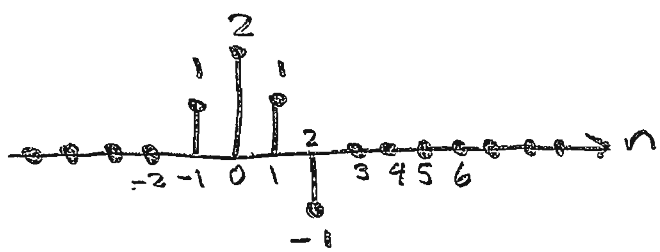
MODULE 3: TIME DOMAIN CONVOLUTION

- Recall from page 2.144 that any discrete-time signal $x[n]$ can be written as a <u>sum</u> of the <u>shifts</u> of $\delta[n]$:

$$x[n] = \cdots + x[-2]\delta[n- -2] + x[-1]\delta[n- -1] + x[0]\delta[n-0]$$
$$+ x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \qquad (*)$$

- For example,

$$x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$$



→ For this $x[n]$, we have:

$x[-1] = 1$
$x[0] = 2$
$x[1] = 1$
$x[2] = -1$

→ and $x[k]=0$ for all the rest of the k's.

- To save writing, we usually write eq. $(*)$ above using a "capital Σ do loop":

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad (**)$$

⇒ This is equation (5.29) in the book.

- In equations (*) and (**) on page 3.1, it is <u>very important</u> to remember that the $x[k]$ are <u>numbers</u>.
  - → k is the loop counter
  - → n is the independent (time) variable

  ⇒ For any given signal, like the $x[n]$ on page 3.1, the $x[k]$ are just numbers.

- Now suppose we have an LTI system H with impulse response $h[n]$:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

- If the input is our signal $x[n]$ from page 3.1

$$x[n] = 1\delta[n+1] + 2\delta[n] + 1\delta[n-1] - 1\delta[n-2]$$

- Then the output signal $y[n]$ is given by:

$$y[n] = H\{x[n]\} = H\{\delta[n+1] + 2\delta[n] + \delta[n-1] - \delta[n-2]\}.$$

→ Because the system is <u>linear</u>, the action of the system commutes with sums and we have:

$$y[n] = H\{1\delta[n+1]\} + H\{2\delta[n]\} + H\{1\delta[n-1]\} - H\{1\delta[n-2]\}$$

→ Also because the system is <u>linear</u>, the action of the system commutes with scalar multiplication and we have:

$$y[n] = 1\,H\{\delta[n+1]\} + 2\,H\{\delta[n]\} + 1\,H\{\delta[n-1]\} - 1\,H\{\delta[n-2]\}.$$

→ Now, because the system is also time invariant, we know that for any $k \in \mathbb{Z}$, $H\{\delta[n-k]\} = h[n-k]$. I.e., the action of the system commutes with time shifts. Applying this, we have:

$$y[n] = 1\,h[n+1] + 2\,h[n] + 1\,h[n-1] - 1\,h[n-2].$$

⇒ In other words, when we write the input signal as a linear combination of the shifts of $\delta[n]$, it's easy to see that the output signal $y[n]$ is given by the <u>same</u> <u>linear</u> <u>combination</u> of the shifts of $h[n]$.

☆☆☆

- More generally, <u>any</u> input signal $x[n]$ can be written as a sum of the natural basis (the shifts of $\delta[n]$) according to:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k], \qquad (5.29)$$

where the $x[k]$ are numbers.

- If we put $x[n]$ into the input of an LTI system H with impulse response $h[n]$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

→ Then the output signal $y[n]$ is given by

$$
\begin{aligned}
y[n] &= H\{x[n]\} \\
&= H\Big\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\Big\} \\
&= \sum_{k=-\infty}^{\infty} H\{x[k]\,\delta[n-k]\} && \text{(because } H \text{ is } \underline{\text{linear}}\text{)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,H\{\delta[n-k]\} && \text{(because } H \text{ is } \underline{\text{linear}}\text{)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \qquad (5.31) && \text{(because } H \text{ is } \underline{\text{time}}\ \underline{\text{invariant}}\text{)}
\end{aligned}
$$

- The last equation on page 3.4 is called the <u>discrete-time convolution</u> of $x[n]$ and $h[n]$.

- This comes up so often that we have a special symbol for it: the asterisk "$*$".

- We write:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (5.31)$$

FACT: if the signal $x[n]$ is input to an LTI system H with impulse response $h[n]$,

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$

then the output signal $y[n]$ is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

"Convolution"

<u>Note:</u> we say:

- "The output of an LTI system is given by the convolution of the input signal with the impulse response."

- "To find the output, we <u>convolve</u> the input signal with the impulse response."

(The verb is <u>convolve</u> ... <u>not</u> <u>convolute</u>)

- The output signal

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

is just like any other discrete-time signal... we model it with a function $y[n]$ that has domain $\mathbb{Z}$ and range $\mathbb{R}$ or $\mathbb{C}$.

- So the basic problem is:
  - Given $x[n]$ and $h[n]$, find $y[n]$.
  - This is called <u>discrete-time convolution</u>.
  - To solve it, we have to say what the number $y[n]$ is for every $n \in \mathbb{Z}$.
  - Usually, we do this by writing down a formula for the function $y[n]$.
  - We will talk a lot more about that in a minute.

- But first, we are going to go over a couple of properties of the convolution operation itself...
  - These properties can be useful for actually working out convolutions.

<u>FACT</u>: the convolution sum $\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$ can be thought of as an operation (or "operator") that gobbles up two signals ($x[n]$ and $h[n]$, for example) and combines them to make a third signal ($y[n]$ for example).

- Often, this comes up when we need to compute the output signal of an LTI system:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n]$$

- But sometimes we also think of convolution as an operation that can combine two signals to make a third signal, even if there's not any system involved.
  - So, sometimes we'll just say "let $x_1[n]$ and $x_2[n]$ be two signals and convolve them"...
  - This means: find a third signal $x_3[n]$ such that

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]$$

- To work the convolution, we must find the number $x_3[n]$ for every n.

- How do we compute this?

  - Well, for <u>each</u> n,

$$x_3[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

  - So, for each n, we have to execute the "do loop" $\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$

    - To execute the loop, we have to "work on" or "run" the loop counter... which is $\underline{\underline{\underline{k}}}$

    - And, in general, we have to do this for <u>each</u> value of n.

  - For example, we have to find the number $x_3[7]$ for $n=7$. It is given by

$$x_3[7] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[7-k]$$

$\longrightarrow$

- To understand how to do that, rewrite it as:

$$x_3[7] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[-k - -7]$$

$\Rightarrow$ So, assuming that $x_1[n]$ and $x_2[n]$ are real-valued signals... so that conjugation doesn't do anything to them, we see that the number $x_3[7]$ is given by the <u>dot product</u> of $x_1[k]$ with $x_2[-k--7]$.

$\Rightarrow$ The graph of $x_1[k]$ is obtained by taking the graph of $x_1[n]$ and simply changing the n's to k's.

$\Rightarrow$ The graph of $x_2[-k--7]$ has a <u>scale</u> by -1 (a flip) and a shift by $n_0 = -7$.

- More generally, for each n,

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[-k--n]$$

$\longrightarrow$

... is a <u>number</u> that is given by the dot product of $x_1[k]$ with a "flipped and shifted" version of x_2 ... specifically $x_2[-k - -n]$.

shift by "$-n$"

Scale by -1.

⇒ Always remember to do the shift first, then the scale... as we said back on page 2.100. ☆☆☆

- But before we talk any more about the details of how to actually compute a convolution, let's go over the two convolution properties I mentioned back on the bottom of page 3.6...

① <u>FACT</u>: Convolution is <u>commutative</u>:

$$x_1[n] * x_2[n] = x_2[n] * x_1[n] \quad (5.33)$$

book

⇒ In other words:

$$\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=-\infty}^{\infty} x_2[k]\, x_1[n-k]$$

⟶

- What this means to you from a practical standpoint:

⇒ If you are given two signals $x_1[n]$ and $x_2[n]$, and you are asked to compute the convolution

$$x_3[n] = x_1[n] * x_2[n],$$

⇒ Then you can put the "k" on x_1 and the "n-k" on x_2 to work it as

$$x_3[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

→ In other words, you can "flip and shift" x_2 . . .

⇒ OR you can do it the other way. . . . you can put the "n-k" on x_1 and put the "k" on x_2 to work it as

$$x_3[n] = \sum_{k=-\infty}^{\infty} x_2[k]\, x_1[n-k]$$

("flip and shift" x_1 instead)

☆ Because convolution is commutative, you will get the exact same signal $x_3[n]$ either way.

→ The same number for every n.

Proof that convolution is commutative:

- Let $x_1[n]$ and $x_2[n]$ be two discrete-time signals.

- Let $x_3[n] = x_1[n] * x_2[n]$.

- Then: $$x_3[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

Let $m = n-k$

Then $k = n-m$

$\rightarrow$ when $k \rightarrow -\infty$, $m \rightarrow \infty$

when $k \rightarrow \infty$, $m \rightarrow -\infty$

So $$x_3[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$= \sum_{m=\infty}^{-\infty} x_1[n-m]\, x_2[m]$$ (change variables from k to m)

$$= \sum_{m=-\infty}^{\infty} x_1[n-m]\, x_2[m]$$ (order of adding doesn't matter)

$$= \sum_{m=-\infty}^{\infty} x_2[m]\, x_1[n-m]$$ (order of multiplying doesn't matter)

$$= \sum_{k=-\infty}^{\infty} x_2[k]\, x_1[n-k]$$ (write "k" for the loop counter instead of "m")

$$= x_2[n] * x_1[n]$$ ///

QED

(2A) FACT: convolution is associative:

$$(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n]) \quad (5.37)$$

- What it means to you:
  - → If you have to convolve three signals together, it doesn't matter which two you do first.
  - → The answer turns out the same whichever two you choose to do first.

- The proof of this is not hard, but it involves a lot of detailed steps that aren't important to us right now.
  - So we're going to skip it. You will prove it in ECE 3793 (Signals & Systems).

2B FACT: convolution is <u>distributive</u>:

$$\left(x_1[n] + x_2[n]\right) * x_3[n] = \left(x_1[n] * x_3[n]\right) + \left(x_2[n] * x_3[n]\right) \qquad (5.38)$$

- What it means to you:

→ you can add $x_1[n]$ and $x_2[n]$ first, then convolve with $x_3[n]$...

→ OR you can convolve each of $x_1[n]$ and $x_2[n]$ with $x_3[n]$ first... and then add those two...

⇒ The final result signal is <u>exactly</u> <u>the</u> <u>same</u> either way.

- As with associativity, the proof of this is not hard, but we're going to skip it because we don't want to get bogged down in the details.

  - you will prove it in ECE 3793.

CONVOLUTION WITH DELTAS

- In a few minutes, we are going to learn a powerful general method for doing convolution that will always work.
    - But it may seem a little bit complicated at first.
- There are certain cases where a simpler method will also work.
    - The simpler method will work when we have to convolve two signals that can both be written as short sums of the shifts of $\delta[n]$.
- For example, if

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

and

$$h[n] = \tfrac{1}{2}\delta[n] - \tfrac{1}{2}\delta[n-1]$$

$\Rightarrow$ Both $x[n]$ and $h[n]$ are short sums of the shifts of $\delta[n]$

$\Rightarrow$ The simpler method will work for finding the convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k].$$

- To understand how the simpler method works, we need a few simple facts.

<u>FACT 1</u> : For any discrete-time signal $x[n]$,

$$x[n] * \delta[n] = x[n]. \quad ///$$

<u>Proof</u> : $x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$ (by definition)

$$= \sum_{k=-\infty}^{\infty} \delta[k]\,x[n-k] \quad \left(\text{because convolution is commutative}\right)$$

$$= \cdots + \underbrace{\overset{\text{zero}}{\delta[-2]}\,x[n+2]}_{k=-2 \text{ term}} + \underbrace{\overset{\text{zero}}{\delta[-1]}\,x[n+1]}_{k=-1 \text{ term}} + \underbrace{\overset{\text{one}}{\delta[0]}\,x[n-0]}_{k=0 \text{ term}}$$

$$+ \underbrace{\overset{\text{zero}}{\delta[1]}\,x[n-1]}_{k=1 \text{ term}} + \underbrace{\overset{\text{zero}}{\delta[2]}\,x[n-2]}_{k=2 \text{ term}} + \cdots$$

$$= \text{just the } k=0 \text{ term} \quad \left(\text{because } \delta[k] = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}\right)$$

$$= 1\,x[n-0]$$

$$= x[n] \quad ///$$

☆☆☆

> In words: convolution with $\delta[n]$ <u>does</u> <u>not</u> <u>change</u> <u>the</u> <u>signal</u> !!

