MODULE 4: Frequency Domain

Recall: Notes page 2.145: <u>time</u> <u>domain</u> means that we think of our discrete-time signals $x[n]$ as a sum of the natural basis $\{\delta[n-k]\}_{k\in\mathbb{Z}}$.

→ In other words, we think of $x[n]$ as being a sum of shifted versions of $\delta[n]$.

→ That's what "time domain" means for discrete-time signals $x[n]$.

→ Thinking in the time domain led us to the conclusion that:

- for a discrete-time LTI system H with impulse response $h[n]$,
- and input $x[n]$,

⇒ The system output signal $y[n]$ is given by the convolution of $x[n]$ with $h[n]$:

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k].
\end{aligned}
$$

- While we did not do it, it turns out to be true that any continuous-time signal $x(t)$ can be written as a sum of the shifts of the Dirac delta $\delta(t)$.

    - You will study this in detail in ECE 3793.

    - When you think of your continuous-time signals $x(t)$ as sums of the shifts of $\delta(t)$, it is called "time domain."

- Now, a <u>real</u>-<u>valued</u> continuous-time signal $x(t)$ can alternatively be written instead as a sum of <u>cosines</u>.

    - You already know this intuitively.

    - When you are stopped at a red light and a car pulls up beside you with a crazy booming stereo system that shakes your whole car,

        - You may think that you wish you could turn down the bass on their stereo.

→

- What does this really mean?
  - It means that you are thinking of the audio signal as a sum of cosines!
  - And you wish that you could "lower" the bass by reducing the sizes of the <u>low frequency</u> cosines in the sum.

- Similarly, you probably have (or at least you have seen) a stereo or an MP3 player that has "tone controls" like bass and treble controls... or bass, treble, and midrange controls... or even a graphic equalizer.
  - Intuitively, these controls enable you to <u>adjust</u> how much bass, treble, midrange, etc. is in the audio signal.
  - In other words, you think of the audio signal as being a sum of cosines.
    - → The bass control adjusts how much the low frequency cosines get added into the sum.
    - → The treble control adjusts how much the high frequency cosines get added into the sum.

- Recall Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

- if $\theta = \omega_0 t$, then we can apply Euler's formula at every time t to get

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

<u>FACT</u>: a real continuous-time signal $x(t)$ can be written as a sum of complex exponentials $e^{j\omega t}$.

- we just have to set up the sum so that all the imaginary parts cancel out.
- Then writing $x(t)$ as a sum of exponentials $e^{j\omega t}$ is no different from writing him as a sum of cosines.
- You will learn how to do this in ECE 3793.

<u>FACT</u>: a complex continuous-time signal $x(t)$ can also be written as a sum of complex exponentials $e^{j\omega t}$.

→ The only difference from the <u>real</u> case is that, when $x(t)$ is complex, the imaginary parts don't have to all cancel out in the sum.

☆☆ When you think of your continuous-time signals $x(t)$ as being sums of complex exponentials $e^{j\omega t} = \cos\omega t + j\sin\omega t$, it is called the <u>frequency</u> <u>domain</u>.

<u>FACT</u>: a real-valued discrete-time signal $x[n]$ can similarly be written as a sum of discrete-time cosines.

<u>FACT</u>: a real-valued discrete-time signal $x[n]$ can alternatively be written as a sum of discrete-time complex exponentials $e^{j\omega n} = \cos\omega n + j\sin\omega n$.

→ we just need to set up the sum so that all the imaginary parts cancel out.

→ we're going to learn how to do that in just a minute here.

<u>FACT</u>: a complex-valued discrete-time signal $x[n]$ can also be written as a sum of discrete-time complex exponentials $e^{j\omega n}$.

→ The only difference from the real case is that, when $x[n]$ is complex, the imaginary parts don't have to all cancel out in the sum.

☆☆ Similar to the continuous-time case,

→ when you think of your discrete-time signals $x[n]$ as being sums of complex exponentials $e^{j\omega n} = \cos\omega n + j\sin\omega n$, it is called the <u>frequency</u> <u>domain</u>.

- But <u>why</u> would we want to do this?

- Why would we want to write our discrete-time signals $x[n]$ as sums of complex exponentials ... instead of writing them as sums of the shifts of $\delta[n]$?

- To answer that question, we need one very important idea from linear algebra.
  - You will learn a lot about it in math 3333.

→ But for ECE 2713, we don't need all the details... we just need the big idea.

→

Eigenvalues and Eigenvectors

- When you multiply a matrix times a vector, you get a vector.

EX

Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$, a matrix.

Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, a vector.

Then $A\vec{v} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$,

a vector.

- In general, when you multiply a matrix times a vector,
  - it does two things to the vector:
    - It changes the length of the vector
    - It changes the direction of the vector.

- But for any matrix A, there will generally be <u>special</u> vectors $\vec{u}$ such that

  - multiplying A times $\vec{u}$ <u>only changes the length</u> ...

    - it does <u>not</u> change the direction.

- In other words, for these <u>special</u> vectors $\vec{u}$, you get

$$A\vec{u} = \lambda \vec{u},$$

where λ is a <u>number</u>.

- $\vec{u}$ is called an <u>eigenvector</u> of the matrix A.

- λ is called the associated <u>eigenvalue</u>.

- Given a matrix A, you will learn how to find the eigenvectors and eigenvalues in Math 3333.

  - But we won't need to worry about that in ECE 2713.

- Here is an example:

For our same matrix A from the example on page 4.7, let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

- we get

$$A\vec{u} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4\vec{u}$$

- So $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector of the matrix A and the associated eigenvalue is $\lambda = 4$.

So What?

- It turns out that for certain discrete-time systems,

  → there may be special input signals $x[n]$ such that the output is just a number times $x[n]$:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = \lambda x[n]$$

where λ = a number

- when this happens, $x[n]$ is called an <u>eigenfunction</u> of the system H.

  - λ is called the <u>eigenvalue</u> associated with $x[n]$.

