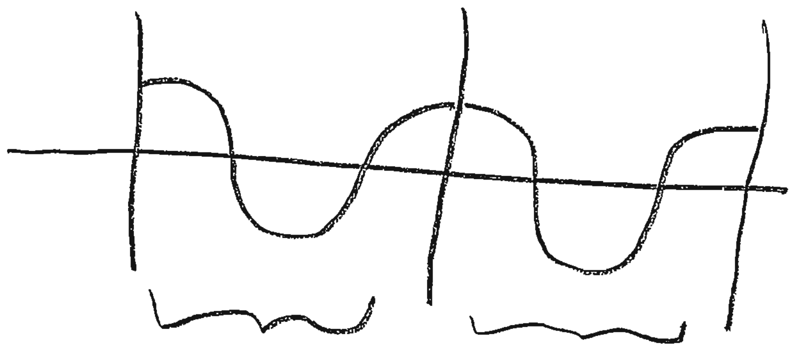
$$\frac{\omega_0}{2\pi} = \frac{m}{N} \quad \text{(reduced form)}$$

$$m, N \in \mathbb{Z}.$$

☆☆ - The discrete-time sinusoidal signal has fundamental period N.

- The graph of the <u>continuous</u> waveform goes through m cycles to make <u>one</u> <u>period</u> of the discrete waveform.

→ For example, if $m=2$, then



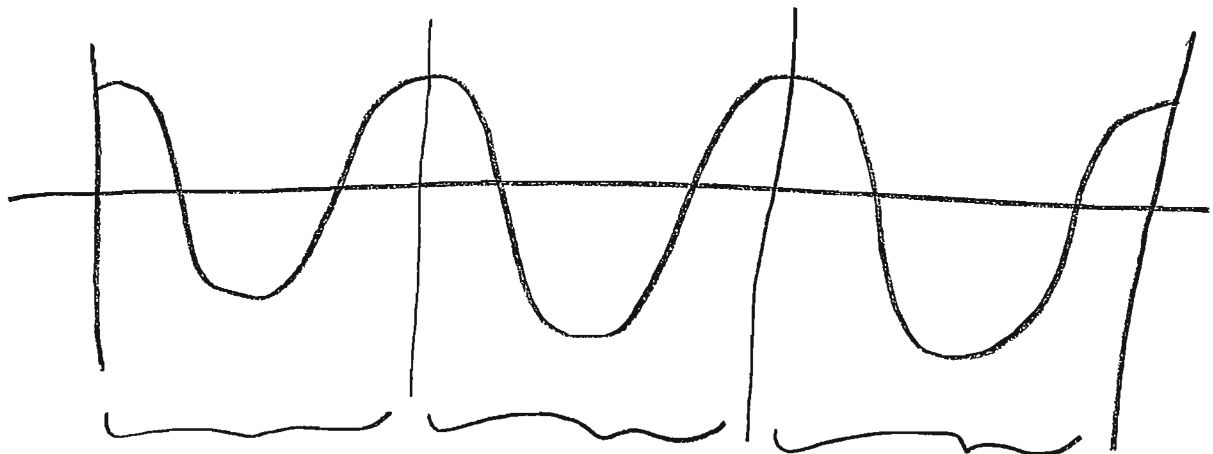
you get one batch of samples here

you get a <u>different</u> batch of samples here

But then this <u>block</u> repeats in the discrete-time signal

$\underline{\underline{m=2}}$.

→ If m=3, then



you get one batch of samples here

you get a second different batch of samples here

and you get a third different batch of samples here.

But then this whole <u>block</u> repeats in the discrete-time sinusoidal signal.

This block, which includes m=3 cycles of the continuous waveform, <u>is</u> the fundamental period N of the discrete-time signal.

EX: $x[n] = \cos\left(\frac{4\pi}{9} n\right)$

$$\omega_0 = \frac{4\pi}{9}$$

$$\frac{\omega_0}{2\pi} = \frac{4\pi/9}{2\pi} = \frac{2}{9} = \frac{m}{N}$$

- Fundamental period $= N = 9$
- Each period of $\cos(\omega_0 n)$ goes through $m=2$ cycles of the continuous waveform.

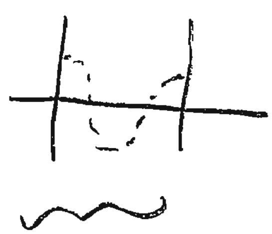
EX: $x[n] = \cos\left(\frac{\pi}{6} n\right)$

one period

$$\omega_0 = \pi/6$$

$$\frac{\omega_0}{2\pi} = \frac{\pi/6}{2\pi} = \frac{1}{12} = \frac{m}{N}$$

- Fundamental period $= N = 12$
- Each period of $\cos(\omega_0 n)$ looks like 1 cycle of the continuous cosine waveform



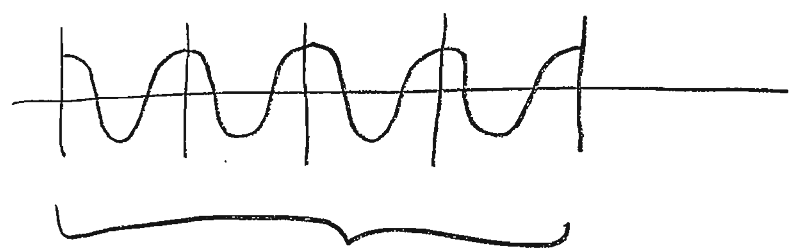
one period

EX: $x[n] = \cos\left(\frac{8\pi}{31} n\right)$

$$\omega_0 = \frac{8\pi}{31}$$

$$\frac{\omega_0}{2\pi} = \frac{8\pi/31}{2\pi} = \frac{4}{31} = \frac{m}{N}$$

- Fundamental period $= N = 31$
- Each period of $\cos(\omega_0 n)$ looks like 4 cycles of the continuous waveform



one period of the discrete signal

EX: $x[n] = \cos\left(\frac{1}{6} n\right)$

$$\omega_0 = \frac{1}{6}$$

$$\frac{\omega_0}{2\pi} = \frac{1/6}{2\pi} = \frac{1}{12\pi} \notin \mathbb{Q} \;!!$$

$\rightarrow$ $\frac{\omega_0}{2\pi}$ is <u>not</u> a rational number

$\rightarrow$ It is <u>not</u> a ratio of two integers.

$\Longrightarrow$ $x[n]$ is <u>NOT</u> periodic.