FACT 2: For any discrete-time signal $x[n]$,

$$x[n] * \delta[n-5] = x[n-5]$$

☆ In words: convolution with $\delta[n-5]$ shifts the signal to the right by 5.

Proof: $x[n] * \delta[n-5] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[(n-5)-k]$

$$= \sum_{k=-\infty}^{\infty} \delta[k-5]\, x[n-k] \quad \left(\begin{array}{c}\text{because convolution} \\ \text{is commutative}\end{array}\right)$$

→ There is only <u>one</u> nonzero term in the sum

→ It is the $k=5$ term

→ On that term $k=5$ and $\delta[k-5] = \delta[0] = 1$

= just the "$k=5$" term

$$= 1 \cdot x[n-5]$$

$$= x[n-5] \quad ///$$

FACT 3 : More generally, for any discrete time signal $x[n]$ and any integer shift amount $n_0 \in \mathbb{Z}$,

$$x[n] * \delta[n-n_0] = x[n-n_0] \qquad (5.36)$$

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In words: convolution with a shifted delta shifts the signal by the same amount.

Proof:

$$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[(n-n_0)-k]$$

$$= \sum_{k=-\infty}^{\infty} \delta[k-n_0]\,x[n-k] \qquad \left(\text{because convolution is commutative}\right.$$

$$= \text{just the "}k=n_0\text{" term}$$

$$= 1\,x[n-n_0]$$

$$= x[n-n_0]. \quad ///$$

⇒ This works for any integer shift amount n_0, including n_0 positive, n_0 negative, and $n_0 = 0$.

EXAMPLES : Good for any $x[n]$

$$x[n] * \delta[n-2] = x[n-2] \qquad (n_0 = 2)$$

$$x[n] * \delta[n+3] = x[n+3] \qquad (n_0 = -3)$$

$$x[n-1] * \delta[n-2] = x[n-3]$$

→ In this case, $x[n]$ was already shifted by 1. Convolution with $\delta[n-2]$ shifts him by two more.

$$x[n+3] * \delta[n-2] = x[n+1]$$

$$x[n-1] * \delta[n+4] = x[n+3]$$

Some more Specific Examples:

$$\delta[n] * \delta[n] = \delta[n]$$

$$\delta[n] * \delta[n-1] = \delta[n-1]$$

$$\delta[n-1] * \delta[n-1] = \delta[n-2]$$

$$\delta[n-2] * \delta[n] = \delta[n-2]$$

$$\delta[n-2] * \delta[n-1] = \delta[n-3]$$

$$\left\{\left(\tfrac{1}{2}\right)^n u[n]\right\} * \delta[n-1] = \left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$\left\{\left(\tfrac{1}{2}\right)^n u[n]\right\} * \delta[n+2] = \left(\tfrac{1}{2}\right)^{n+2} u[n+2]$$

FACT 4 : Convolution is linear.

- For any three discrete-time signals $x_1[n]$, $x_2[n]$, and $x_3[n]$
- And any two constants $a, b \in \mathbb{C}$,

$$(a x_1[n] + b x_2[n]) * x_3[n]$$

$$= a(x_1[n] * x_3[n]) + b(x_2[n] * x_3[n]).$$

→ This is a slightly more advanced version of the distributive property (2B) that we had back on page 3.14.

Proof : $(a x_1[n] + b x_2[n]) * x_3[n]$

$$= \sum_{k=-\infty}^{\infty} (a x_1[k] + b x_2[k]) \, x_3[n-k]$$

$$= \sum_{k=-\infty}^{\infty} a x_1[k] x_3[n-k] + b x_2[k] x_3[n-k]$$

$$= a \sum_{k=-\infty}^{\infty} x_1[k] x_3[n-k] + b \sum_{k=-\infty}^{\infty} x_2[k] x_3[n-k]$$

$$= a(x_1[n] * x_3[n]) + b(x_2[n] * x_3[n])$$

NOTE: FACT 1 from page 3.16 ($x[n] * \delta[n] = x[n]$)

and FACT 2 from page 3.17 ($x[n] * \delta[n-5] = x[n-5]$)

are both really just special cases of FACT 3 from page 3.18: $x[n] * \delta[n-n_0] = x[n-n_0]$.

→ If we take $n_0 = 0$, then FACT 3 becomes $x[n] * \delta[n] = x[n]$

→ If we take $n_0 = 5$, then FACT 3 becomes $x[n] * \delta[n-5] = x[n-5]$.

⇒ So now let's use FACT 3 and FACT 4 to work the example convolution problem we had all the way back on page 3.15.

GIVEN: $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$

FIND: the signal $y[n] = x[n] * h[n]$

→

SOLUTION :

$$y[n] = x[n] * h[n]$$

$$= x[n] * \left(\tfrac{1}{2}\delta[n] - \tfrac{1}{2}\delta[n-1]\right)$$

$$= \tfrac{1}{2}\left(x[n] * \delta[n]\right) - \tfrac{1}{2}\left(x[n] * \delta[n-1]\right)$$

$$= \tfrac{1}{2}x[n] - \tfrac{1}{2}x[n-1]$$

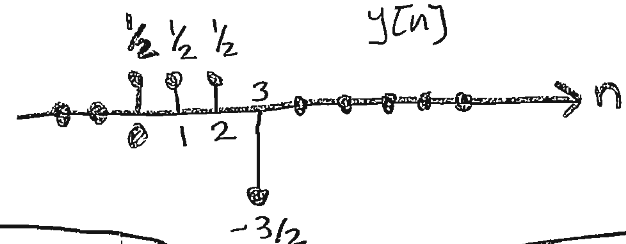
$$= \tfrac{1}{2}\Big(\underbrace{\delta[n] + 2\delta[n-1] + 3\delta[n-2]}_{x[n]}\Big)$$

$$- \tfrac{1}{2}\Big(\underbrace{\delta[n-1] + 2\delta[n-2] + 3\delta[n-3]}_{x[n-1]}\Big)$$

$$= \tfrac{1}{2}\delta[n] + \delta[n-1] + \tfrac{3}{2}\delta[n-2]$$
$$- \tfrac{1}{2}\delta[n-1] - \delta[n-2] - \tfrac{3}{2}\delta[n-3]$$

$$= \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{2}\delta[n-2] - \tfrac{3}{2}\delta[n-3]$$

$$\boxed{y[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{2}\delta[n-2] - \tfrac{3}{2}\delta[n-3]}$$



- Here's another example:

A discrete-time LTI system H has impulse response

$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2].$$

The system input is given by

$$x[n] = \delta[n+1] + 3\delta[n] + \delta[n-1].$$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

<u>FIND</u> the output signal $y[n]$.

<u>SOLUTION</u>

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= x[n] * \left(\delta[n] - 2\delta[n-1] + \delta[n-2]\right) \\
&= \left(x[n] * \delta[n]\right) - 2\left(x[n] * \delta[n-1]\right) + \left(x[n] * \delta[n-2]\right) \\
&= x[n] - 2x[n-1] + x[n-2] \\
&= \delta[n+1] + 3\delta[n] + \delta[n-1] \\
&\qquad - 2\delta[n] - 6\delta[n-1] - 2\delta[n-2] \\
&\qquad + \delta[n-1] + 3\delta[n-2] + \delta[n-3] \\
&= \delta[n+1] + \delta[n] - 4\delta[n-1] + \delta[n-2] + \delta[n-3]
\end{aligned}
$$

<u>Reminder</u> from page 3.15: this "simpler method" works when we have to convolve two signals that can both be written as short sums of the shifts of $\delta[n]$.

- But this simpler "convolution with deltas" method will **not** work at all if either of the two signals is infinite in length... for example, if $h[n] = (\frac{1}{2})^n u[n]$.

- It is also very inconvenient if either of the two signals is too long to be written as a short sum of the shifts of $\delta[n]$.

- For those types of cases, we need the more general method.

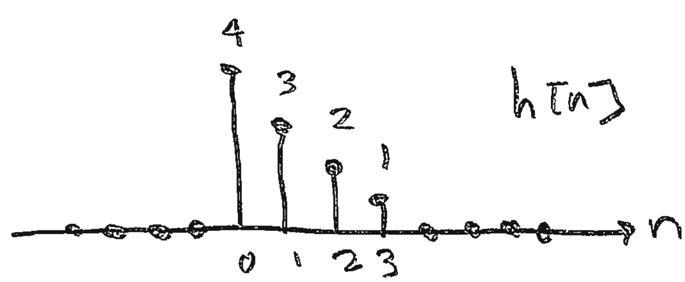
GENERAL METHOD FOR COMPUTING CONVOLUTIONS

- Some people call this method "graphical convolution"
  - The reason is that we will typically need to draw graphs of $x[k]$ and $h[n-k]$ when using this method.

- To understand how it works, lets start by considering a simple example:

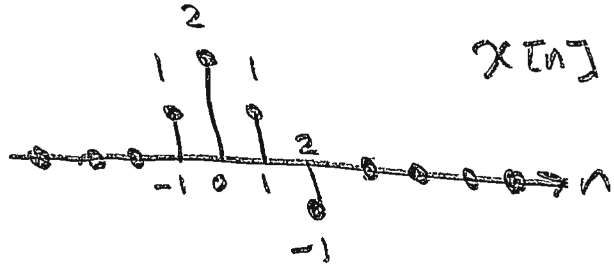
⇒ Let H be a discrete-time LTI system with impulse response

$$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$



- Let the input signal be

$$x[n] = \delta[n+1] + 2\delta[n] + \delta[n-1] - \delta[n-2]$$



→ We could use the simpler "convolution with deltas" method for this problem.

→ But we're not going to do that.

→ Instead, we're going to analyze this example in more detail to develop a greater intuitive understanding of how convolution works.

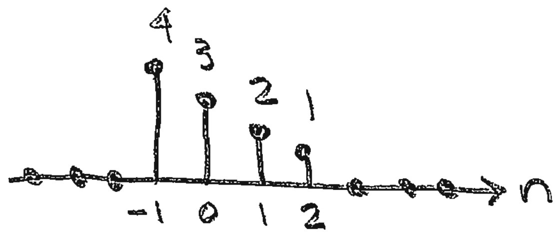
- From linearity, we have that the system output $y[n]$ must be given by

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\{\delta[n+1] + 2\delta[n] + \delta[n-1] - \delta[n-2]\} \\ &= H\{\delta[n+1]\} + 2H\{\delta[n]\} + H\{\delta[n-1]\} - H\{\delta[n-2]\} \end{aligned}$$

- And since H is time invariant, we have that

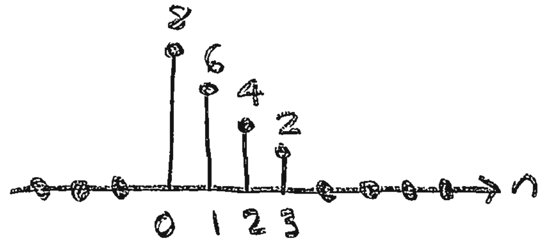
$$y[n] = \underbrace{h[n+1]}_{\text{response due to input term } \delta[n+1]} + \underbrace{2h[n]}_{\text{response due to input term } 2\delta[n]} + \underbrace{h[n-1]}_{\text{response due to input term } \delta[n-1]} - \underbrace{h[n-2]}_{\text{response due to input term } -\delta[n-2]}$$

- So we see that the input term $\delta[n+1]$ arrives at time $n=-1$ and causes a response $h[n+1]$ to start coming out at time $n=-1$:



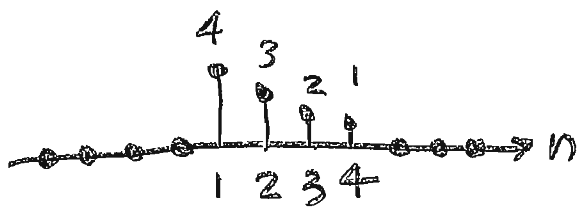
response due to input at $n=-1$

- The input term $2\delta[n]$ arrives at time $n=0$ and causes a response $2h[n]$ to start coming out at time $n=0$:



response due to input at $n=0$

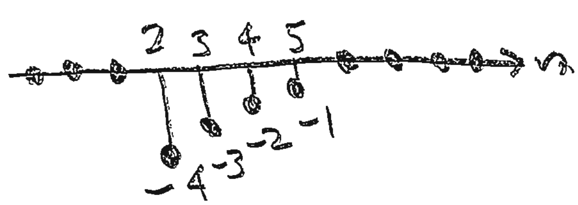
- The input term $\delta[n-1]$ arrives at time $n=1$ and causes a response $h[n-1]$ to start coming out at time $n=1$:



response due to input at $n=1$

- And finally the input term $-\delta[n-2]$ arrives at time $n=2$ and causes a response $-h[n-2]$ to start coming out at time $n=2$:

response due to input at $n=2$



- When the signal $x[n]$ is input to the system H, it causes all of these individual responses to come out simultaneously.