<u>FACT</u>: for any fixed real number ω_0, the discrete-time signal $x[n] = e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$ is an eigenfunction of <u>any</u> discrete-time LTI system.

$\longrightarrow$ when $x[n] = e^{j\omega_0 n}$ is input to the system, the output will be $y[n] = \lambda e^{j\omega_0 n}$, a <u>number</u> times the input signal.

$\longrightarrow$

Proof:

Let H be a discrete-time LTI system with impulse response $h[n]$. Let $\omega_0 \in \mathbb{R}$ be a constant.

Let the input signal be $x[n] = e^{j\omega_0 n}$.

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow y[n]$$

Then the output signal $y[n]$ is given by

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 (n-k)} \\
&= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 n} e^{-j\omega_0 k} \\
&= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}} = \underline{\underline{\lambda x[n]}}
\end{aligned}$$

This is just the input signal $x[n]$

This is a number... there is no "n" in it. so it sums to a <u>number</u> that depends on the particular system ($h[k]$) and on the particular frequency ω_0 ... but it is a <u>number</u>. Call it λ.

→ Notice that the number λ is complex in general.

- So here is the big idea:

  - if we can somehow make a basis out of the signals $e^{j\omega n}$, then every basis vector will be an <u>eigenfunction</u> of <u>any</u> LTI system.

  - if we can then write an arbitrary input signal $x[n]$ as a sum of this basis,

    - Then every term in the sum will just get multiplied by a complex eigenvalue when going through the system.

    - This will be a <u>lot</u> <u>easier</u> than doing the convolution!

- It turns out that it <u>is</u> possible to do all of this.

- When you write a discrete-time signal $x[n]$ as a sum of eigenfunctions $e^{j\omega n}$, it is called the "frequency domain representation" of $x[n]$.

  - It is also called the "spectral" representation.
  - The book calls it the "spectrum representation."

- The first thing we need to do is use the signals $e^{j\omega n}$ to build a basis for representing our discrete-time signals $x[n]$.

- Recall from pages 2.111 through 2.113 that we only need 2π worth of frequencies ω to make all possible graphs of the signal $e^{j\omega n}$.

  → We can make all of the possible graphs by just taking ω's in the range $-\pi \leq \omega < \pi$.

  → If you take ω's outside this range, then you just get repeats of the same graphs you already got with ω's inside the range $-\pi \leq \omega < \pi$.

EX: $\omega_1 = \frac{15\pi}{8}$ is outside the range $[-\pi, \pi)$.

→ But the graph of $x_1[n] = e^{j\omega_1 n} = e^{j\frac{15\pi}{8}n}$ is the same as the graph of

$x_0[n] = e^{j\omega_0 n} = e^{j(-\frac{\pi}{8})n}$, where $\omega_0 = -\frac{\pi}{8}$.

→ And $\omega_0 = -\frac{\pi}{8}$ is inside the range $[-\pi, \pi)$.

→ The graphs are the same because

$$\omega_1 - 2\pi = \frac{15\pi}{8} - 2\pi = \frac{15\pi}{8} - \frac{16\pi}{8} = -\frac{\pi}{8} = \omega_0.$$

$\rightarrow$ $x_1[n] = e^{j\omega_1 n} = e^{j\frac{15\pi}{8}n}$ and $x_0[n] = e^{j\omega_0 n} = e^{j(-\frac{\pi}{8})n}$

are just two different ways of writing the <u>exact</u> <u>same</u> <u>signal</u>.

<u>Proof</u>: let $\omega_1 = \frac{15\pi}{8}$ and $\omega_0 = -\frac{\pi}{8}$.

Let $x_1[n] = e^{j\omega_1 n}$ and $x_0[n] = e^{j\omega_0 n}$.

Then, for every $n \in \mathbb{Z}$,

$$
\begin{aligned}
x_1[n] = e^{j\omega_1 n} &= e^{j\frac{15\pi}{8}n} \\
&= e^{j\left(\frac{16\pi}{8}-\frac{\pi}{8}\right)n} \\
&= e^{j\frac{16\pi}{8}n} e^{j\left(-\frac{\pi}{8}\right)n} \\
&= \underbrace{e^{j2\pi n}}_{\text{ONE}} e^{j\left(-\frac{\pi}{8}\right)n} \\
&= e^{j\left(-\frac{\pi}{8}\right)n} \\
&= x_0[n] \checkmark
\end{aligned}
$$

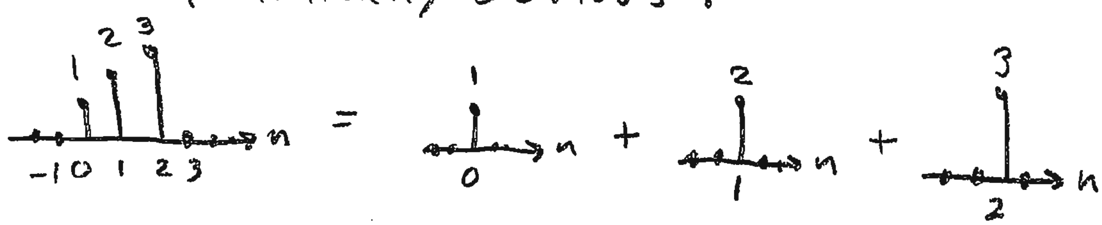
- So, to make our basis of eigenfunctions $e^{j\omega n}$, we only need frequencies ω in the range $[-\pi, \pi)$.
- Taking frequencies outside this range just gives us repeats of the signals we already have.

Thinking About $e^{j\omega n}$ as a Basis Vector

- For the time domain representation, we wrote our discrete-time signals $x[n]$ as a sum of the natural basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$.