TIME DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS

- A vector in $\mathbb{R}^2$ or $\mathbb{C}^2$ is an ordered pair of numbers...
  e.g. $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} 2e^{j\pi/4} \\ 3-j \end{bmatrix}$

- A vector in $\mathbb{R}^3$ or $\mathbb{C}^3$ is an ordered triple. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- A vector in $\mathbb{R}^{1000}$ or $\mathbb{C}^{1000}$ is an ordered 1,000-tuple of numbers:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{1000} \end{bmatrix}.$$

- For bigger vectors, we usually write them on paper using the transpose "T" to save space... in $\mathbb{R}^{1000}$ or $\mathbb{C}^{1000}$,

$$\vec{x} = [x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{1000}]^T.$$

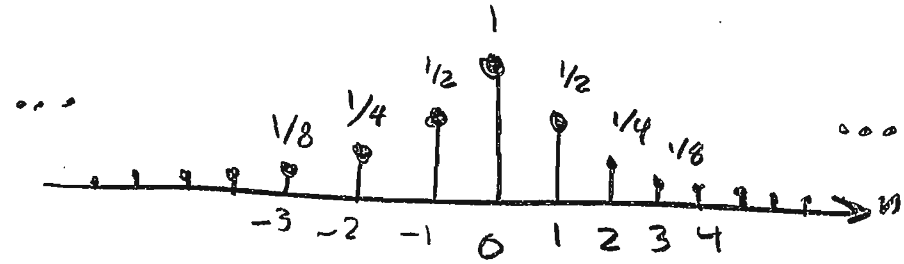
- Our discrete-time signals like $x[n] = \left(\frac{1}{2}\right)^n u[n]$ or $x[n] = \cos(\omega_0 n)$ (or any of the others) can <u>also</u> be thought of as <u>vectors</u>.

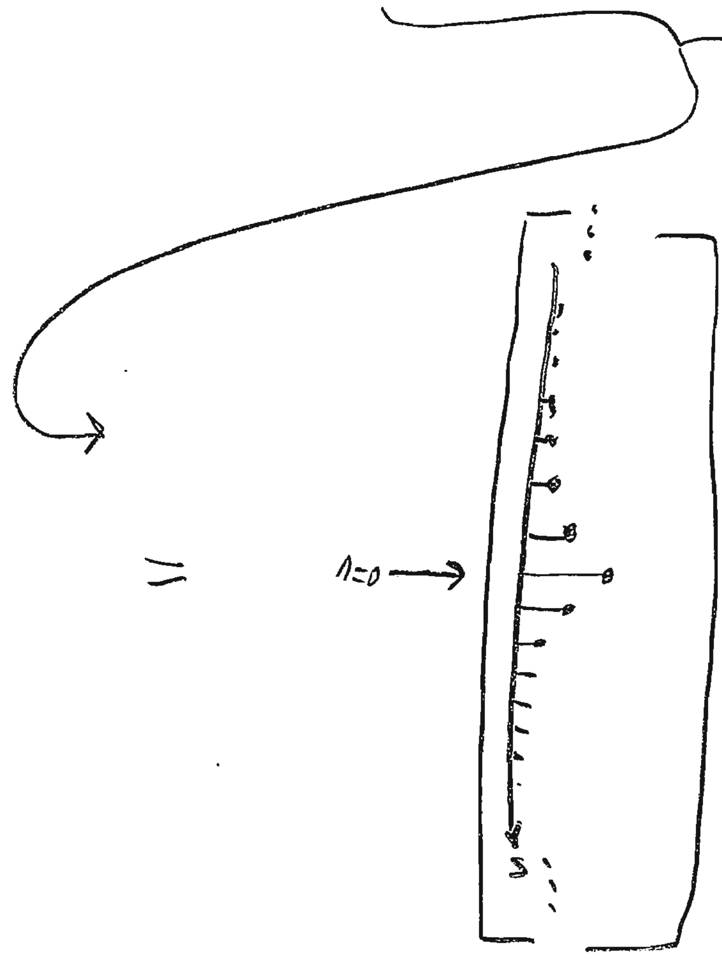
  → Any $x[n]$ is (modeled by) a discrete-time function that is a number at every $n \in \mathbb{Z}$.

  → So the values of the signal... the graph... can be thought of an ordered ∞-tuple of numbers.

— To see this, it may help if you think of "loading up" the values of $x[n]$ into a giant pair of square brackets.

→ For example, suppose $x[n] = \left(\frac{1}{2}\right)^{|n|}$





$$= \begin{bmatrix} \vdots \\ \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \vdots \end{bmatrix}$$

- When we think of two discrete-time signals $x[n]$ and $y[n]$ as <u>vectors</u>, it becomes possible to take their dot product (just like you can for any two vectors with the same dimension).
  - → How does this work?
    - → It works just like <u>always</u> ... just like it does in $\mathbb{R}^2$ or $\mathbb{C}^2$ or $\mathbb{R}^3$ or $\mathbb{C}^3$...
    - → You line up the vectors beside each other
    - → You conjugate the entries of the second vector.
    - → You multiply the entries that are beside each other.
    - → You add it up down the vector to get a number

$\mathbb{C}^2$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^*$$

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1^* + x_2 y_2^*$$

- For two discrete-time signals $x[n]$ and $y[n]$, it works exactly the same way:

$$\begin{bmatrix} \vdots \\ x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \vdots \\ y[-3] \\ y[-2] \\ y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ \vdots \end{bmatrix}^*$$

$$\begin{aligned}\langle x[n], y[n] \rangle = \cdots & \, x[-3]y^*[-3] + x[-2]y^*[-2] \\ & + x[-1]y^*[-1] + x[0]y^*[0] \\ & + x[1]y^*[1] + x[2]y^*[2] \\ & + x[3]y^*[3] + \cdots \end{aligned}$$

(a number)

$\Rightarrow$ A capital Σ do loop saves a <u>lot</u> of writing!

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

- It is very important for you to understand the idea of this..

  - the concept that our discrete-time signals $x[n]$ can be thought of as vectors in $\mathbb{R}^\infty$ or $\mathbb{C}^\infty$,

  - and that the dot product is no different than in $\mathbb{C}^2$ or $\mathbb{C}^3$... there's just more terms in the sum... but the idea is no different.

- It will also sometimes happen that you need to actually work out the dot product between two discrete-time signals.

  - The main tools for doing this are:

    → The exponent rules on page 1.17

    → The sum formulas on page 2 of the course formula sheet.

    → At least for ECE 2713, the first formula will be the one that you use most often:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \alpha \neq 1$$

- Notice that the "k" in this formula is just a loop counter.
  - The formula still works just fine if the loop counter is something else like n or m or anything else... it just has to stand for integers.