- The total response $y[n]$ is the sum of these individual responses.

- Recalling that

$$h[0] = 4 \qquad x[-1] = 1$$
$$h[1] = 3 \qquad x[0] = 2$$
$$h[2] = 2 \quad \text{and} \quad x[1] = 1$$
$$h[3] = 1 \qquad x[2] = -1$$

$\longrightarrow$ We can tabulate the individual responses as shown on the next page...

Input Terms		OUTPUT TERMS $n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$n=-1$	$x[-1]=1$	$x[-1]h[0]=4$	$x[-1]h[1]=3$	$x[-1]h[2]=2$	$x[-1]h[3]=1$	0	0	0
$n=0$	$x[0]=2$	0	$x[0]h[0]=8$	$x[0]h[1]=6$	$x[0]h[2]=4$	$x[0]h[3]=2$	0	0
$n=1$	$x[1]=1$	0	0	$x[1]h[0]=4$	$x[1]h[1]=3$	$x[1]h[2]=2$	$x[1]h[3]=1$	0
$n=2$	$x[2]=-1$	0	0	0	$x[2]h[0]=-4$	$x[2]h[1]=-3$	$x[2]h[2]=-2$	$x[2]h[3]=-1$

At $n=2$,

$x[-1]h[3]$ is still coming out because of the input at $n=-1$
$x[0]h[2]$ is still coming out because of the input at $n=0$
$x[1]h[1]$ is still coming out because of the input at $n=1$
$x[2]h[0]$ just started coming out because of the input at $n=2$

All Together, $y[2] = x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0]$

$$= \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

$\longrightarrow$

- To find the number y[2], we have to add up what is coming out at time n=2 due to each input term. We get:

The numbers $\longmapsto$ $\quad 1 \cdot 1 \quad + \quad 2 \cdot 2 \quad + \quad 1 \cdot 3 \quad + \quad (-1)\ 4$

(*) $\quad y[2] = x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0]$

Input term x[-1] was the first to arrive... so the index of his "x" is -1... the smallest. And his "h" has been coming out the longest. So his "h" has the largest index (3).

Input term x[2] was the last to arrive... so his "x" has the largest index (2). But his "h" just started coming out... so he's only an h[0].

$\Rightarrow$ For these individual contributions to the number y[2], we see that the indices of the "x" and "h" go in opposite order

$\longrightarrow$

- More generally, the number $y[2]$ is given by

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] \qquad (**)$$

$\Rightarrow$ This loop adds up the contribution to $y[2]$ from <u>every</u> one of the input terms $x[k]$.

- For example, on the $k=-5$ term of the sum, we get

$$x[-5]h[2--5]$$
$$= x[-5]h[7]$$

$\rightarrow$ Because the input term $x[-5]$ caused a weighted "h" to start coming out at time $n=-5$, here at time $n=2$ he is up to $x[-5]h[\underline{\underline{7}}]$.

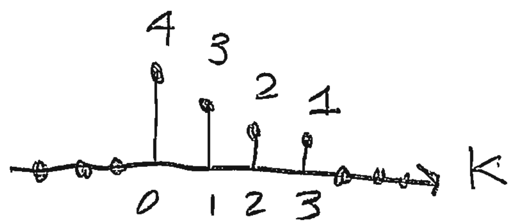
- Now, let's make the graph of $h[2-k]$ as a function of the <u>loop</u> <u>counter</u> <u>k</u> ...

- We have $h[2-k] = h[-k - -2]$

  → There is a <u>shift</u> by -2

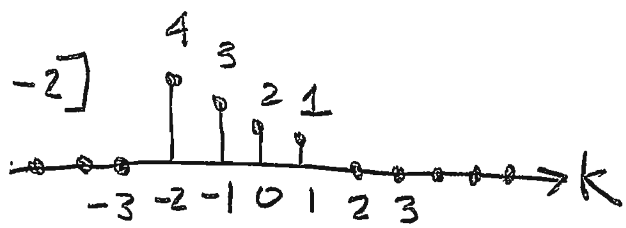
  → There is a <u>scale</u> by -1

$h[k]$



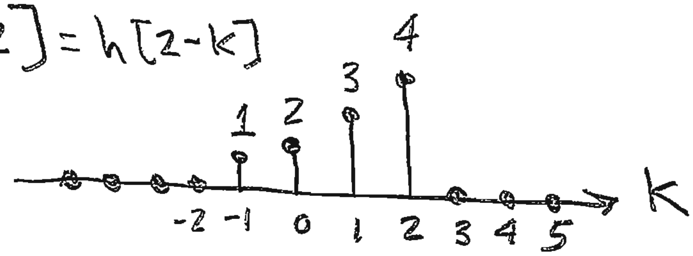
- Do the <u>shift</u> <u>first</u>
  (see page 2.100)

$h[k - -2]$

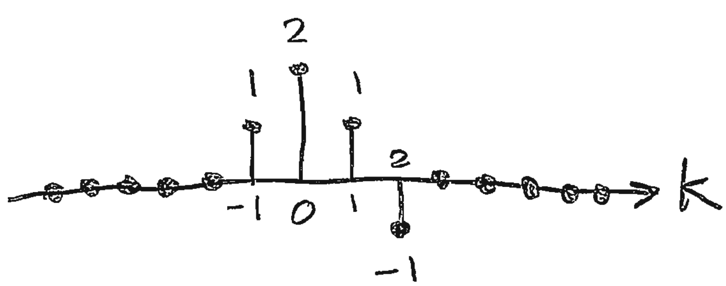


- Then do the scale

$h[-k - -2] = h[2-k]$



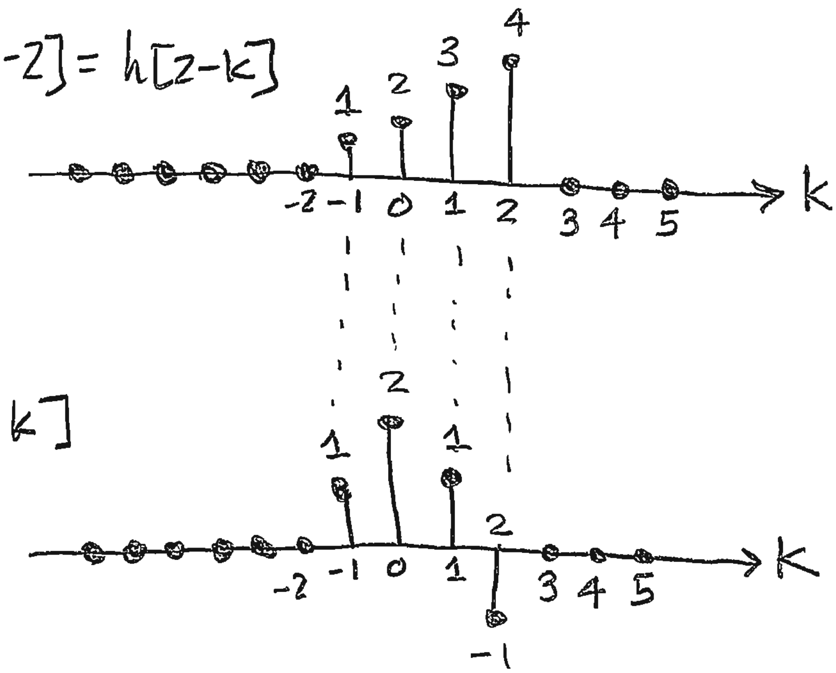
- And here is the graph of $x[k]$: (see page 3.25)



- We now use these graphs from page 3.31 to compute the number $y[2]$ as given in equation $(**)$ on page 3.30:

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k]$$

$h[-k--2] = h[2-k]$



$x[k]$

$$y[2] = \underbrace{1}_{x[-1]}\cdot\underbrace{1}_{h[3]} + \underbrace{2}_{x[0]}\cdot\underbrace{2}_{h[2]} + \underbrace{1}_{x[1]}\cdot\underbrace{3}_{h[1]} + \underbrace{(-1)}_{x[2]}\cdot\underbrace{4}_{h[0]} = \underline{\underline{4}}$$

$\Rightarrow$ Compare this to equation $(*)$ on page 3.29...

$\Rightarrow$ They are the <u>same</u>!!

$\Rightarrow$ This exactly adds up what is coming out of the system at time $n=2$ due to <u>each</u> term of the input signal $x[n]$.

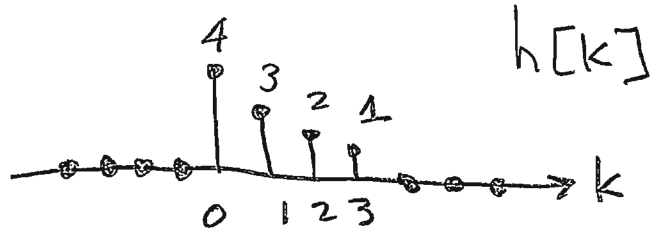
- Now let's see how this works if we compute the convolution the "other way"... i.e. we compute it as (see pages 3.10 and 3.11):

$$y[n] = x[n] * h[n]$$
$$= h[n] * x[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

- Plugging in $n=2$, we get

$$y[2] = \sum_{k=-\infty}^{\infty} h[k]\, x[2-k]$$

→ here is the graph of $h[k]$:

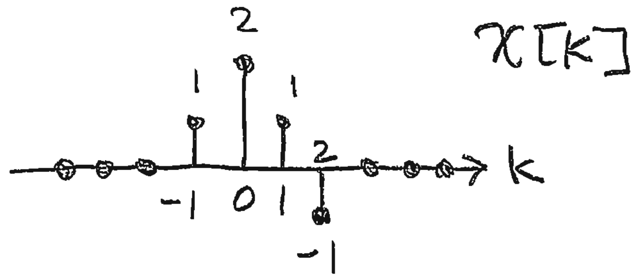


- For $x[2-k]$, we think of it as $x[-k - -2]$

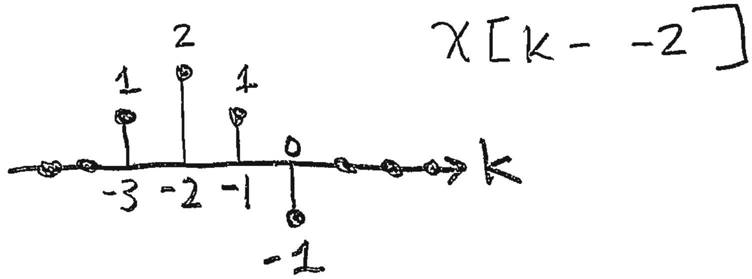
- why? Because we are working the do loop on "k"... so we need the graphs as a function of $\underline{\underline{k}}$ ⟶

- So, for $x[-k - -2]$ we have a shift by -2 and a scale by -1.