    - This was easy to think about because each basis vector $\ldots \delta[n+1], \delta[n], \delta[n-1], \ldots$ was obviously similar to the basis vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

    - And, just like in $\mathbb{R}^2$, the dot products were practically obvious:

$x[n] =$ 

$$= 1\delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

- But recall the <u>rotated</u> basis $\vec{e}_1, \vec{e}_2$ for $\mathbb{R}^2$ that we saw on page 1.104:

$$\vec{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

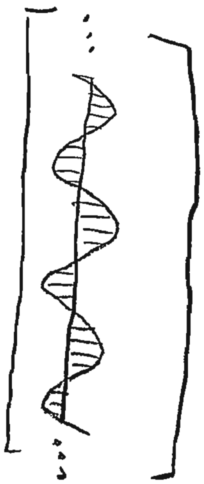
- As we said on page 1.105, we can get this basis by multiplying the natural basis vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ times a rotation matrix.

- The entries of the rotation matrix follow a pattern of sines and cosines.

- So: when we <u>rotate</u> a natural basis, we <u>expect</u> the rotated basis vectors to have entries that follow a pattern of cosines and sines.

- When we take the natural basis $\{\delta[n-k]\}_{k \in \mathbb{Z}}$ and perform an appropriate rotation, we get rotated basis vectors with entries that follow a pattern like $e^{j\omega n}$.

- Rotated basis vector:



← The entries of this vector are numbers that follow the pattern $x[n] = e^{j\omega n}$ for some particular $\omega \in \mathbb{R}$ with $-\pi \le \omega < \pi$.

The Spectral Basis

- So here is our rotated basis:

$$\left\{ e^{j\omega n} \right\}_{\omega \in [-\pi, \pi)}$$

What it means:

- Think of the curly braces as a "bag full of guys".
  - Each "guy" is a signal $e^{j\omega n}$ for some particular ω between $-\pi$ and π.
    - we think of him as a discrete time signal described by his graph or by the function $e^{j\omega n}$.

- At the same time, we also think of him as being a vector in an infinite dimensional vector space.

- And we will write our discrete-time signals $x[n]$ as sums of this basis.

FACT: the spectral basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$

is orthogonal, but it is not orthonormal.

→ Each basis vector has length $\sqrt{2\pi}$ instead of one.

- This means that any two different vectors from the basis are orthogonal... they have a zero dot product with each other.

  - In other words, if $\omega_1 \neq \omega_2$, then

$$\langle e^{j\omega_1 n}, e^{j\omega_2 n} \rangle = \sum_{n=-\infty}^{\infty} e^{j\omega_1 n} e^{-j\omega_2 n} = 0$$

- And the dot product of any basis vector with himself is his length squared, which is $(\sqrt{2\pi})^2 = 2\pi$.

NOTE: These last two facts require 20th century distributional math to prove. They are proved in ECE 4213. We will not prove them in ECE 2713.

- The last two facts can be written together in one equation like this:

$$\langle e^{j\omega_1 n}, e^{j\omega_2 n} \rangle = 2\pi\delta(\omega_1 - \omega_2),$$

where $\delta(\cdot)$ is the <u>Dirac</u> delta distribution described in the notes for Module 2.

<u>Summary:</u> The basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$ is orthogonal but not orthonormal. Each basis vector has length $\sqrt{2\pi}$ (in a distributional sense).

☆☆ Look back at notes pages 1.106 through 1.108 now!!

→ The fact that the spectral basis vectors $e^{j\omega n}$ have length $\sqrt{2\pi}$ instead of one means that:

⇒ When we add up the dot products times the basis vectors to get $x[n]$, we will have to divide by 2π... which is the length of a basis vector squared.

- Now we will write a discrete-time signal $x[n]$ as a sum of the spectral basis vectors $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$.

- The steps are the same as always:

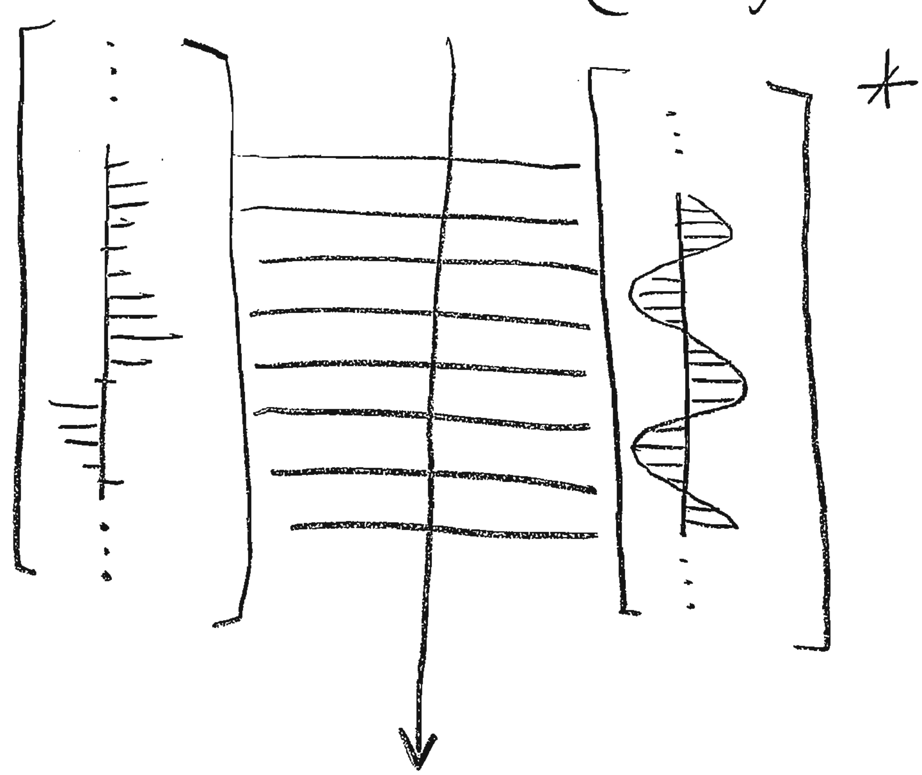
  1. Take the dot product of $x[n]$ with each basis vector. This gives us a <u>number</u> for each basis vector.