- You will be allowed to use the formula sheet for the tests and exam in ECE 2713.
  - But this formula comes up so often that I find it useful to have it memorized.
  - Here is a trick that helps me to remember it:
    - For any real or complex number α, the sequence of numbers α^k for $k = \ldots -2, -1, 0, 1, 2 \ldots$ is called a <u>geometric</u> <u>series</u>.
    - The number α is called the <u>common</u> <u>ratio</u> or "radius" of the series.
  - Now let's take another look at the formula:

The top power

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \alpha \neq 1$$

The radius

The bottom power

- The formula can <u>only</u> be applied if $\alpha \neq 1$!!

- Here is a phrase that I have memorized to help me remember the formula:

  sum = [radius to the bottom power
  minus
  radius to the top power plus one]

  over one minus the radius.

- Now here is an example of using this formula to compute the dot product of the discrete-time signals

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$y[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

$\longrightarrow$ Notice that, for all $n<0$, $x[n]=0$ and $y[n]=0$.

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

$$= \sum_{n=0}^{\infty} x[n]\, y^*[n] \qquad \left(\text{because } \begin{matrix} x[n]=0 \\ y[n]=0 \end{matrix} \ \forall\, n<0\right)$$

$$= \sum_{n=0}^{\infty} x[n]\, y[n] \qquad (\text{because } y[n] \text{ is } \underline{\text{real}}, \text{ so } y^*[n] = y[n])$$

$\longrightarrow$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{3}\right)^n \qquad (\text{because } u[n] = 1 \ \forall n \geq 0)$$

$$= \lim_{N_2 \to \infty} \sum_{n=0}^{N_2} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{3}\right)^n$$

$$= \lim_{N_2 \to \infty} \sum_{n=0}^{N_2} \left(\tfrac{1}{2} \cdot \tfrac{1}{3}\right)^n = \lim_{N_2 \to \infty} \sum_{n=0}^{N_2} \left(\tfrac{1}{6}\right)^n \qquad \left(\begin{array}{c}\text{by the} \\ \text{exponent rules} \\ \text{on p. 1.17}\end{array}\right)$$

$$= \lim_{N_2 \to \infty} \frac{\left(\frac{1}{6}\right)^0 - \left(\frac{1}{6}\right)^{N_2+1}}{1 - 1/6} \qquad (\text{by the sum formula})$$

$$= \lim_{N_2 \to \infty} \frac{1 - \left(\frac{1}{6}\right)^{N_2+1}}{5/6} \qquad \Rightarrow \text{but } \lim_{N_2 \to \infty} \left(\tfrac{1}{6}\right)^{N_2+1} = 0$$

$$= \frac{1 - 0}{5/6} = \frac{1}{5/6} = \underline{\underline{\frac{6}{5}}}$$

– Now, in this example, we had $N_1 = 0$, $N_2 = \infty$, and $\alpha = \frac{1}{6}$ ... so $|\alpha| < 1$.

$\Rightarrow$ This means that we could alternatively have used the <u>second</u> sum formula on page 2 of the course formula sheet:

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

- This second formula would have given us exactly the same answer:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$y[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

$$= \sum_{n=0}^{\infty} x[n]\, y^*[n] \qquad \left(\begin{array}{l}\text{because } x[n]=0 \text{ and} \\ y[n]=0 \quad \forall\ n<0\end{array}\right)$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{3}\right)^n \qquad (\text{because } y[n] \text{ is real})$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{6}\right)^n \qquad (\text{by the exponent rules on p. 1.17})$$

$$= \frac{1}{1 - 1/6} \qquad \left(\begin{array}{l}\text{by the } 2^{nd} \text{ sum formula} \\ \text{on page 2 of the course} \\ \text{formula sheet}\end{array}\right)$$

$$= \frac{1}{5/6}$$

$$= \frac{6}{5}$$

- But you should realize that the <u>second</u> sum formula on page 2 of the formula sheet is really just a special case of the first formula.

→ because if $|a| < 1$, then $a \neq 1$.

→ So, for any complex number a such that $|a| < 1$, we can evaluate $\sum_{k=0}^{\infty} a^k$ using the <u>first</u> formula...

→ we have $N_1 = 0$; $N_2 = \infty$, and $a \neq 1$...

→ so, using the first formula, we get:

$$\sum_{k=0}^{\infty} a^k = \lim_{N_2 \to \infty} \sum_{k=0}^{N_2} a^k$$

$$= \lim_{N_2 \to \infty} \frac{a^0 - a^{N_2+1}}{1-a}$$

$$= \lim_{N_2 \to \infty} \frac{1 - a^{N_2+1}}{1-a}$$

$$= \frac{1-0}{1-a} = \frac{1}{1-a} \checkmark$$

⟹ Moral of the story: the second formula, that

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \text{ if } |a| < 1,$$

isn't really needed. It follows from the first formula.

→ You can always get the answer using just the first formula.

- Here's another example.

  - If we're going to think of discrete-time signals $x[n]$ as vectors, then we ought to be able to compute the length or <u>norm</u>.

  - Recall from page 1.93 that we defined the norm as

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

→ For vectors with real entries, this is the <u>same</u> as the Euclidean norm that you are used to:

→ to find the length (norm), you square the entries of the vector, add up the squares, and take the square root.

<u>EX</u>: $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\text{length} = \|\vec{x}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

→ But the definition above, $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, also works for vectors with complex entries.

$\longrightarrow$

- Now suppose we have $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

- Then $\langle x[n], x[n] \rangle = \sum_{n=-\infty}^{\infty} x[n] x^*[n]$

$$= \sum_{n=0}^{\infty} x[n] x^*[n] \quad \left(\text{because } x[n]=0 \ \forall n<0\right)$$

$$= \sum_{n=0}^{\infty} x[n] x[n] \quad \left(\text{because } x[n] \text{ is } \underline{\text{real}} \text{ so } x^*[n] = x[n]\right)$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n \quad \left(\text{by the rules of exponents from p. 1.17: } a^c b^c = (ab)^c\right)$$

$$= \lim_{N_2 \to \infty} \sum_{n=0}^{N_2} \left(\tfrac{1}{4}\right)^n$$

$$= \lim_{N_2 \to \infty} \frac{\left(\frac{1}{4}\right)^0 - \left(\frac{1}{4}\right)^{N_2+1}}{1 - 1/4} \quad \left(\text{by the first sum formula on p. 2 of course formula sheet}\right)$$

$$= \frac{1-0}{1-1/4}$$

$$= \frac{1}{3/4} = \frac{4}{3}$$

- So we get:

$$\text{length of } x[n] = \| x[n] \|$$

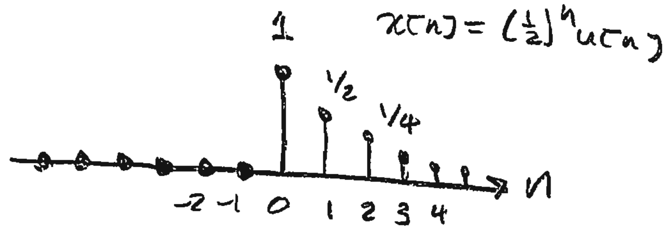
$$= \sqrt{\langle x[n], x[n] \rangle} = \sqrt{\tfrac{4}{3}} = \frac{2}{\sqrt{3}} \ ///$$