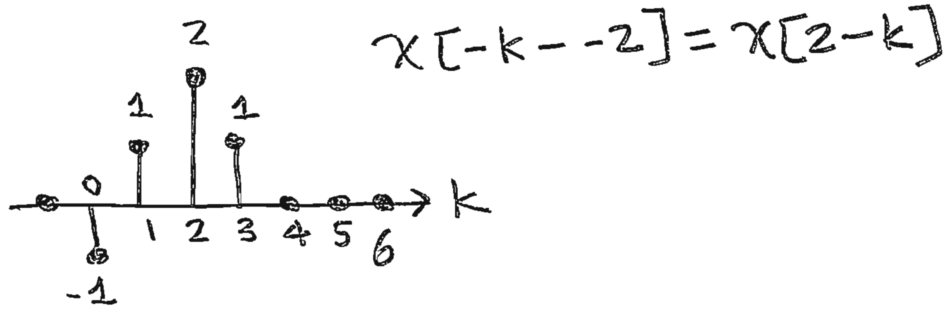
- Following the rule from pages 2.21 and 2.100, we do the shift first, then the scale:



SHIFT by -2 :

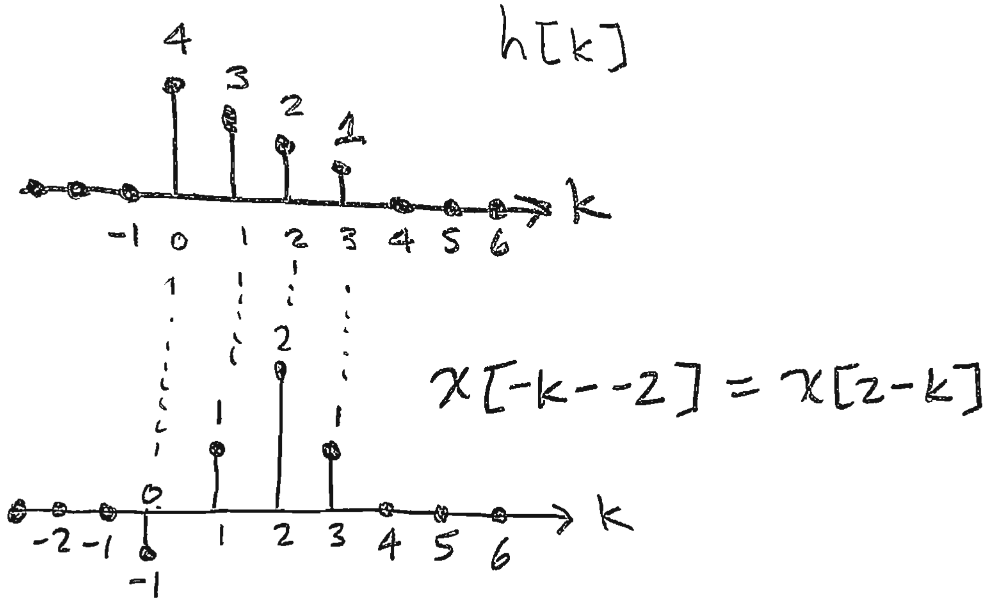


SCALE by -1 :



– So now let's use the graphs on pages 3.33 & 3.34 to compute the number $y[2]$ according to

$$y[2] = \sum_{k=-\infty}^{\infty} h[k]\, x[2-k]$$



$$y[2] = \underset{x[2]}{(-1)}\cdot\underset{h[0]}{4} + \underset{x[1]}{1}\cdot\underset{h[1]}{3} + \underset{x[0]}{2}\cdot\underset{h[2]}{2} + \underset{x[-1]}{1}\cdot\underset{h[3]}{1} = \underline{\underline{4}}$$

→ It's <u>exactly</u> the same thing we got before on p. 3.32 <u>and</u> p. 3.29

→ It adds up what is coming out of the system at $n=2$ due to each term of the input signal $x[n]$.

⇒ But compared to how we did it on p. 3.32, this way adds up the individual terms in backwards order.

SUMMARY :

- On pages 3.24 and 3.25, we had an LTI system H with impulse response

$$h[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

- The input was (page 3.25)

$$x[n] = \delta[n+1] + 2\delta[n] + \delta[n-1] - \delta[n-2]$$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

- On pages 3.30 – 3.32, we computed the number $y[2]$ using the formula

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\Rightarrow$ we got $y[2] = 4$.

- On pages 3.33 – 3.35, we computed the same number $y[2]$ using the formula

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$\Rightarrow$ we again got $y[2] = 4$ ✓

- In general, to find the <u>whole</u> signal $y[n]$, we have to do this for <u>every</u> <u>n</u>.

  ⇒ This is called <u>graphical</u> <u>convolution</u>.

- However, there is <u>good</u> <u>news</u>:

  - usually we don't have to do every n individually.

  - Usually we can do the n's in <u>batches</u>.

    ⇒ In other words, usually we can find just <u>a</u> <u>few</u> expressions for $y[n]$, where each expression is good for a whole batch of n's.

  - In most books, these <u>expressions</u>, each one good for a whole batch of n's, are called <u>"regions"</u> or <u>"cases"</u>.

☆☆ HOW TO PERFORM GRAPHICAL CONVOLUTION ON A TEST WITHOUT MAKING MISTAKES

Given: H is a discrete-time LTI system with impulse response $h[n]$ (given) and input signal $x[n]$ (given).

Find: the output signal $y[n]$.

① choose which signal gets the "k" and which one gets the "n-k", i.e.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

OR

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

→ you can always get the right answer either way. But one way might be slightly easier to work out.

→ Usually it's best to pick the one with the simpler expression to get the "n-k".

⇒ For the following steps, assume we picked the first way... i.e., $x[k]$ and $h[n-k]$.

(2) Make a graph of $x[k]$.

(3) Make a graph of $h[n-k]$ ... as a function of $\underline{\underline{k}}$.

→ To avoid making mistakes, always do this in <u>three</u> <u>steps</u>:

(3A) Graph $h[k]$

(3B) Shift the graph in (3A) to the right by $-n$ to obtain the graph of
$$h[k-(-n)] = h[n+k]$$

(3C) Flip the graph in (3B) with respect to the k-axis to obtain the graph of
$$h[-k-(-n)] = h[n-k].$$

(4) To find $y[n]$, you must multiply the graphs in (2) and (3C) pointwise and then add up the product graph from $k=-\infty$ to $k=+\infty$.

⇒ The product graph generally depends on "n".

→ you will see that different batches of n's result in different limits for the sum.

→ These different batches are the so-called "cases" or "regions"!

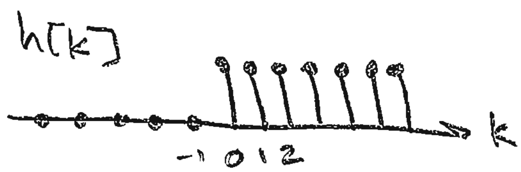
EX: $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$ (for example, α could be $1/2$)

$h[n] = u[n]$

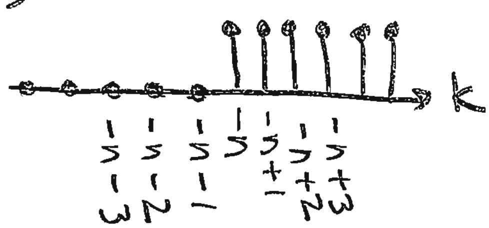
→ since $x[n]$ has the more complicated expression, we will put the "k" on x and the "n-k" on h.

- Graph $x[k]$:

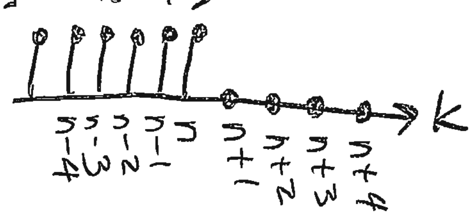
- Graph $h[n-k]$ in three steps:



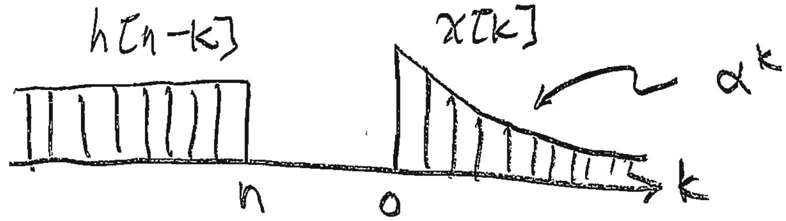
$h[k - -n] = h[n+k]$



$h[-k - -n] = h[n-k]$

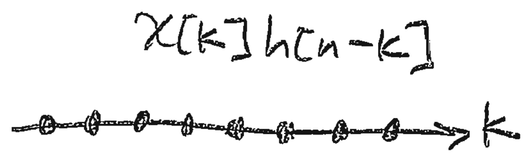


- when $n<0$, we have:



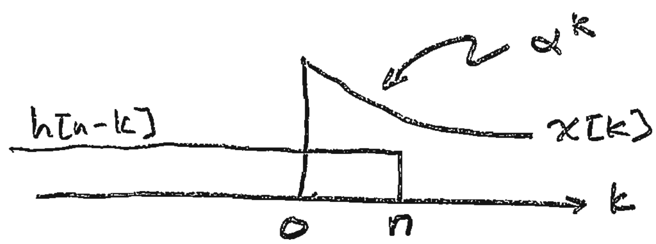
- when $n<0$, for every k either $x[k]=0$ or $h[n-k]=0$.

- So the product graph is all zeros when $n<0$.



- So $y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$ when $n<0$.

- when $n \geq 0$, we have:

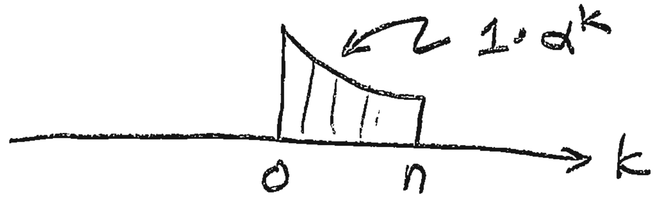


→ In this case $(n \geq 0)$, the product graph is nonzero from $k=0$ to $k=n$

→ In this range of k's, we have $x[k] = \alpha^k$ and $h[n-k] = 1$

→

- So, when $n \geq 0$ the product graph looks like this:



- We get

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k$$

$$= \frac{\alpha^0 - \alpha^{n+1}}{1-\alpha} \qquad \text{(sum formula)}$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha} .$$

- For this problem, these two cases cover <u>all</u> of the n's.

- So, putting them together, the answer is:

$$y[n] = \begin{cases} 0 , & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha} , & n \geq 0 \end{cases}$$

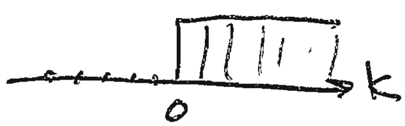
$$= \frac{1-\alpha^{n+1}}{1-\alpha}\, u[n] \qquad ///$$

- Now let's work that same problem the other way... to show that we will get the same answer.
- And let's write it the way we would on a test!!

$$x[n] = \alpha^n u[n] \quad , \quad h[n] = u[n]$$

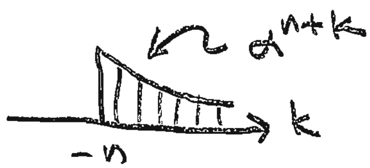
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$h[k]$

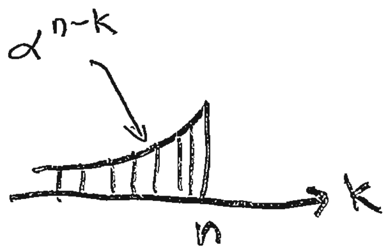


$x[k]$

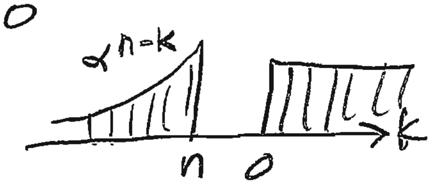
$x[k \to -n] = x[n+k]$



$x[-k \to -n] = x[n-k]$

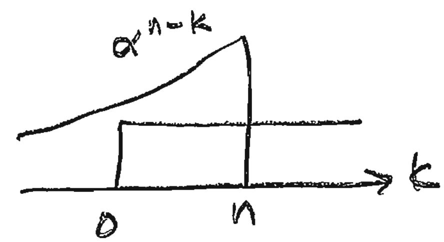


Case I) $n < 0$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq 0$



$$y[n] = \sum_{k=0}^{n} \alpha^{n-k} \cdot 1 = \sum_{k=0}^{n} \alpha^n \alpha^{-k}$$

$$= \alpha^n \sum_{k=0}^{n} \alpha^{-k} = \alpha^n \sum_{k=0}^{n} \left(\frac{1}{\alpha}\right)^k$$

$$= \alpha^n \frac{\left(\frac{1}{\alpha}\right)^0 - \left(\frac{1}{\alpha}\right)^{n+1}}{1 - \frac{1}{\alpha}}$$

$$= \alpha^n \frac{1 - \alpha^{-n-1}}{1 - \alpha^{-1}} \cdot \underbrace{\frac{\alpha}{\alpha}}_{\text{one}}$$

$$= \alpha^n \frac{\alpha - \alpha^{-n}}{\alpha - 1} = \frac{\alpha^{n+1} - 1}{\alpha - 1} \cdot \underbrace{\frac{-1}{-1}}_{\text{one}}$$

$$= \frac{1 - \alpha^{n+1}}{1 - \alpha} \longrightarrow$$

<u>All Together</u> :

$$y[n] = \begin{cases} 0, & n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & n \geq 0 \end{cases}$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha} u[n] \quad ///$$

$\Rightarrow$ SAME answer as we got doing it the first way.