  2. Add up the dot products (numbers) times the basis vectors.
     - Because the basis vectors have length $\sqrt{2\pi}$ instead of one, we have to divide by 2π.
     - This gives us $x[n]$.

- To take the dot product of $x[n]$ with a basis vector, we follow the same steps as we did in $\mathbb{R}^2$:

  - line up the two vectors beside each other
  - conjugate the entries of the second vector
  - Multiply the entries that are beside each other
  - Add it up down the vector

$x[n]$ $\qquad\qquad$ $(e^{j\omega n})^* = e^{-j\omega n}$



$$\langle x[n], e^{j\omega n} \rangle = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n}$$

an entry from the first vector

the conjugate of the corresponding entry of the 2nd vector

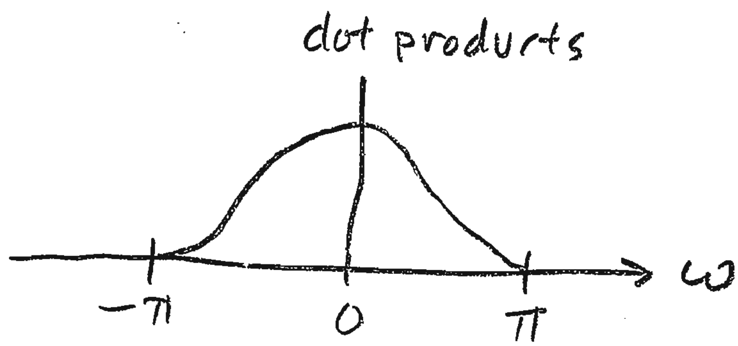
- This gives us a <u>number</u> for each basis vector

- There is one basis vector for each $\omega \in [-\pi, \pi)$.

  → So we get a <u>number</u> for each ω from $-\pi$ to π.

- These numbers are complex in general, even if $x[n]$ is real.

- To keep track of all these numbers (dot products), we make an ω axis... and we plot the dot products on the ω axis:



→ Notice that this makes a <u>function</u> of ω.

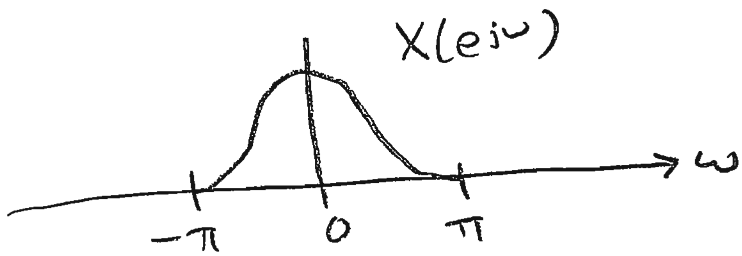
- Because there is one basis vector for <u>every</u> real ω from $-\pi$ to π, this is "like" a continuous-time function in the sense that ω is a continuous variable... like t... not discrete like n.

- This graph of the dot products is a function of ω.

  - It would make sense to call it $X(\omega)$.
  - But that is <u>not</u> what we do.
  - For reasons that I will explain later, we instead call it $X(e^{j\omega})$.

<u>Summary of taking the dot product of $x[n]$ with the spectral basis vectors:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$



☆☆ This is called the <u>discrete-time Fourier Transform</u> or DTFT.

- It is given in the book in eq. (7.2) on page 241.

<u>NOTE</u>: the book writes $\hat{\omega}$ for the frequency variable instead of ω. This is <u>not</u> standard!! Most people write "ω" and so will we!!

- Why does the book write $\hat{\omega}$?
  - often, you get a discrete-time signal $x[n]$ by sampling a continuous-time signal $x(t)$ every T_s seconds.
    - This gives you $\underset{\uparrow \text{ integers}}{x[n]} = x(\underset{\uparrow \text{ times... not integers in general.}}{nT_s})$
    - T_s is called the "sampling interval."
  - If $x(t) = A\cos(\omega t + \phi)$,

    then $x[n]$ is also a cosine, but the frequency is not ω.
    - The sampling times are at $t = nT_s$.
    - So we get

$$
\begin{aligned}
x[n] &= x(nT_s) \\
&= A\cos(\omega n T_s + \phi) \\
&= A\cos(\underbrace{\omega T_s}_{\hat{\omega}} n + \phi)
\end{aligned}
$$

  - So, for a continuous-time frequency of ω, we get a discrete-time frequency of

$$\hat{\omega} = \omega T_s$$

    - This is eq. (4.3) on p. 105 of the book.

- Since the continuous and discrete frequencies are not the same, you need to use different symbols for them.

- The book uses ω for continuous frequency and $\hat{\omega}$ for discrete frequency.

- But this is not standard.

    - The ECE 3793 book uses ω for continuous frequency and Ω for discrete frequency.

    - The ECE 4213 book does it backwards from that: it uses Ω for continuous frequency and ω for discrete frequency.

<u>Moral of the Story</u>: any time you look in a new book, make sure to pay attention to what symbols are being used for continuous frequency and discrete frequency.

There's not any standard notation and it could be different in every book you look at.

Okay, now back to the main story:

- We want to write the discrete-time signal $x[n]$ as a sum of the spectral basis

$$\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)} .$$

- We took the dot product of $x[n]$ with a basis vector like this:

$$\langle x[n], e^{j\omega n} \rangle = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  - This gives us a number for each basis vector... a number for each ω.

  - $X(e^{j\omega})$ is a function of ω.

  - If you plug in $\omega = \frac{\pi}{17}$, you get a number that is the dot product of $x[n]$ with the basis vector $e^{j\frac{\pi}{17}n}$.