Natural Basis for Dirscrete-Time Signals

- We think of our discrete-time signals $x[n]$ as being described or modeled as a function like

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

- we also think of them as being described by a graph like

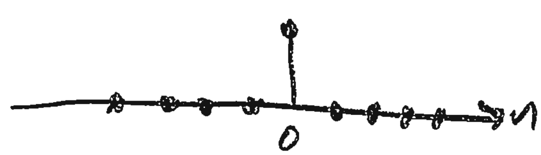


- And, at the same time, we also think of them as being vectors from a very large vector space:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] = \begin{bmatrix} \vdots \\ 0 \\ \underline{1} \\ 1/2 \\ 1/4 \\ 1/8 \\ \vdots \end{bmatrix}$$

- Now recall the discrete-time unit impulse $\delta[n]$ from p. 2.82:

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$

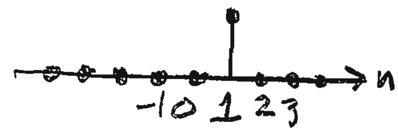


→ Turned on at $n=0$

→ zero everywhere else

- The shifted signal $\delta[n-1]$ is given by

$$\delta[n-1] = \begin{cases} 1, & n=1 \\ 0, & \text{otherwise} \end{cases}$$



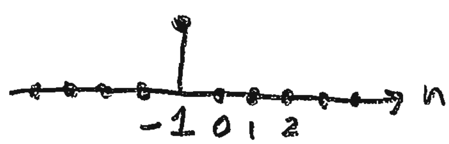
→ Turned on at $n=1$

→ zero everywhere else

- Such a shifted version is called a "translate" of $\delta[n]$ ... because "translation" is a synonym for "shifting"!

- The shifted signal $\delta[n+1] = \delta[n - -1]$ is given by

$$\delta[n+1] = \delta[n - -1] = \begin{cases} 1, & n=-1 \\ 0, & \text{otherwise} \end{cases}$$

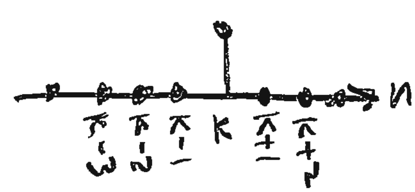


→ turned on at $n=-1$

→ zero everywhere else

- And more generally, for any integer $k \in \mathbb{Z}$, the shifted signal $\delta[n-k]$ is given by

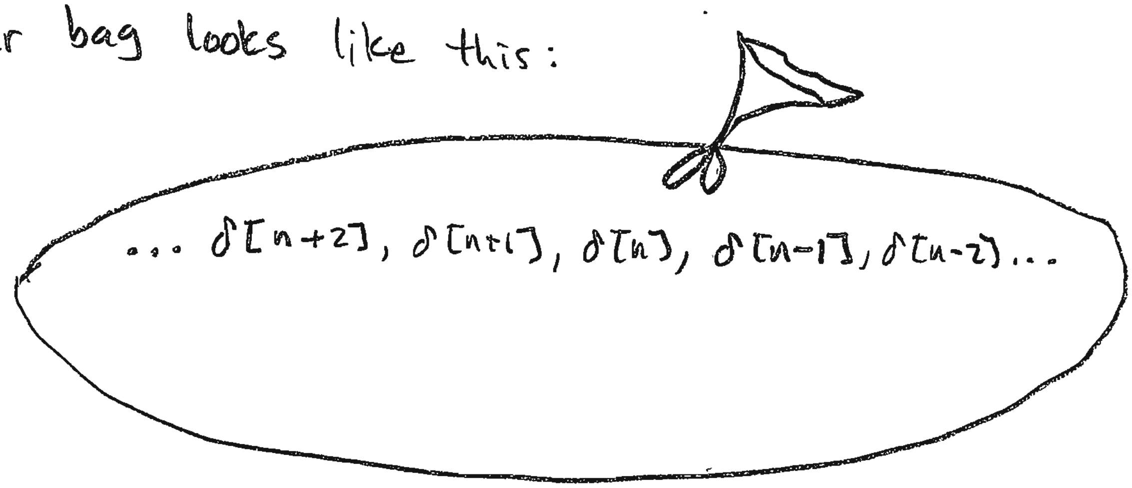
$$\delta[n-k] = \begin{cases} 1, & n=k \\ 0, & \text{otherwise} \end{cases}$$



→ Turned on at $n=k$

→ zero everywhere else

- So now imagine that we take a bag,
  - and we throw into the bag <u>all</u> of the integer translates of $\delta[n]$... including the one for $k=0$, which is just $\delta[n]$...
  - In other words, we throw the guy $\delta[n-k]$ into the bag for <u>each</u> integer $k \in \mathbb{Z}$.
  - our bag looks like this:



- In math, this is written as follows:

$$\left\{ \delta[n-k] \right\}_{k \in \mathbb{Z}}$$

  - The curly braces mean "set"... which is our bag... a set of signals.
  - In the bag, we've got the guy $\delta[n-k]$ for <u>each</u> $k \in \mathbb{Z}$.
  - In other words, we've got $\delta[n--2]$ and $\delta[n--1]$ and $\delta[n]$ and $\delta[n-1]$ and $\delta[n-2]$ and $\delta[n-100]$ and $\delta[n-k]$ for every $k \in \mathbb{Z}$.

- Each guy in our bag is zero everywhere except at one place where he is turned on and is equal to 1.

- And in the bag, we have exactly one guy who is turned on at each place n.