<u>EX</u> :

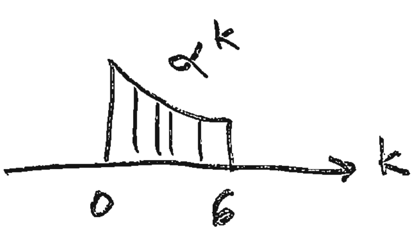
$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-5]$$

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases} = \alpha^n \{u[n] - u[n-7]\}$$

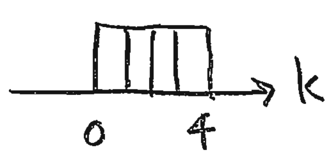
$0 < \alpha < 1$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

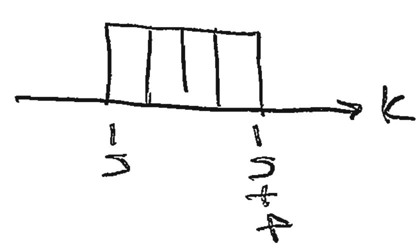
$h[k]$



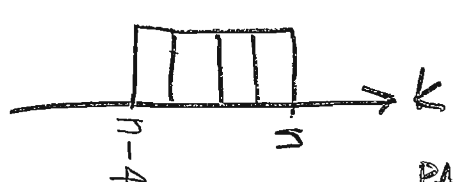
$x[k]$



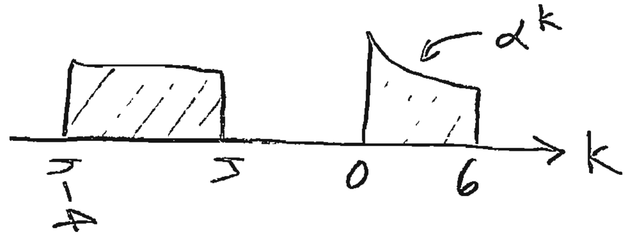
$x[k--n] = x[n+k]$



$x[-k--n] = x[n-k]$



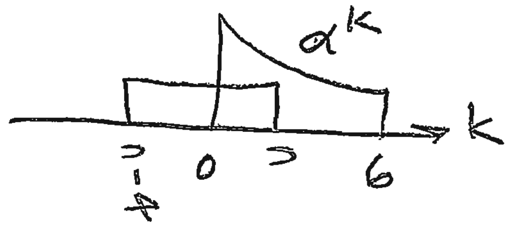
Case I) $n<0$;



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq 0$ and $n-4<0$: $0 \leq n < 4$

- The product graph has something in it from $k=0$ to $k=n$



→ In other words, the product graph is "turned on" from $k=0$ to $k=n$.

→ $h[k]$ turns it on at $n=0$

→ $x[n-k]$ turns it off <u>after</u> $k=n$

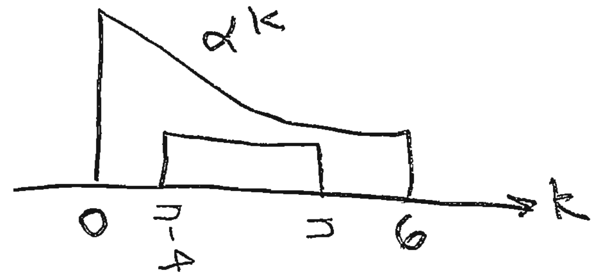
<u>NOTE</u> : any time there is a <u>change</u> in who turns it on and who turns it off, it will signal the start of a <u>new</u> region or "case".

- here in case II, we have:

$$y[n] = \sum_{k=0}^{n} \alpha^k \cdot 1 = \sum_{k=0}^{n} \alpha^k = \frac{\alpha^0 - \alpha^{n+1}}{1-\alpha}$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha}$$

Case III) $n-4 \geq 0$ and $n < 6$: $4 \leq n < 6$:

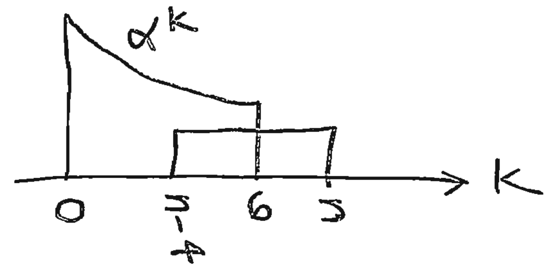


- In this case, $x[n-k]$ turns the product graph on at $k=n-4$.

- And $x[n-k]$ also turns the product graph off after $k=n$ (i.e., the product graph is zero starting at $k=n+1$).

$$y[n] = \sum_{k=n-4}^{n} \alpha^k = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

Case IV) $n \geq 6$ and $n-4 < 7$: $6 \leq n < 11$



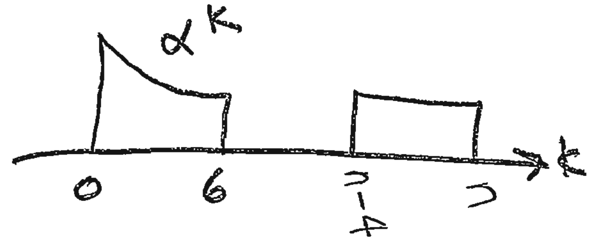
- In this case, $x[n-k]$ turns the product graph on at $k=n-4$
- $h[k]$ turns the product graph off for $k>6$

$$y[n] = \sum_{k=n-4}^{6} \alpha^k = \frac{\alpha^{n-4} - \alpha^7}{1-\alpha}$$

$\longrightarrow$

Case V) $n-4 \geq 7$: $n \geq 11$:

→ The product graph is <u>all</u> <u>zeros</u>



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

All Together :

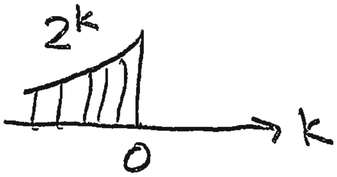
$$y[n] = \begin{cases} 0 & , \quad n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & , \quad 0 \leq n < 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & , \quad 4 \leq n < 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha} & , \quad 6 \leq n < 11 \\ 0 & , \quad n \geq 11 \end{cases}$$

EX : $x[n] = 2^n u[-n]$

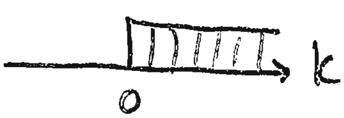
$h[n] = u[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$x[k]$

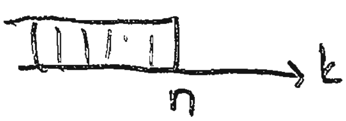


$h[k]$

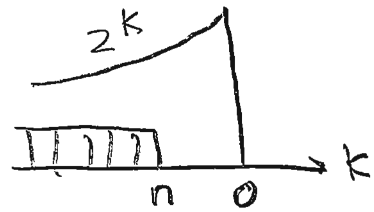


$h[k--n] = h[n+k]$

$h[-k--n] = h[n-k]$



case I) $n < 0$

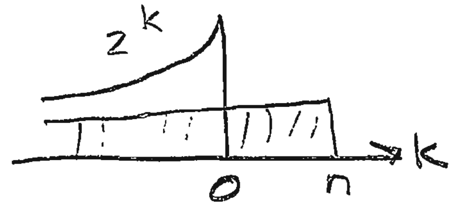


$$y[n] = \sum_{k=-\infty}^{n} 2^k$$

$$= \lim_{A\to\infty} \sum_{k=-A}^{n} 2^k = \lim_{A\to\infty} \frac{2^{-A} - 2^{n+1}}{1-2}$$

$$= \lim_{A\to\infty} \frac{(\frac{1}{2})^A - 2^{n+1}}{-1} = \frac{0 - 2^{n+1}}{-1} = 2^{n+1}$$

case II) $n \geq 0$



$$y[n] = \sum_{k=-\infty}^{0} 2^k$$

$$= \lim_{A\to\infty} \sum_{k=-A}^{0} 2^k = \lim_{A\to\infty} \frac{2^{-A} - 2^1}{1-2}$$

$$= \lim_{A\to\infty} \frac{(\frac{1}{2})^A - 2}{-1} = \frac{0-2}{-1} = 2$$

All Together :

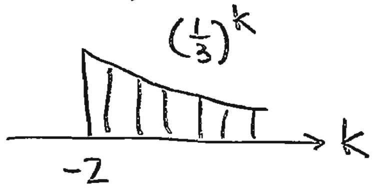
$$y[n] = \begin{cases} 2^{n+1}, & n < 0 \\ 2, & n \geq 0 \end{cases}$$

EX: $x[n] = \left(\frac{1}{3}\right)^n u[n+2]$

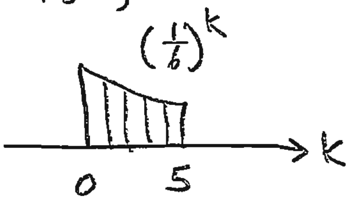
$$h[n] = \left(\frac{1}{6}\right)^n \{u[n] - u[n-6]\} = \begin{cases} \left(\frac{1}{6}\right)^n, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

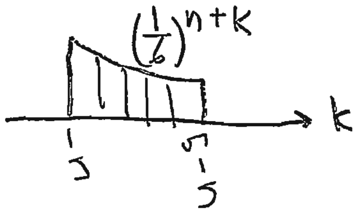
$x[k]$



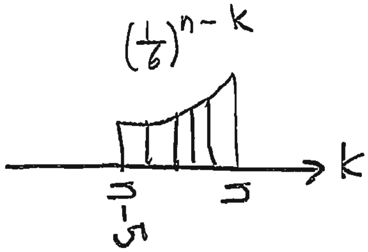
$h[k]$



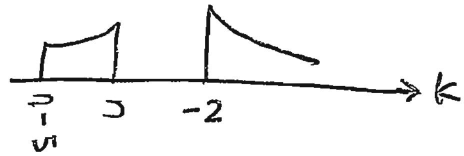
$h[k - -n] = h[n+k]$



$h[-k - -n] = h[n-k]$



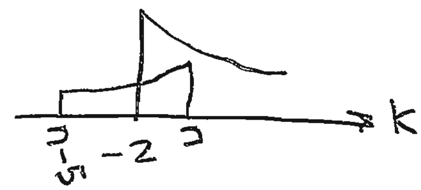
Case I) $n < -2$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq -2$ and $n-5 < -2$: $-2 \le n < 3$

$$y[n] = \sum_{k=-2}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k}$$



$$= \sum_{k=-2}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^n \left(\tfrac{1}{6}\right)^{-k}$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(\tfrac{1}{3}\right)^k 6^k = \left(\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} 2^k$$

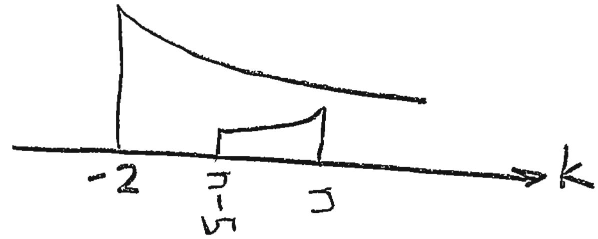
$$= \left(\tfrac{1}{6}\right)^n \frac{2^{-2} - 2^{n+1}}{1-2} = \left(\tfrac{1}{6}\right)^n \frac{\frac{1}{4} - 2 \cdot 2^n}{-1}$$

$$= \left(\tfrac{1}{6}\right)^n \frac{2 \cdot 2^n - \frac{1}{4}}{1} = 2\left(\tfrac{1}{6}\right)^n 2^n - \tfrac{1}{4}\left(\tfrac{1}{6}\right)^n$$

$$= 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{4}\left(\tfrac{1}{6}\right)^n$$