- Now we need to add up the dot products times the basis vectors to get $x[n]$.
  - Because the basis vectors have length $\sqrt{2\pi}$ instead of one, we have to divide by 2π.
  - Because there is one basis vector for every real number between $-\pi$ and π, we cannot use capital Σ adding
    - Capital Σ adding is a do loop.
    - The loop counter has to be integer
    - But here, our ω is real, not integer.
    - So we have to use capital $\int$ adding instead.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{X(e^{j\omega})}_{\text{dot product of } x[n] \text{ with the } \omega^{th} \text{ basis vector}} \underbrace{e^{j\omega n}}_{\text{the } \omega^{th} \text{ basis vector}} d\omega$$

fix for length of basis vectors

add it up over the basis

adding over the basis

- This is called the <u>inverse Discrete-Time Fourier Transform</u>, or IDTFT.

- It is given in eq. (7.8) of the book on page 246.

  - NOTE: the book writes $\hat{\omega}$ instead of ω.

<u>Summary of DTFT and IDTFT:</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- We say that $x[n]$ and $X(e^{j\omega})$ are a "DTFT pair" or a "Fourier transform pair".

- Here are some different ways that you can write "$x[n]$ and $X(e^{j\omega})$ are a DTFT pair":

$$X(e^{j\omega}) = \text{DTFT}\{x[n]\} \qquad x[n] = \text{IDTFT}\{X(e^{j\omega})\}$$

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} \qquad x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

- Now, in the inverse DTFT equation, we wrote $x[n]$ as a sum of basis functions $e^{j\omega n}$ with frequencies ω going from $-\pi$ to π:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

→ But if you wanted to, you could instead use basis functions $e^{j\omega n}$ with frequencies going from 0 to 2π

- or from π to 3π
- or from <u>any</u> other interval of length 2π

→ This is because you get <u>all</u> the basis functions... i.e., you get <u>all</u> the possible graphs of $e^{j\omega n}$... as long as you take "2π worth" of frequencies.

→ The basis functions $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$

have the same graphs as the basis functions $\{e^{j\omega n}\}_{\omega \in [0, 2\pi)}$

and the same graphs as the basis functions $\{e^{j\omega n}\}_{\omega \in [\pi, 3\pi)}$

→ They are all just different names for the <u>same</u> basis functions.

- In other words, they are all just different ways of writing the same basis functions.

- So, for the inverse DTFT on page 4.28, we could have written instead:

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

or

$$x[n] = \frac{1}{2\pi}\int_{\pi}^{3\pi} X(e^{j\omega n})\, e^{j\omega n}\, d\omega$$

- For this reason, many books write the IDTFT like this:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- The symbol "$\int_{2\pi}$" means that you can integrate on any interval of length 2π... $x[n]$ will turn out the same no matter how you do it...
  - because you will get the same basis functions no matter how you do it.

☆ And the function $X(e^{j\omega})$ is actually <u>defined</u> for all $\omega \in \mathbb{R}$ <u>and</u> it is 2π-periodic in ω.

Proof that $X(e^{j\omega})$ is 2π-periodic:

Let $\omega_0 \in \mathbb{R}$ such that $-\pi \leq \omega_0 < \pi$.

Let $k \in \mathbb{Z}$ be any integer.

Let $\omega_1 = \omega_0 + 2\pi k$.

Then $X(e^{j\omega_0}) = X(e^{j\omega_1})$ ... they are the same number.

Here's how to show it:

$$X(e^{j\omega_1}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_1 n}$$

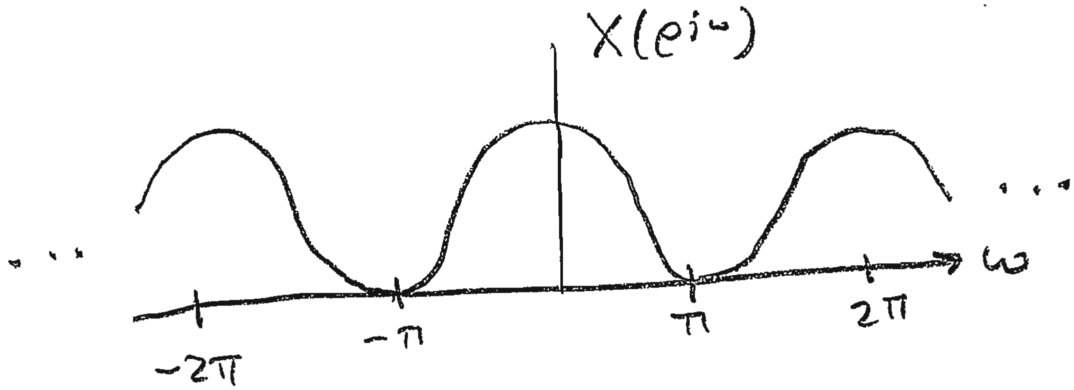
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega_0 + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n} \underbrace{e^{-j2\pi kn}}_{\text{ONE}}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n}$$

$$= X(e^{j\omega_0}) \quad \checkmark$$

- So the graph of $X(e^{j\omega})$ is actually 2π-periodic.
  - For example, it might look like this:



- But for convenience, we usually just graph it from $-\pi$ to π like we did on page 4.23
  - Sometimes you may also see it graphed from 0 to 2π instead.

→ But even though we usually only graph just one period,

⇒ You must always remember that $X(e^{j\omega})$ ... i.e. <u>any</u> DTFT...

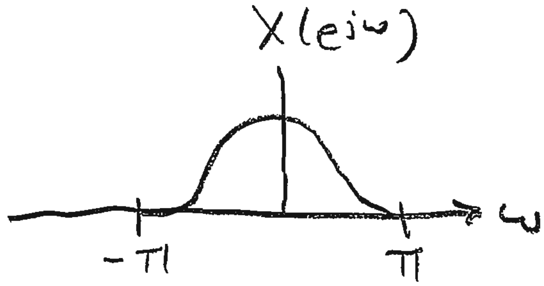
is <u>always</u> a 2π-periodic

function. ☆☆☆

- In ECE 2713, we will always take the IDTFT from $-\pi$ to π like this:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- And we will usually graph $X(e^{j\omega})$ from $-\pi$ to π like this:



- Now, for each ω, $X(e^{j\omega})$ is actually a complex number in general.