- Have you ever heard of such a thing before?
  - → You have!
  - → How about in $\mathbb{R}^2$ when we had the natural basis $\{\vec{i}, \vec{j}\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?
    - → Each guy is zero everywhere except one place where he is equal to 1.
    - → In the bag, we have one guy who is turned on at each place.
  - → And similarly in $\mathbb{R}^3$, the natural basis was

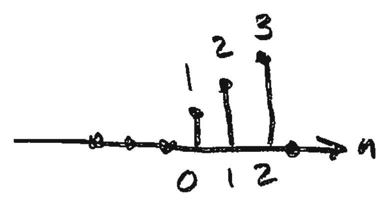
$$\{\vec{i}, \vec{j}, \vec{k}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

    - → Each guy is zero everywhere but one place where he is equal to 1.
    - → We have one guy who is turned on at each place.

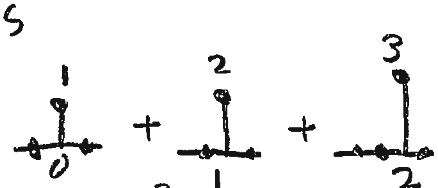
FACT: the set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ of integer translates of $\delta[n]$ is the natural basis for our vector space of discrete-time signals $x[n]$.

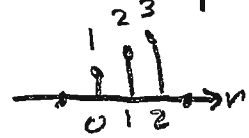
- For our vector space of discrete-time signals, the set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ plays the same role that $\{\vec{i}, \vec{j}\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ plays in $\mathbb{R}^2$ (or $\mathbb{C}^2$),
  - and the same role that
  $$\{\vec{i}, \vec{j}, \vec{k}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$
  plays in $\mathbb{R}^3$ (or $\mathbb{C}^3$).

FACT: every one of our discrete-time signals $x[n]$ can be written as a sum (a linear combination) of the set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

- For example, the signal $x[n] =$ 

can be written as

$1\,\delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2] =$ 

$=$ 

FACT: The basis $\{\delta[n-k]\}_{k\in\mathbb{Z}}$ is orthonormal.

→ Each guy $\delta[n-k]$... like $\delta[n--1]$, $\delta[n]$, $\delta[n-1]$, etc... has unit length.

→ And they are all mutually orthogonal.

★ Unit Length: consider $\delta[n-1]$ for example.

→ His dot product with himself is

$$\langle \delta[n-1], \delta[n-1] \rangle = \sum_{n=-\infty}^{\infty} \delta[n-1]\,\delta[n-1]$$

(no conjugation because everything is real)

→ There is only one nonzero term in the sum. It is the term when $n=1$.

→ On that term, we have $\delta[n-1]=1$, so we get $\delta[n-1]\,\delta[n-1] = 1\cdot 1 = 1$.

→ So, for the whole sum, we get

$$\langle \delta[n-1], \delta[n-1] \rangle = \sum_{n=-\infty}^{\infty} \delta[n-1]\,\delta[n-1]$$

$$= \cdots + \underbrace{0\cdot 0}_{n=-1} + \underbrace{0\cdot 0}_{n=0} + \underbrace{1\cdot 1}_{n=1} + \underbrace{0\cdot 0}_{n=2} + \underbrace{0\cdot 0}_{n=3} + \cdots$$

$$= 1$$

– So the length (norm) is $\sqrt{\langle \delta[n-1], \delta[n-1] \rangle} = \sqrt{1} = \underline{\underline{1}}$

- The same is true for all of the guys $\delta[n-k]$: for any integer $k \in \mathbb{Z}$, we get $\langle \delta[n-k], \delta[n-k] \rangle = 1$, so the length (norm) of $\delta[n-k]$ is <u>one</u> for any integer k.

☆ Mutually orthogonal: any two <u>different</u> guys from the set $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ have dot product zero... they are orthogonal to each other.

- This is because any two different guys like $\delta[n-1]$ and $\delta[n-2]$ are turned on at <u>different</u> places.
- So every single term in the sum

$$\langle \delta[n-1], \delta[n-2] \rangle = \sum_{n=-\infty}^{\infty} \delta[n-1]\delta[n-2]$$

is <u>zero</u>.

- Specifically the dot product is given by

$$\langle \delta[n-1], \delta[n-2] \rangle = \sum_{n=-\infty}^{\infty} \delta[n-1]\delta[n-2]$$

$$= \cdots \underbrace{0 \cdot 0}_{n=-1} + \underbrace{0 \cdot 0}_{n=0} + \underbrace{1 \cdot 0}_{n=1} + \underbrace{0 \cdot 1}_{n=2} + \underbrace{0 \cdot 0}_{n=3} + \cdots$$

$$= 0 \quad ///$$

- Recall from page 1.100 that any vector $\vec{v}$ from $\mathbb{R}^2$ can be written as

$$\vec{v} = \langle \vec{v}, \vec{i} \rangle \vec{i} + \langle \vec{v}, \vec{j} \rangle \vec{j}$$

where $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- But, when we represent our vectors in terms of the natural basis, we don't really have to do all the linear algebra...

→ Because it's practically obvious, for example, that

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

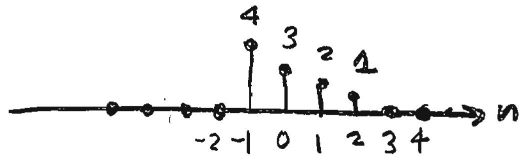
can be written as $2\vec{i} + 3\vec{j}$

$$= 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

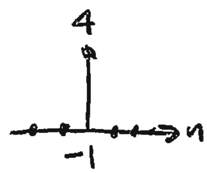
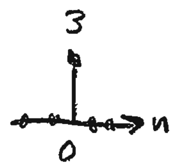
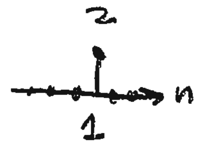
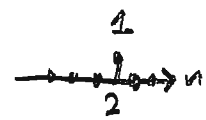
→ That's the beauty of the natural basis

- The <u>same</u> <u>thing</u> is true for our discrete-time signals $x[n]$ when we use the natural basis $\left\{ \delta[n-k] \right\}_{k \in \mathbb{Z}}$.