$\longrightarrow$

Case III) $n-5 \geq -2 \ : \quad n \geq 3$



$$y[n] = \sum_{k=n-5}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^{n-k}$$

$$= \sum_{k=n-5}^{n} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{6}\right)^n \left(\tfrac{1}{6}\right)^{-k}$$

$$= \left(\tfrac{1}{6}\right)^n \sum_{k=n-5}^{n} \left(\tfrac{1}{3}\right)^k 6^k = \left(\tfrac{1}{6}\right)^n \sum_{k=n-5}^{n} 2^k$$

$$= \left(\tfrac{1}{6}\right)^n \frac{2^{n-5} - 2^{n+1}}{1-2} = \left(\tfrac{1}{6}\right)^n \frac{2^n 2^{-5} - 2 \cdot 2^n}{-1}$$

$$= \left(\tfrac{1}{6}\right)^n \left[2 \cdot 2^n - \left(\tfrac{1}{32}\right) 2^n\right] = \left(\tfrac{1}{6}\right)^n 2^n \left[2 - \tfrac{1}{32}\right]$$

$$= \left(\tfrac{2}{6}\right)^n \left[\tfrac{64}{32} - \tfrac{1}{32}\right] = \frac{63}{32}\left(\tfrac{1}{3}\right)^n$$

All Together:

$$y[n] = \begin{cases} 0 & , \quad n < -2 \\ 2\left(\tfrac{1}{3}\right)^n - \tfrac{1}{4}\left(\tfrac{1}{6}\right)^n & , \quad -2 \leq n < 3 \\ \tfrac{63}{32}\left(\tfrac{1}{3}\right)^n & , \quad n \geq 3 \end{cases}$$

MORE ON HOW TO FIND THE "REGIONS" OR "CASES"

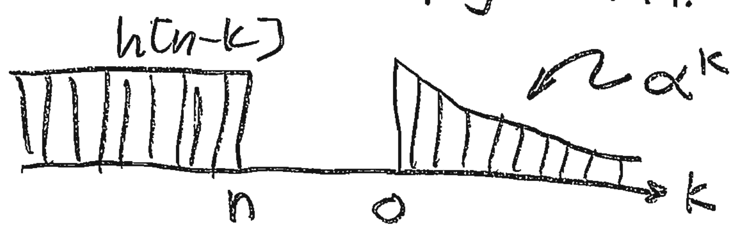
$$x[n] \rightarrow \boxed{H} \rightarrow y[n] = x[n] * h[n]$$

- To solve for the output signal, we have to say what the number $y[n]$ is for every n.
- For <u>each</u> n, the number $y[n]$ is given by the sum of the product graph:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

- For any particular problem, $x[n]$ and $h[n]$ will be given... and the product graph will look different for different batches of n's.

$\Rightarrow$ More specifically, there may be batches of n's where the product graph has nothing in it and $y[n]=0$ ... as we saw, e.g., on page 3.41.



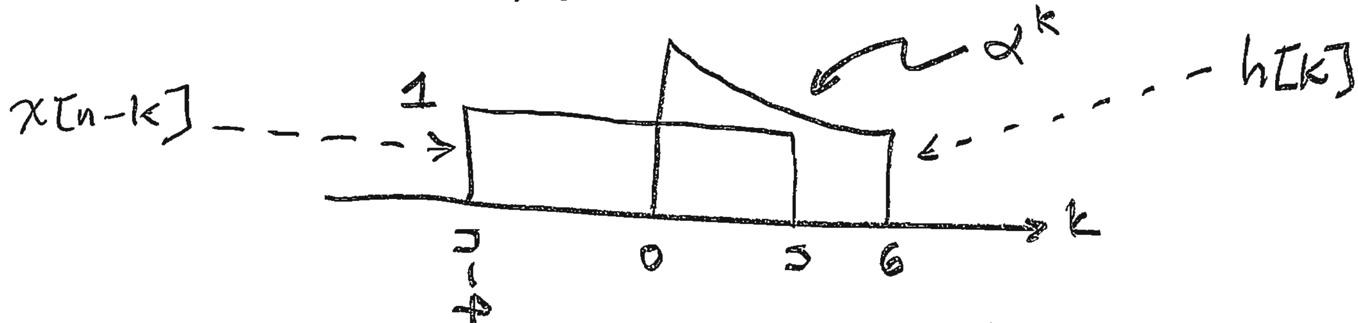
$\longrightarrow$ product graph is <u>all</u> <u>zeros</u>.

⇒ For other batches of n's, the product graph will have something in it...

⇒ And to find $y[n]$ for these n's, we have to add up what's in the product graph.

⇒ The limits of the sum are determined by <u>who</u> turns the product graph <u>on</u> ... and <u>who</u> turns the product graph <u>off</u>.

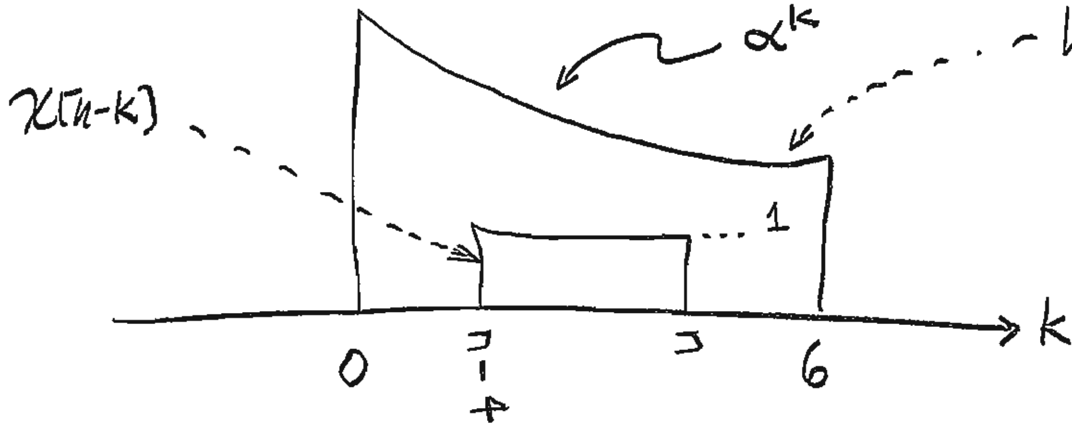
<u>EX</u>: case II from page 3.45:



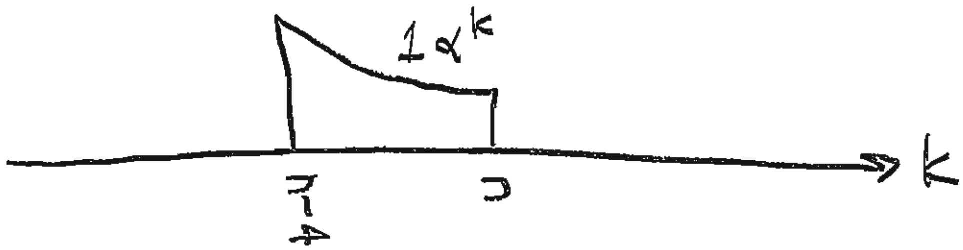
→ $h[k]$ turns it <u>on</u> at $k=0$.

→ $x[n-k]$ turns it <u>off</u> after $k=n$.

EX: Case III from page 3.46:



product graph:



→ $x[n-k]$ turns it <u>on</u> at $k=n-4$

→ And $x[n-k]$ <u>also</u> turns it <u>off</u> after $k=n$.

— To find $y[n]$, you must consider <u>all</u> of the n's... you must find $y[n]$ $\forall\ n \in \mathbb{Z}$.

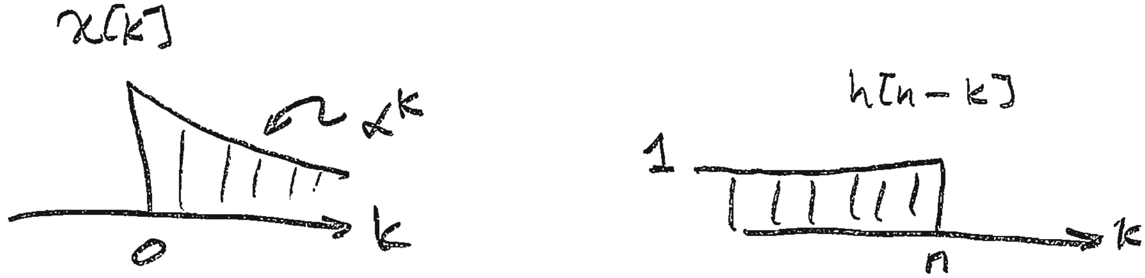
— Start by considering gigantically negative n's... like $n=-10^{40}$. Figure out what the product graph looks like... and what are the limits of the sum (i.e., who turns it <u>on</u> and who turns it <u>off</u>).

⟶

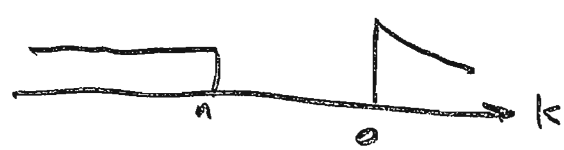
- Now consider larger and larger values of n.
  - $\Rightarrow$ Any time there is a change in who turns it <u>on</u> and/or who turns it <u>off</u>, it changes the limits of the sum...
    - $\Rightarrow$ This means that you have hit the start of a <u>new</u> <u>region</u>.
- You must continue in this way until you have done <u>all</u> <u>of</u> <u>the</u> <u>n's</u>.

- Let's re-examine the first example from page 3.40 and pay more attention to how the regions are determined.

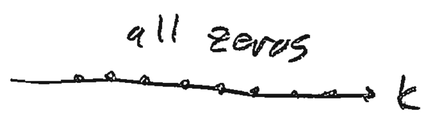
- We've got:



→ when n is gigantically negative, these graphs don't overlap... and so the product graph is <u>all</u> <u>zeros</u> for the gigantically negative n's:



product graph.



→ So this is our first "region": <u>nobody</u> turns it <u>on</u> and <u>nobody</u> turns it <u>off</u>..

⇒ Case I) $y[n] = 0$

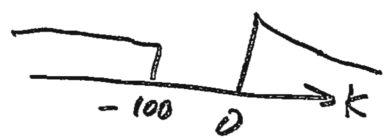
⟶

- How long is this good for?

→ In other words, what n's are included in case I ?

→ It starts at $n = -\infty$.

→ When $n = -100$, we've still got

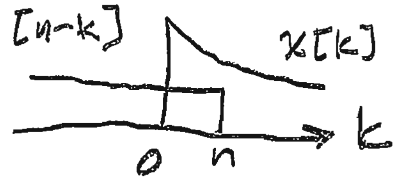


- nobody turns it on, nobody turns it off

→ when $n = -1$, we've still got

-nobody turns it on, nobody turns it off

⇒ but when $n = 0, 1, 2, 3 \ldots$ things change



→ starting with $n=0$, the product graph has something in it.

→ Because, starting with $n=0$,

⇒ $x[k]$ turns it on at $k=0$.

⇒ $h[n-k]$ turns it off after $k=n$.

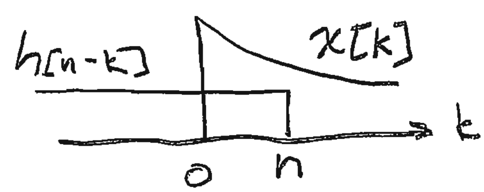
⇒ So the <u>last</u> n that's included in case I (the "nobody turns it on, nobody turn it off" case) is $n = -1$.

⇒ Case I is for n's going $-\infty$ to -1.

⇒ Case I is $\underline{\underline{n < 0}}$.