  - So we actually need two graphs:

    - one graph for the real part $\mathrm{Re}\{X(e^{j\omega})\}$

    - and a second graph of the imaginary part $\mathrm{Im}\{X(e^{j\omega})\}$

- Since $X(e^{j\omega})$ is a complex number for each ω, we can also convert all those numbers to polar form.
  - This gives us a function $|X(e^{j\omega})|$ that has all the magnitudes,
  - And a function $\arg X(e^{j\omega})$ that has all the angles.
    - $\arg X(e^{j\omega})$ is sometimes written as $\measuredangle X(e^{j\omega})$.

- Here is the math for converting between the rectangular and polar forms of the function $X(e^{j\omega})$:

$$|X(e^{j\omega})| = \sqrt{(\mathrm{Re}\{X(e^{j\omega})\})^2 + (\mathrm{Im}\{X(e^{j\omega})\})^2}$$

$$\arg X(e^{j\omega}) = \arctan\left(\frac{\mathrm{Im}\{X(e^{j\omega})\}}{\mathrm{Re}\{X(e^{j\omega})\}}\right)$$

$$\mathrm{Re}\{X(e^{j\omega})\} = |X(e^{j\omega})| \cos(\arg X(e^{j\omega}))$$

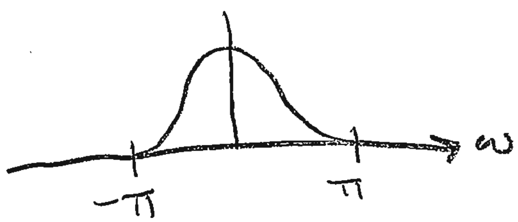
$$\mathrm{Im}\{X(e^{j\omega})\} = |X(e^{j\omega})| \sin(\arg X(e^{j\omega}))$$

→ The meaning is that, conceptually at least, you do these calculations at every ω.

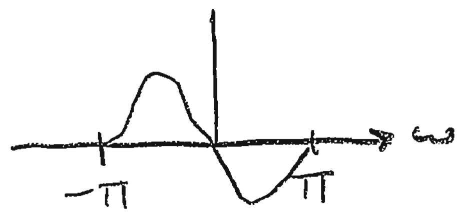
- So instead of graphing the real and imaginary parts of $X(e^{j\omega})$,
  - we can alternatively graph the magnitude and angle.

__EX__:

$Re\{X(e^{j\omega})\}$

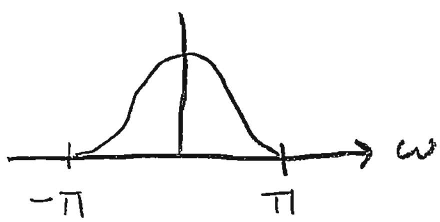


$Im\{X(e^{j\omega})\}$

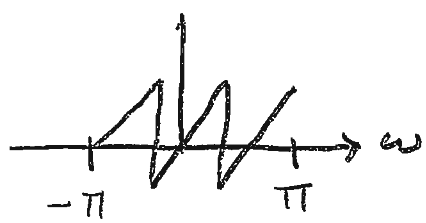


— or —

$|X(e^{j\omega})|$



$\arg X(e^{j\omega})$



- Often it is more __informative__ to look at the graphs of $|X(e^{j\omega})|$ and $\arg X(e^{j\omega})$ ... i.e., to look at the "polar version".

- The function $|X(e^{j\omega})|$ is called the <u>spectral magnitude</u> of the signal $x[n]$.

- The function $\arg X(e^{j\omega})$ is called the <u>spectral phase</u> of the signal $x[n]$.

- For an LTI system H, the DTFT of the impulse response $h[n]$ is given by

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

  - $H(e^{j\omega})$ is called the <u>frequency response</u> of the system.

  - The spectral magnitude $|H(e^{j\omega})|$ and spectral phase $\arg H(e^{j\omega})$ tell you a <u>lot</u> about the system.

    - we will spend a lot of time talking about this later.

  - For now, take a look back at page 4.11.

    - look carefully at the expression we got for the eigenvalues of the system H.

- we got that, for the input eigenfunction $x[n] = e^{j\omega_0 n}$, the associated eigenvalue was a number given by

$$\lambda = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}$$

- but "k" is just a loop counter. We used "k" because "n" was already being used for something else (the independent time variable) on page 4.11.

- But here on page 4.38, we could just as well write "n" for the loop counter instead. Then the eigenvalue for $x[n] = e^{j\omega_0 n}$ looks like this:

$$\lambda = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega_0 n}$$

- And, moreover, this equation is good for any real pick of the number ω_0... it's good for <u>any</u> $\omega \in \mathbb{R}$.

- So, for any $\omega \in \mathbb{R}$, the signal $x[n] = e^{j\omega n}$ is an eigenfunction of <u>any</u> LTI system H... and the associated eigenvalue is given by

$$\sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \quad !!$$

- But this is just the DTFT of the impulse response $h[n]$!!

- In other words: the <u>frequency</u> <u>response</u> $H(e^{j\omega})$ is a function that gives us <u>all</u> <u>the</u> <u>eigenvalues</u> of the LTI system H.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

→ plug in any real number ω_0 and you get the eigenvalue for the input eigenfunction $x[n] = e^{j\omega_0 n}$.

★★★★★

THIS IS <u>SUPER</u> <u>SUPER</u>

<u>SUPER</u> <u>IMPORTANT</u> !!!