- For example, it's practically obvious that the signal

$x[n] =$ 

Can be written as

$x[n] =$  $+$  $+$  $+$ 

$$= \underbrace{4}_{x[-1]}\delta[n+1] + \underbrace{3}_{x[0]}\delta[n] + \underbrace{2}_{x[1]}\delta[n-1] + \underbrace{1}_{x[2]}\delta[n-2]$$

think of this as $\delta[n--1]$

Think of this as $\delta[n-0]$

- But we <u>could</u> use dot products to get the same thing if we wanted to.
  - → For example,

$$\delta[n-1] = \begin{cases} 1, & n=1 \\ 0, & \text{other} \end{cases}$$

$$\langle x[n], \delta[n-1] \rangle = \sum_{n=-\infty}^{\infty} x[n]\,\delta[n-1]$$

→ There is only one nonzero term in the sum

$$= \cdots + \underbrace{x[-1]\cdot 0}_{n=-1} + \underbrace{x[0]\cdot 0}_{n=0} + \underbrace{x[1]\cdot 1}_{n=1} + \underbrace{x[2]\cdot 0}_{n=2} + \cdots$$

$$= x[1] = 2 \checkmark$$

- More generally, for <u>any</u> discrete time signal $x[n]$ and <u>any</u> integer k, we get

$$\langle x[n], \delta[n-k] \rangle = \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = \underbrace{x[k]}_{\text{a number}} \qquad (*)$$

→ it's the value of the signal $x[n]$ where $\delta[n-k]$ is turned on.

- And <u>any</u> discrete-time signal $x[n]$ can be written as

$$\cdots + x[-2]\delta[n--2] + x[-1]\delta[n--1] + x[0]\delta[n-0]$$
$$+ x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

where $x[-2], x[-1], x[0], x[1], x[2]$, etc... are <u>numbers</u>.

- We can save writing by using a capital Σ do loop for this:

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{numbers}} \delta[n-k] \qquad (**)$$

- Combing equations (*) and (**) from page 2.144, we see that,

→ Just like $\vec{v} = \langle \vec{v}, \vec{i} \rangle \vec{i} + \langle \vec{v}, \vec{j} \rangle \vec{j}$ in $\mathbb{R}^2$,

→ Our discrete-time signals $x[n]$ can be written as:

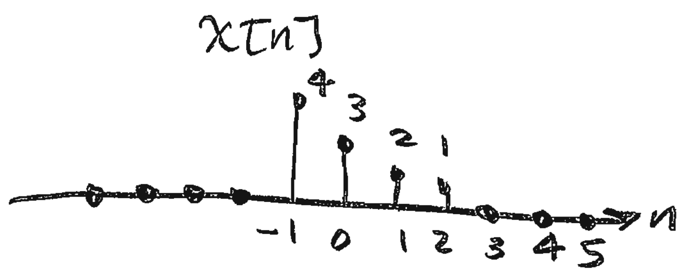
$$x[n] = \sum_{k=-\infty}^{\infty} \langle x[n], \delta[n-k] \rangle \delta[n-k]$$

$$= \cdots + \langle x[n], \delta[n--1] \rangle \delta[n--1] + \langle x[n], \delta[n-0] \rangle \delta[n-0] + \langle x[n-1], \delta[n-1] \rangle \delta[n-1] + \cdots$$

$$= \cdots + x[-1]\delta[n--1] + x[0]\delta[n-0] + x[1]\delta[n-1] + \cdots$$

☆☆☆ __TIME__ __DOMAIN__ means that we think of our signals this way... as a sum of the natural basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

- You can think of the time domain <u>graph</u> of a discrete-time signal $x[n]$ as a visual depiction of the dot products between the signal and all of the basis vectors (basis signals) from the <u>natural basis</u>.



$$\langle x[n], \delta[n+1] \rangle = x[-1] = 4$$
$$\langle x[n], \delta[n] \rangle = x[0] = 3$$
$$\langle x[n], \delta[n-1] \rangle = x[1] = 2$$
$$\langle x[n], \delta[n-2] \rangle = x[2] = 1$$

and $\langle x[n], \delta[n-k] \rangle = 0$ for all the rest of the k's ...

- These numbers can be computed using the dot product or they can be read directly off of the graph.
  - → They are sometimes called the <u>coordinates</u> of the signal $x[n]$ with respect to the natural basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

⇒ By contrast, if we instead write our discrete signals $x[n]$ in terms of a certain rotated basis, then it is called frequency domain.

- This is like a change of basis or change of coordinates.
- The graph of the coordinates of $x[n]$ with respect to the rotated basis is different from the time domain graph.
- we will discuss the frequency domain in detail later.

Discrete-Time Systems

- A system inputs one signal and outputs another signal.
- Useful systems transform the input signal into the output signal in a predictable way that can be designed and/or analyzed.

- Usually, but not always, the input signal is called $x[n]$ and the output signal is called $y[n]$.
- The system is usually designated with a capital letter.
  - → The most common letters used are F, G, and H.
  - → H is used most often.

- Here is how we draw a system H with input signal $x[n]$ and output signal $y[n]$:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

- Mathematically, we model a system with a function that specifies the relationship between the input signal and the output signal.
  - → usually, this is specified by an equation that tells how the input signal gets turned into the output signal
  - → This equation is called the "input-output equation" or "I/O equation."

EX: $y[n] = 100x[n]$ (a pure amplifier)

EX: $y[n] = x[n-1]$ (a pure delayer)

EX: $y[n] = x^2[n]$ (square law system)

- This is often written using "operator notation" like this:

$$y[n] = H\{x[n]\}$$

  - what it means: $y[n]$ is the output signal of the system H when $x[n]$ is the input.
  - In English, this is read "$y[n]$ equals H of $x[n]$."

Examples:

$$y[n] = H\{x[n]\} = 100\,x[n]$$

→ if the input is $x[n] = \left(\frac{1}{2}\right)^n u[n]$, then the output is $y[n] = 100\left(\frac{1}{2}\right)^n u[n]$.

$$y[n] = H\{x[n]\} = x[n-1]$$

→ if the input is $x[n] = \left(\frac{1}{2}\right)^n u[n]$, then the output is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$y[n] = \mathcal{H}\{x[n]\} = x^2[n]$$

→ if the input is $x[n] = \left(\frac{1}{2}\right)^n u[n]$,

then the output is

$$y[n] = \left(\left(\tfrac{1}{2}\right)^n u[n]\right)^2$$

$$= \left(\tfrac{1}{2}\right)^n u[n] \left(\tfrac{1}{2}\right)^n u[n]$$

$$= \left(\tfrac{1}{2}\right)^{2n} u[n]$$

$$= \left(\tfrac{1}{4}\right)^n u[n]$$

- So a system is modeled by a function that maps each input signal to one and only one "buddy"... the corresponding output signal.
  - The domain is a vector space of input signals.
  - The range is a vector space of output signals.