- - - - - - - - - - - - - - - -

— Starting with $n = 0$, we are in a <u>new</u> case or "region" (Case II) where $x[k]$ turns it <u>on</u> and $h[n-k]$ turns it <u>off</u>:



product graph:

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

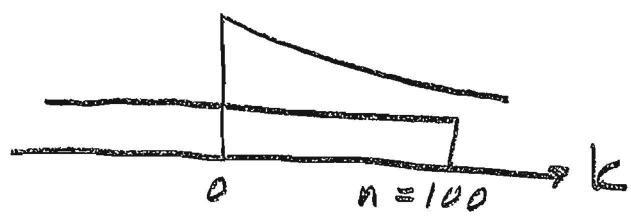
$$= \sum_{k=0}^{n} \alpha^{k}$$

- How long is this good for?

→ In other words, case II starts with $n=0$, but how far does it go?

⇒ case II is where $x[k]$ turns the product graph on at $k=0$ and $h[n-k]$ turns the product graph off after $k=n$.

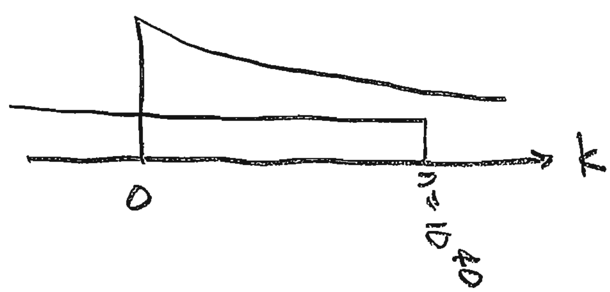
- when $n=100$, we have



⇒ Still the same

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

- when $n=10^{40}$, we have



⇒ Still the same

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

- So Case II is good for <u>all</u> the rest of the n's.

⇒ We've got:

case I) $n < 0$

case II) $n \geq 0$

⇒ This takes care of <u>all</u> the n's.

⇒ So there are only two cases (or regions) in this problem.

- On page 3.42, we got the answer for $y[n]$ as

$$y[n] = \begin{cases} 0 & , \ n < 0 \quad \leftarrow \text{case I} \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & , \ n \geq 0 \quad \leftarrow \text{case II} \end{cases}$$

⇒ Notice that our answer says what $y[n]$ is for <u>all</u> of the n's. ★★★

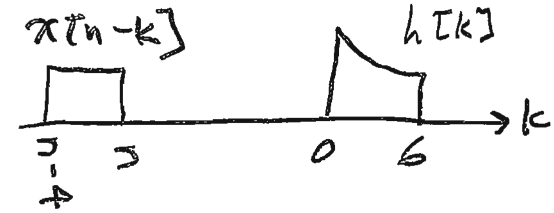
⇒ Your answer must include cases that cover <u>all</u> of the n's.

⇒ In other words, <u>every</u> <u>n</u> must be included in one of your cases. ★★★★

- Now let's take another look at the example that started on page 3.44... and pay special attention to how we determine the "regions."

- At the bottom of page 3.44, we got

→ Start with the gigantically negative n's... like $n = -10^{40}$:



→ The product graph is <u>all</u> <u>zeros</u>.

→ $y[n] = 0$

→ As in the last example, this case is for "nobody turns it on, nobody turns it off."

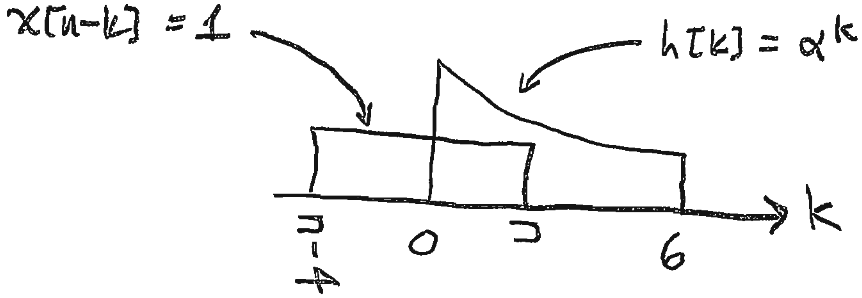
→ This is good for $n = -10^{40}$, $n = -100$, and $n = -1$.

→ But, at $n = 0$ things change!

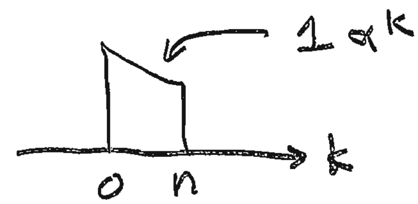
⇒ So: Case I) $n < 0$ ///

→

- Starting at $n=0$, we get into a new case where the product graph has something in it.



product graph :



→ $h[k]$ turns the product graph <u>on</u> at $k=0$.

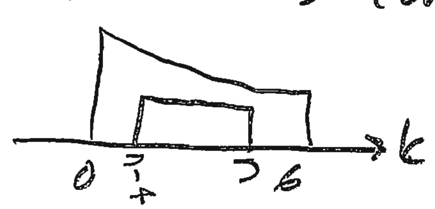
→ $x[n-k]$ turns the product graph <u>off</u> after $k=n$.

⇒ we've got

$$y[n] = \sum_{k=0}^{n} \alpha^k \qquad (\text{case II})$$

→ *How long is this good for?*

- Well, when $n-4=0$ (i.e., when $n=4$), things change!
- Starting at $n-4=0$, we get that $x[n-k]$ turns the product graph <u>on</u> and $x[n-k]$ turns the product graph <u>off</u> :



– So, starting at $n=4$ we are in a <u>new</u> region (case III).

– So case II is for: $n \geq 0$ <u>and</u> $n-4 < 0$

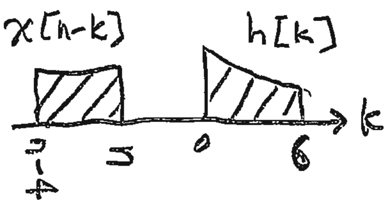
$\Rightarrow 0 \leq n < 4$ ///

$\Rightarrow$ And case III starts with $n=4$.

– We have to continue in this way until we have covered <u>all</u> of the n's.

$\Rightarrow$ For this problem, we get the following cases:

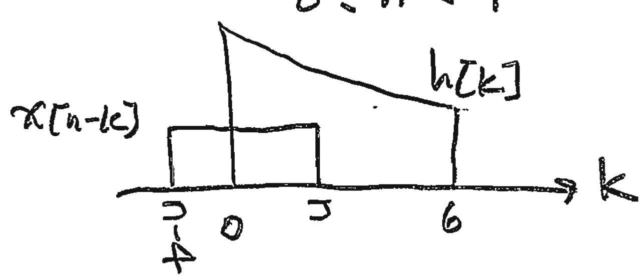
Case I) $n<0$:



→ Nobody turns the product graph on, nobody turns the product graph off.

Case II) $n \geq 0$ and $n-4<0$

$0 \leq n < 4$



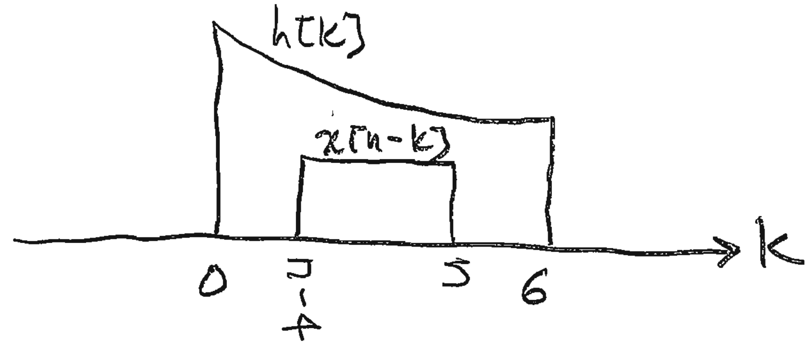
→ $h[k]$ turns the product graph <u>on</u> at $k=0$.

→ $x[n-k]$ turns the product graph <u>off</u> after $k=n$.

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

⟶

Case III) $n-4 \geq 0$ and $n<6$: $4 \leq n < 6$

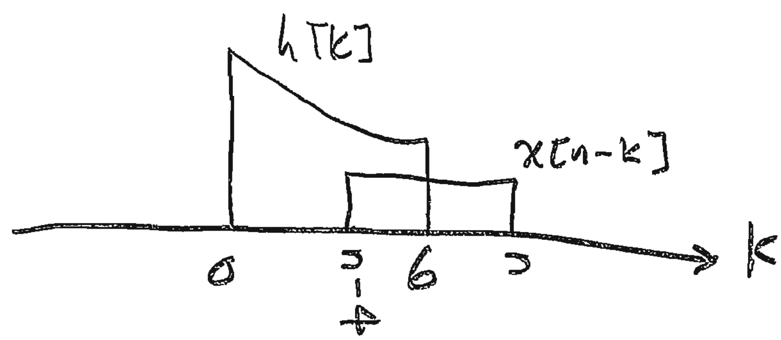


→ $x[n-k]$ turns the product graph <u>on</u> at $k=n-4$.

→ $x[n-k]$ turns the product graph <u>off</u> after $k=n$.

$$y[n] = \sum_{k=n-4}^{n} \alpha^k$$

Case IV) $n \geq 6$ and $\underbrace{n-4 < \underline{\underline{7}}}_{\text{Same as } n-4 \leq 6}$: $6 \leq n < 11$



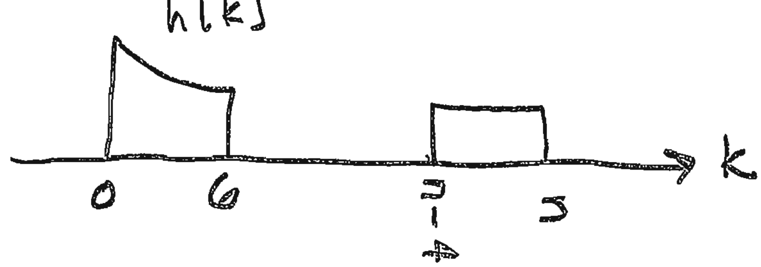
– $x[n-k]$ turns the product graph <u>on</u> at $k=n-4$

– $h[k]$ turns the product graph <u>off</u> after $k=6$

$$y[n] = \sum_{k=n-4}^{6} \alpha^k$$

⟶

Case V) $n-4 \geq 7$: $n \geq 11$



→ Nobody turns the product graph on.
→ Nobody turns the product graph off.

⇒ The product graph is once again <u>all</u> <u>zeros</u>.

$$y[n] = 0$$

- As in our final answer on page 3.47, our cases for this problem are:

case I) $n < 0$

case II) $0 \leq n < 4$

case III) $4 \leq n < 6$

case IV) $6 \leq n < 11$

case V) $n \geq 11$

⇒ <u>All</u> of the n's are covered.

⇒ Every n is included in one of our cases.

A Note About the Boundary Points Between the Cases:

- Let's take another closer look at the example we have been analyzing.

- It's:

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-5]$$

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases} = \alpha^n \{u[n] - u[n-7]\}$$

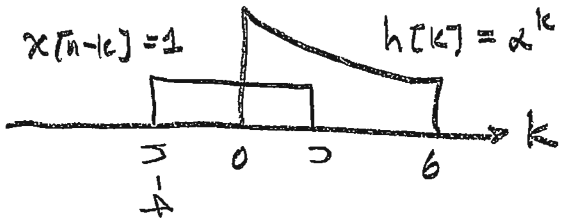
⇒ we just analyzed the "regions" or "cases" for this problem on pages 3.60 – 3.64

⇒ we also solved it on pages 3.44 – 3.47.

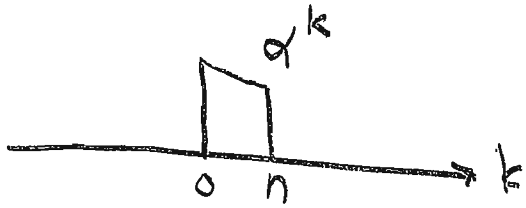
→ The answer we got for $y[n]$ on page 3.47 was

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n < 4 \\ \dfrac{\alpha^{n+4}-\alpha^{n+1}}{1-\alpha}, & 4 \le n < 6 \\ \dfrac{\alpha^{n+4}-\alpha^{7}}{1-\alpha}, & 6 \le n < 11 \\ 0, & n \ge 11 \end{cases}$$

- On page 3.45 and on page 3.61, we saw that case II for this problem looks like:



product graph:

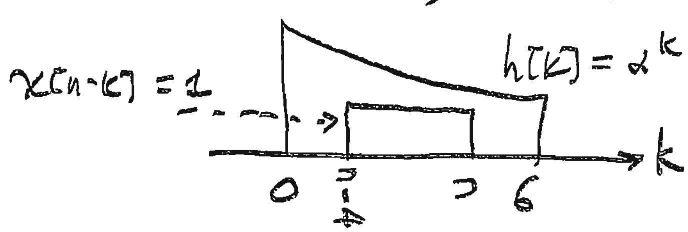


→ $h[k]$ turns the product graph <u>on</u> at $k=0$.
→ $x[n-k]$ turns the product graph <u>off</u> after $k=n$.