★★★★★

- In other words, the dot product of $h[n]$ with $e^{j\omega n}$:

$$\langle h[n], e^{j\omega n} \rangle = H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

→ gives you the <u>projection</u> of the vector $h[n]$ onto the vector $e^{j\omega n}$,

→ And that is <u>exactly</u> the eigenvalue of the LTI system H that is associated with the eigenfunction $e^{j\omega n}$

→ So when we write an arbitrary input signal $x[n]$ as a sum of eigenfunctions,

⇒ The frequency response $H(e^{j\omega})$ gives us <u>all</u> the eigenvalues that we need to find the output signal $y[n]$.

- Now let's work out a DTFT from the definition:

EX: $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]\, e^{-j\omega n} \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{2}\right)^n$$

→ I could have written this as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2e^{j\omega}}\right)^n$$

→ But we usually write DTFT's in terms of $e^{-j\omega}$ ... because that's how all the formulas are usually written...

→ So I left the summand as $\left(\frac{e^{-j\omega}}{2}\right)^n$ instead.

→ Note that, no matter how you pick ω, $\frac{e^{-j\omega}}{2} \neq 1$. So we can apply the sum formula to get →

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}e^{-j\omega}\right)^n$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \left(\tfrac{1}{2}e^{-j\omega}\right)^n$$

apply sum formula with $\alpha = \frac{1}{2}e^{-j\omega}$

$$= \lim_{N\to\infty} \frac{\left(\frac{1}{2}e^{-j\omega}\right)^0 - \left(\frac{1}{2}e^{-j\omega}\right)^{N+1}}{1-\frac{1}{2}e^{-j\omega}}$$

[Now: for any $\omega \in \mathbb{R}$, $\frac{1}{2}e^{-j\omega}$ is a complex number with magnitude $\left|\frac{1}{2}e^{-j\omega}\right| = \frac{1}{2} < 1$.

$\to$ so $\lim_{N\to\infty} \left(\frac{1}{2}e^{-j\omega}\right)^{N+1} = 0$]

$$X(e^{j\omega}) = \frac{1-0}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

- To write $X(e^{j\omega})$ in rectangular form,
  - in other words, to find the real and imaginary parts,
  - you have to clear the j out of the denominator.
- Just like always, you do this by multiplying $X(e^{j\omega})$ by <u>one</u> in the form $1 = \frac{\text{conjugate of denominator}}{\text{conjugate of denominator}}$
  - The denominator is $1-\frac{1}{2}e^{-j\omega}$
  - The conjugate of the denominator is $1-\frac{1}{2}e^{j\omega}$

So: $$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \underbrace{\frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}}_{\text{ONE}}$$

$$= \frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{4}\underbrace{e^{-j\omega}e^{j\omega}}_{\text{one}}}$$

$$= \frac{1-\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}\cos\omega-\frac{1}{2}j\sin\omega-\frac{1}{2}\cos(-\omega)-\frac{1}{2}j\sin(-\omega)+\frac{1}{4}}$$

$\longrightarrow$

$$\ldots X(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}\cos\omega - \frac{1}{2}\cos\omega \underbrace{- \frac{1}{2}j\sin\omega + \frac{1}{2}j\sin\omega}_{\text{zero}} + \frac{1}{4}}$$

$$= \frac{1 - \frac{1}{2}\cos\omega - \frac{1}{2}j\sin\omega}{\frac{5}{4} - \cos\omega}$$

$$= \frac{1 - \frac{1}{2}\cos\omega}{\frac{5}{4} - \cos\omega} - j\,\frac{\frac{1}{2}\sin\omega}{\frac{5}{4} - \cos\omega}$$

So:

$$\mathrm{Re}\{X(e^{j\omega})\} = \frac{1 - \frac{1}{2}\cos\omega}{\frac{5}{4} - \cos\omega}$$

$$\mathrm{Im}\{X(e^{j\omega})\} = \frac{-\frac{1}{2}\sin\omega}{\frac{5}{4} - \cos\omega}$$

To find the spectral magnitude $|X(e^{j\omega})|$ and spectral phase $\arg X(e^{j\omega})$, we could apply the formulas on page 4.35 ... it would be <u>messy</u>.

- In practice, we don't often have to work out DTFTs, by hand.

- Instead, we look them up in a Table.

- The formula sheet for Test II has a table of DTFT pairs.

  - If you look in the table, you will find the DTFT pair:

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

  - plugging in $a=\frac{1}{2}$, we get the one that we just worked out:

$$\left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad \checkmark$$

FACT: if $x[n]$ is <u>real</u>, then $X(e^{j\omega})$ is <u>conjugate</u> <u>symmetric</u>:

$$X(e^{j\omega}) = X^*(e^{-j\omega}).$$

- It follows from this that:
  - The real part of $X(e^{j\omega})$ is an <u>even</u> function of ω.
  - The imaginary part of $X(e^{j\omega})$ is an <u>odd</u> function of ω.
  - The magnitude of $X(e^{j\omega})$ is an <u>even</u> function of ω.
  - The angle (spectral phase or just "phase") of $X(e^{j\omega})$ is an <u>odd</u> function of ω.

- Back near the middle of page 4.5, we said that:

a real-valued discrete-time signal $x[n]$ can be written as a sum of discrete-time exponentials

$$e^{j\omega n} = \cos\omega n + j\sin\omega n$$

→ But that, to make this work out, we have to fix up the sum so that all the imaginary parts cancel out.

→ The fact that $X(e^{j\omega})$ is <u>always</u> conjugate symmetric when $x[n]$ is real is <u>exactly</u> what makes this work.

- Suppose that $x[n]$ is real, and suppose that for $\omega = \frac{\pi}{7}$, $X(e^{j\omega})$ is some complex number $X(e^{j\frac{\pi}{7}}) = A + jB$ where $A \in \mathbb{R}$ and $B \in \mathbb{R}$.

  - Then, because $x[n]$ is real, $X(e^{j\omega})$ is conjugate symmetric

  - So $X(e^{-j\frac{\pi}{7}}) = X^*(e^{j\frac{\pi}{7}}) = A - jB$

→

- So when we write $x[n]$ as the sum

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega,$$

- The sum (integral actually) will contain a term $\frac{1}{2\pi}X(e^{j\frac{\pi}{7}})\,e^{j\pi/7\,n}$ and also a term $\frac{1}{2\pi}X(e^{-j\pi/7})\,e^{-j\pi/7\,n}$.