- Next, we need to develop two adjectives to describe systems.
  - These adjectives are:
    - Time Invariant
    - Linear

Time Invariance

- Time invariance means that the action of the system commutes with time shifts.
  - In other words, you can put the input signal through the system to get the output signal, and then shift the output signal,
  - OR you can shift the input signal first, and then put it through the system.

$\Rightarrow$ If the result is the same either way,

$\Rightarrow$ And this holds true for every possible input signal $x[n]$ and every possible integer shift amount n_0,

$\Rightarrow$ Then the system is called time invariant.

- This is also sometimes called "shift invariant" or "translation invariant."

- Here is another way to look at it:
  - Let H be a system.
  - Let $x_1[n]$ be an input signal
  - Let $n_0 \in \mathbb{Z}$ be a shift amount
  - Let $y_1[n]$ be the output signal when $x_1[n]$ is the input signal. In math, we write this as $y_1[n] = H\{x_1[n]\}$.

- For the first way, we put $x_1[n]$ through the system and then we shift the output signal $y_1[n]$ by n_0.
  - In pictures:

<u>First Way</u>

$$\underbrace{x_1[n] \longrightarrow \boxed{H} \longrightarrow y_1[n]}_{\text{Through the system}} \; \underbrace{\xmapsto{\text{SHIFT}} y_1[n-n_0]}_{\text{Then Shift}}$$

- For the second way, we <u>shift first</u>:
  - Let $x_2[n] = x_1[n-n_0]$. Then put $x_2[n]$ through the system to get $y_2[n] = H\{x_2[n]\}$.
  - In pictures:

<u>Second Way</u>

$$\underbrace{x_1[n] \xmapsto{\text{SHIFT}} x_2[n] = x_1[n-n_0]}_{\text{Shift First}} \; \underbrace{\longrightarrow \boxed{H} \longrightarrow y_2[n]}_{\text{Through the system second}}$$

$\Rightarrow$ If $y_2[n] = y_1[n-n_0]$, and this works for <u>all possible</u> input signals $x_1[n]$ and <u>all possible</u> integer shift amounts n_0,

$\Rightarrow$ Then the system H is time invariant.

- Here is how to write the definition of <u>time invariant</u> in math:

<u>DEF</u>: Let H be a system with input-output equation $y[n] = H\{x[n]\}$.

If $H\{x[n-n_0]\} = y[n-n_0]$ $\forall$ input signals $x[n]$ and $\forall\, n_0 \in \mathbb{Z}$, then H is called <u>time invariant</u>.

<u>EX</u>: Let H be a pure amplifier with I/O equation

$$y[n] = H\{x[n]\} = 100\,x[n].$$

First way: through the system, then shift:

Let $x_1[n]$ be the input signal. Then the output signal is $y_1[n] = 100\,x_1[n]$. Now shift to get $y_1[n-n_0] = 100\,x_1[n-n_0]$.

Second way: shift first, then through the system:

Let $x_2[n] = x_1[n-n_0]$. Now put $x_2[n]$ through the system $\longrightarrow$

$$y_2[n] = H\{x_2[n]\}$$
$$= 100\, x_2[n]$$
$$= 100\, x_1[n-n_0]$$
$$= y_1[n-n_0] \checkmark$$

→ This shows that we will always get the same thing both ways. So H is a time invariant system.

Linear

- Linearity means that the action of the system commutes with linear combinations.
  - In other words, you can take two (or more) input signals, put them through the system, and then take a linear combination of the resulting output signals,
  - OR you can take the same linear combination of the input signals first, then put it through the system.

⇒ If the result is the same either way,

$\Rightarrow$ And this holds true for <u>all</u> <u>possible</u> input signals and <u>all</u> <u>possible</u> linear combinations,

$\Rightarrow$ Then the system is called <u>linear</u>.

- It is sufficient to look at linear combinations of just two input signals. If it works for all possible linear combinations of two input signals, then it will also work for all possible linear combinations of three or more input signals.

- How to look at it:

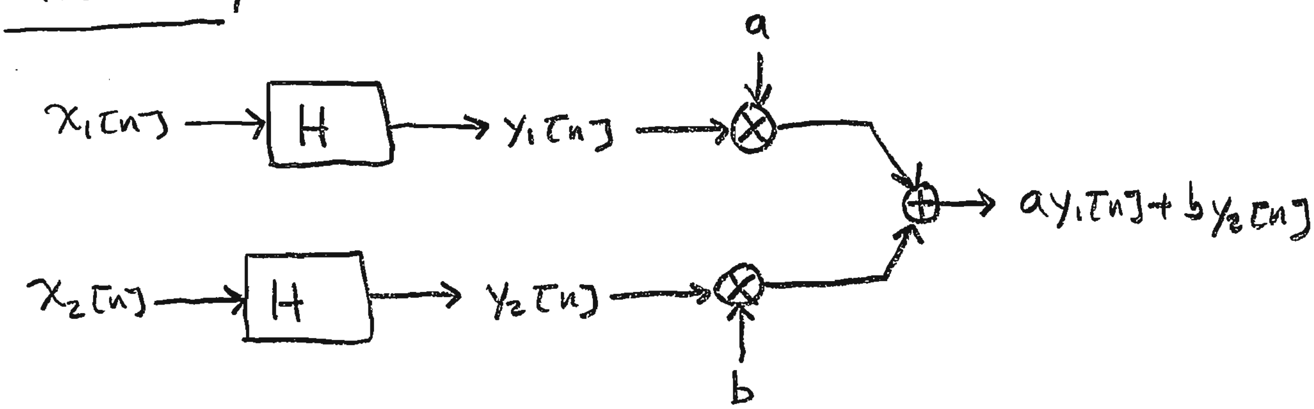
  - Let H be a system.
  - Let $x_1[n]$ and $x_2[n]$ be two input signals.
  - Let a and b be two constants (complex in general).
  - Let $y_1[n] = H\{x_1[n]\}$ be the output signal when $x_1[n]$ is the input signal.
  - Let $y_2[n] = H\{x_2[n]\}$ be the output signal when $x_2[n]$ is the input signal.

$\longrightarrow$

- For the first way, we put $x_1[n]$ through the system to get $y_1[n]$ and we put $x_2[n]$ through the system to get $y_2[n]$. Then we take the linear combination $a y_1[n] + b y_2[n]$.