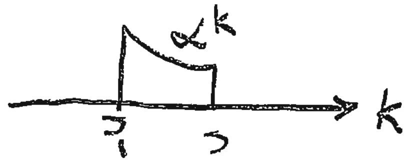
$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{(case II)}$$

⇒ when $n-4=0$ ... so that $n=4$, we considered it to be the start of Case III.

- on pages 3.46 and 3.63, we set up Case III as:



product graph:



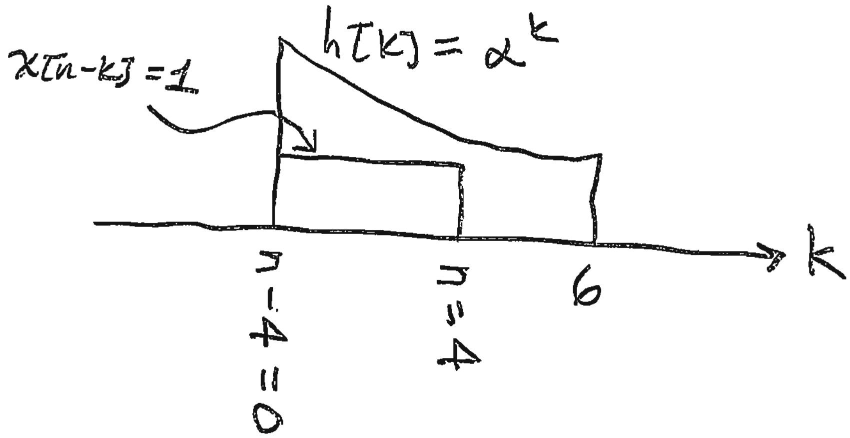
→ $x[n-k]$ turns the product graph <u>on</u> at $k=n-4$.
→ $x[n-k]$ turns the product graph <u>off</u> after $k=n$.

$$y[n] = \sum_{k=n-4}^{n} \alpha^k = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

⟼

- But now lets consider the "n" value that is right on the boundary between these two regions...

  - It is when $n-4=0$, or $n=4$.
  - For $n=4$, the picture looks like this:



⇒ From this picture, it's clear that we can consider $n=4$ to be the first time that $x[n-k]$ turns the product graph on at $k=n-4$... as we did on pages 3.46 and 3.63...

⇒ OR we can consider $n=4$ to be the last time that $h[k]$ turns the product graph on at $k=0$ (since $n-4=0$ when $n=4$)

⇒ Either way is correct ⟶

- In other words, we can consider the boundary point $n=4$ to be the <u>last</u> "n" in case II...

- or we can consider it to be the <u>first</u> "n" in case III.

- But does either way really work??

⇒ If we put $n=4$ in case II, then we are saying that $y[4]$ should use the case II formula.

→ We get

$$y[4] = \left.\frac{1-\alpha^{n+1}}{1-\alpha}\right|_{n=4} = \frac{1-\alpha^5}{1-\alpha}$$

⇒ If we put $n=4$ in case III instead, then we are saying that $y[4]$ should use the case III formula instead... we get

$$y[4] = \left.\frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}\right|_{n=4} = \frac{\alpha^0-\alpha^5}{1-\alpha} = \underline{\underline{\frac{1-\alpha^5}{1-\alpha}}}$$

⟹ It's the <u>same</u>.

☆☆
☆ For any "n" that is a boundary point between two regions... where the product graph has something in both regions...

→ In other words, for any "n" that is the boundary point between two regions where the product graph is <u>not</u> all zeros,

⇒ The formulas for these two regions will <u>always</u> <u>agree</u> at the boundary point.

⇒ The "n" that is at the boundary can always be considered to be:

- The <u>last</u> "n" in the "left" case,
- Or the <u>first</u> "n" in the "right" case.

⟶

- How we choose this determines which case gets the "=" in the inequalities.

  - If we do it the way we did on pages 3.45 and 3.46 when we first worked this problem,

    - which is the same as how we did it on pages 3.62 and 3.63 when we further analyzed the cases for this problem,

      - Then we put $n=4$ as the <u>first</u> "n" in case III, and we get:

$$\text{case II)} \quad 0 \le n < 4$$

$$\text{case III)} \quad 4 \le n < 6$$

→ $n=4$ is the <u>first</u> "n" in case III.

- If we instead put $n=4$ as the <u>last</u> "n" in case II, then we get:

now the "=" is here

$$\text{Case II)} \quad 0 \le n \le 4$$

$$\text{Case III)} \quad 4 < n < 6$$

no "=" here

→ Either way is actually correct.

- What is important :

☆☆ However you decide to do it, you must make sure that every "n" is included in one of your cases!!!

⇒ This would be incorrect :

case I) $0 \le n < 4$ ✗

case II) $4 < n < 6$ ✗

⇒ Incorrect because $n=4$ is not included in either case... $n=4$ is left out ✗✗✗

- You must make sure that your cases include all the n's.

- As a matter of convention, I usually pick my boundary points so that the cases look like $a \le n < b$

equal "=" going here

not here

as we did in all of the examples on pages 3.40 - 3.50.

- This makes the solution look a little bit more "elegant."

- From what we have said about how to find the regions on pages 3.51 – 3.71, you might be starting to think that maybe the first region, or first case, which always starts at $n = -\infty$,
  - Also ~~always~~ has $y[n] = 0$. X X

  $\Rightarrow$ But that's <u>wrong</u> !!!

$\rightarrow$ If either $x[n]$ is a <u>left-sided</u> signal or $h[n]$ is a <u>left-sided</u> signal,

$\Rightarrow$ Then the first case (which includes $n = -\infty$) will <u>not</u> have $y[n] = 0$!!

$\Rightarrow$ This is illustrated by the example we did back on page 3.48 ... where we got

$$y[n] = \begin{cases} 2^{n+1}, & n < 0 \\ 2, & n \geq 0 \end{cases}$$

$\Rightarrow$ The first case does <u>not</u> always have $y[n] = 0$!!!

WHAT TO DO IF ONE OF THE SIGNALS IS DEFINED "PIECEWISE"

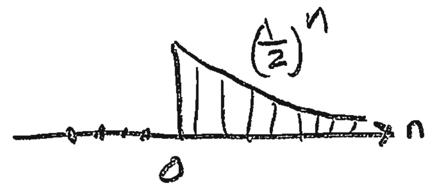
- Sometimes, especially in ECE 3793, you may get a convolution problem

$$x[n] \rightarrow \boxed{H} \rightarrow y[n] = x[n] * h[n]$$

$\Rightarrow$ where one of the signals $x[n]$ or $h[n]$ has "two parts" and is defined "piecewise".

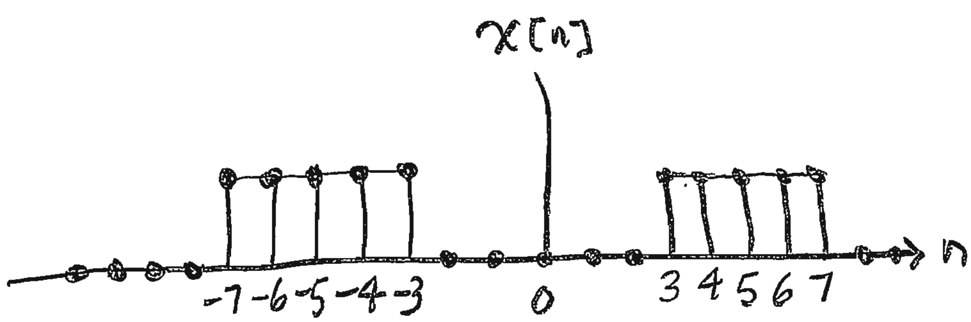
- For example, you might get

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

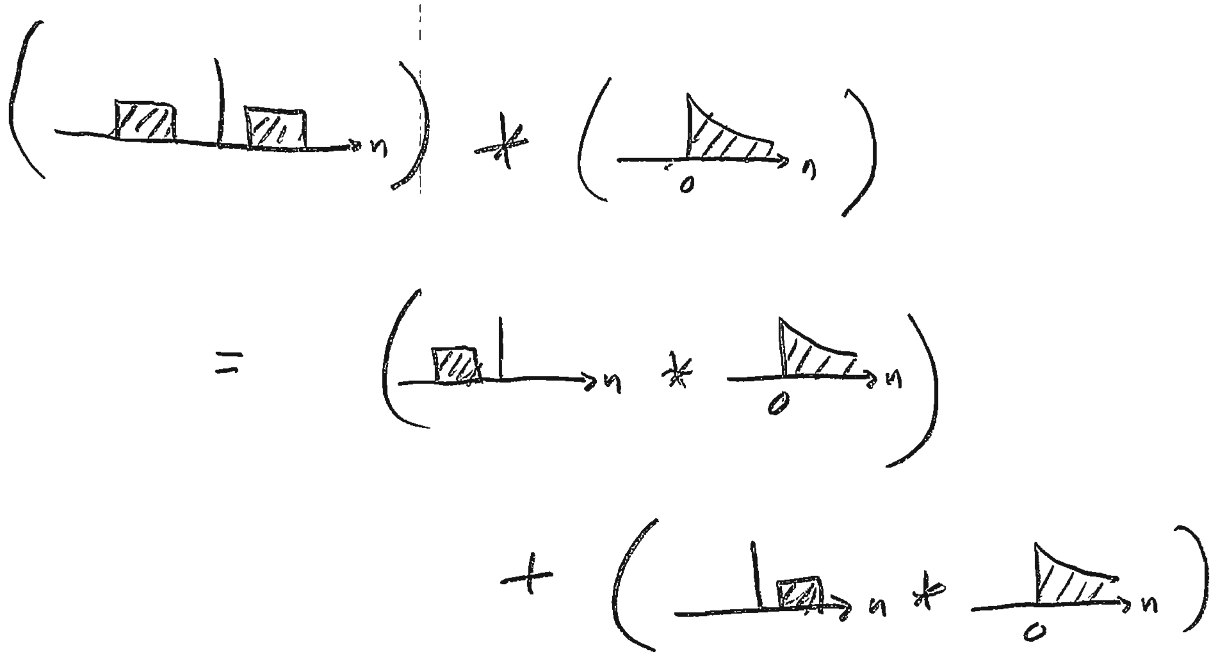


and

$$x[n] = \begin{cases} 1, & -7 \le n \le -3 \\ 1, & 3 \le n \le 7 \\ 0, & \text{otherwise} \end{cases} = \begin{array}{l} \left(u[n+7] - u[n+2]\right) \\ + \left(u[n-3] - u[n-8]\right) \end{array}$$



- You can handle these problems by using the distributive property to break the problem into two simpler convolutions... i.e.,



- For the "example" problem on page 3.73, this would give you:

$$y[n] = x[n] * h[n]$$

$$= \left\{ (u[n+7] - u[n+2]) + (u[n-3] - u[n-8]) \right\} * \left(\tfrac{1}{2}\right)^n u[n]$$

$$= \underbrace{(u[n+7] - u[n+2]) * \left(\tfrac{1}{2}\right)^n u[n]} + \underbrace{(u[n-3] - u[n-8]) * \left(\tfrac{1}{2}\right)^n u[n]}$$

work each part using the techniques we have learned in this module.

MODULE 3 SUMMARY

- We learned that the output of an LTI system is given by the convolution of the input with the impulse response:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \end{aligned}$$

- We learned that when $x[n]$ and $h[n]$ can both be written as short sums of the shifts of $\delta[n]$, then we can find $y[n]$ using the simpler "convolution with deltas" method.

- We learned the more general "graphical convolution" method that can always be used to find $y[n]$.