- In pictures:

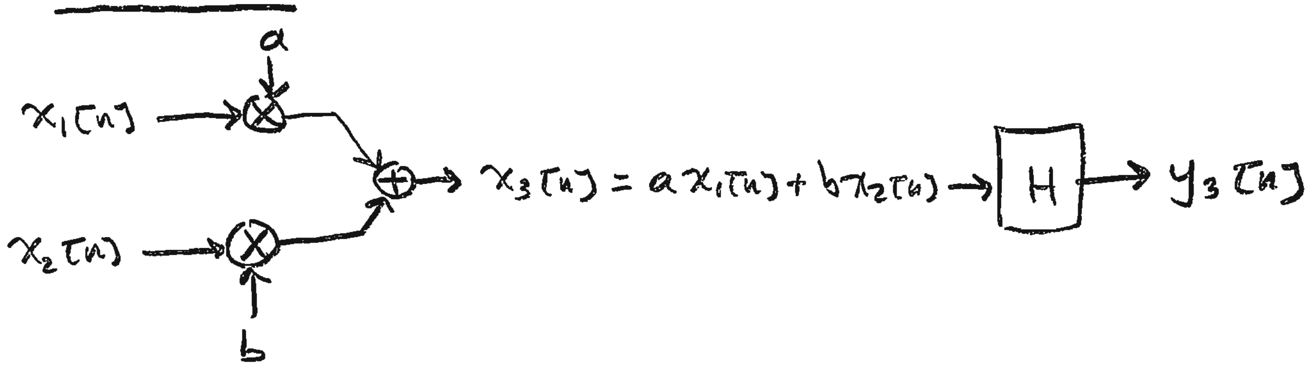
First Way



- For the second way, we take the linear combination first. Let $x_3[n] = a x_1[n] + b x_2[n]$. Then we put $x_3[n]$ through the system to get $y_3[n]$.

- In pictures:

Second Way



$\Rightarrow$ If $y_3[n] = a y_1[n] + b y_2[n]$, and this works for all possible input signals $x_1[n]$ and $x_2[n]$ and all possible constants a, b

$\Rightarrow$ Then the system H is linear.

- Here is how to write the definition of linear in math:

DEF: Let H be a system with input-output equation

$$y[n] = H\{x[n]\}.$$

If $H\{a x_1[n] + b x_2[n]\} = a y_1[n] + b y_2[n]$

$\forall$ input signals $x_1[n]$, $x_2[n]$ and $\forall$ $a, b \in \mathbb{C}$,

then H is called linear.

EX: Let H be a pure amplifier with I/O equation

$$y[n] = H\{x[n]\} = 100\, x[n].$$

Let $x_1[n]$ and $x_2[n]$ be two arbitrary input signals.

Let $a, b \in \mathbb{C}$ be two arbitrary constants.

$\longrightarrow$

First way: through the system, then take linear combination:

- put $x_1[n]$ through the system to get $y_1[n] = 100\,x_1[n]$.
- put $x_2[n]$ through the system to get $y_2[n] = 100\,x_2[n]$.
- For constants $a, b \in \mathbb{C}$, take the linear combination of the output signals to get $a\,y_1[n] + b\,y_2[n] = 100a\,x_1[n] + 100b\,x_2[n]$.

Second way: take the linear combination of the input signals first, then put it through the system:

- Let $x_3[n] = a\,x_1[n] + b\,x_2[n]$
- Now put $x_3[n]$ through the system to get

$$\begin{aligned} y_3[n] &= 100\,x_3[n] \\ &= 100\left(a\,x_1[n] + b\,x_2[n]\right) \\ &= 100a\,x_1[n] + 100b\,x_2[n] \; \checkmark \end{aligned}$$

→ This shows that we will always get the same thing both ways. So H is a linear system.

NOTE: some people like to break up linear into two properties called homogeneity and superposition.

DEF: Let H be a system with input-output equation $y[n] = H\{x[n]\}$.

If $H\{a\,x[n]\} = a\,y[n]$ $\forall$ input signals $x[n]$ and all constants $a \in \mathbb{C}$, then H is called homogeneous.

→ It means that the action of the system commutes with scalar multiplication.

→ You can do the multiplication first, then put it through the system,

→ or you can put it through the system first and then do the multiplication.

→ For a homogeneous system, you get the same thing either way.

<u>DEF</u>: Let H be a system with input-output equation

$$y[n] = H\{x[n]\}.$$

If $H\{x_1[n] + x_2[n]\} = y_1[n] + y_2[n]$

$\forall$ input signals $x_1[n]$ and $x_2[n]$, then the system H is said to have the <u>superposition property</u>.

→ It means that the action of the system commutes with sums.

→ You can put the signals through the system first and then take the sum,

→ or you can take the sum first and then put it through the system

→ For a system that has the superposition property, you get the same thing either way.

☆☆☆ Superposition plus homogeneity is <u>equivalent</u> to linear

- A system that is <u>both</u> linear <u>and</u> time invariant is called a "linear time invariant system" or "LTI system."

    - Sometimes also called "LSI" for "linear shift invariant."

- The class of LTI systems is <u>super</u> <u>super</u> <u>super</u> <u>important</u>.

<u>FACT</u>: in general, the input signal $x[n]$ and output signal $y[n]$ of an LTI system are related by a constant coefficients linear difference equation.

→ This means that a linear combination of the shifts of the input signal is equal to a linear combination of the shifts of the output signal.

<u>EX</u>:

$$y[n] = -\tfrac{1}{2}x[n] + x[n-1] - \tfrac{1}{2}x[n-2]$$

<u>EX</u>:

$$y[n] - y[n-1] = x[n] + \tfrac{1}{2}x[n-1] - \tfrac{1}{2}x[n-2]$$

DEF: The <u>impulse</u> <u>response</u> of a discrete-time LTI system is the output signal that is obtained when the input signal is $\delta[n]$.

→ The impulse response is denoted by a lower case letter the same as the upper case letter that is used for the system.

$$\delta[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ H\end{array}} \longrightarrow h[n]$$

"impulse response"

$$\delta[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ F\end{array}} \longrightarrow f[n]$$

$$\delta[n] \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ G\end{array}} \longrightarrow g[n]$$

- Because an LTI system is <u>time invariant</u>,
  - if you put in a shifted impulse, you get out a shifted impulse response:

$$\delta[n-2] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow h[n-2]$$

$$\delta[n+7] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow h[n+7]$$

- Because an LTI system is <u>both</u> linear <u>and</u> time invariant, if you put in a linear combination of shifted impulses, you get out the same linear combination of shifted impulse responses:

$$3\delta[n+1] - 2\delta[n] + 4\delta[n-1] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow 3h[n+1] - 2h[n] + 4h[n-1]$$

<u>We are done with Module 2!!</u>