- when these two terms get added by the integral, it makes

$$\frac{1}{2\pi}\left[X(e^{j\pi/7})\,e^{j\pi/7\,n} + X(e^{-j\pi/7})\,e^{-j\pi/7\,n}\right]$$

$$= \frac{1}{2\pi}\Big[(A+jB)\left(\cos\tfrac{\pi}{7}n + j\sin\tfrac{\pi}{7}n\right)$$
$$+ (A-jB)\left(\cos(-\tfrac{\pi}{7}) + j\sin(-\tfrac{\pi}{7}n)\right)\Big]$$

$$= \frac{1}{2\pi}\left[(A+jB)\left(\cos\tfrac{\pi}{7}n + j\sin\tfrac{\pi}{7}n\right) + (A-jB)\left(\cos\tfrac{\pi}{7}n - j\sin\tfrac{\pi}{7}n\right)\right]$$

$$= \frac{1}{2\pi}\Big[A\cos\tfrac{\pi}{7}n + jA\sin\tfrac{\pi}{7}n + jB\cos\tfrac{\pi}{7}n - B\sin\tfrac{\pi}{7}n$$
$$+ A\cos\tfrac{\pi}{7}n - jA\sin\tfrac{\pi}{7}n - jB\cos\tfrac{\pi}{7}n - B\sin\tfrac{\pi}{7}n\Big]$$

$$= \frac{1}{2\pi}\left[2A\cos\tfrac{\pi}{7}n - 2B\sin\tfrac{\pi}{7}n\right] = \text{REAL}$$

- And the same thing happens at all the other frequencies ω from $-\pi$ to π.

- So, when $x[n]$ is real, $X(e^{j\omega})$ is conjugate symmetric,

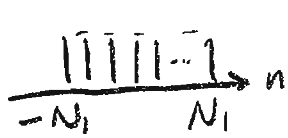
  - and this makes all the imaginary parts in the inverse DTFT cancel out so that

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \underline{\underline{\text{REAL}}}$$

  - as it must!

- Now, some of the entries in our DTFT table (see test 2 formula sheet on the course web site) look pretty easy to understand, like:

$$a^n u[n], \; |a| < 1 \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad \frac{1}{1 - a e^{-j\omega}}$$

$x[n] =$ 

$\overset{\text{DTFT}}{\longleftrightarrow} \quad \dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin \omega/2}$

$$\delta[n] \overset{\text{DTFT}}{\longleftrightarrow} 1 \qquad \text{etc...}$$

- But some of the other ones look hard because they have "capital sigma do loops" in them.

- Here's how to understand that:

First, recall that it's easy to understand in $\mathbb{R}^2$ that $\begin{bmatrix} 3 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix}$...

i.e., it's easy to understand that $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ is a sum of two basis vectors $\vec{i}$ and $\vec{j}$ with weights (dot products) 3 and 7.

Second, recall that, from Euler's formula,

$$\cos \omega n = \tfrac{1}{2} e^{j\omega n} + \tfrac{1}{2} e^{-j\omega n}$$

$\rightarrow$ But $e^{j\omega n}$ and $e^{-j\omega n}$ <u>are</u> DTFT basis functions.

$\rightarrow$ So $\cos \omega n$ is a sum of exactly two DTFT basis functions, both with weight (dot product) $\frac{1}{2}$.

$\longrightarrow$

- For example, if $\omega = \frac{\pi}{8}$, then

$$\cos\frac{\pi}{8}n = \frac{1}{2}e^{j\pi/8\,n} + \frac{1}{2}e^{-j\pi/8\,n}$$

- Because the IDTFT is an <u>integral</u>, for everything to work out we need the transform of $\cos\frac{\pi}{8}n$ to be

$$X(e^{j\omega}) = \pi\delta(\omega - \tfrac{\pi}{8}) + \pi\delta(\omega + \tfrac{\pi}{8})$$

where $\delta(\cdot)$ is the <u>Dirac</u> delta.

- This follows from the "sifting property" of the Dirac delta that we saw on page 2.63.

- with $X(e^{j\omega}) = \pi\delta(\omega - \frac{\pi}{8}) + \pi\delta(\omega + \frac{\pi}{8})$, we get

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\pi\delta(\omega - \tfrac{\pi}{8}) + \pi\delta(\omega + \tfrac{\pi}{8})\right]e^{j\omega n}\,d\omega$$

$$= \frac{\pi}{2\pi}\int_{-\pi}^{\pi}\delta(\omega - \tfrac{\pi}{8})e^{j\omega n}\,d\omega + \frac{\pi}{2\pi}\int_{-\pi}^{\pi}\delta(\omega + \tfrac{\pi}{8})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\delta(\omega - \tfrac{\pi}{8})e^{j\omega n}\,d\omega + \frac{1}{2}\int_{-\pi}^{\pi}\delta(\omega + \tfrac{\pi}{8})e^{j\omega n}\,d\omega$$

$\longrightarrow$

– But $\delta(\omega - \frac{\pi}{8})$ is "turned on" at $\omega = \frac{\pi}{8}$.

– So $\int_{-\pi}^{\pi} \delta(\omega - \frac{\pi}{8}) e^{j\omega n}\, d\omega =$ Value of $e^{j\omega n}$ at $\omega = \frac{\pi}{8}$

$$= e^{j\frac{\pi}{8}n}$$

– So $\frac{1}{2}\int_{-\pi}^{\pi} \delta(\omega - \frac{\pi}{8}) e^{j\omega n}\, d\omega = \frac{1}{2} e^{j\pi/8\, n}$

– And $\delta(\omega + \frac{\pi}{8})$ is "turned on" at $\omega = -\frac{\pi}{8}$.

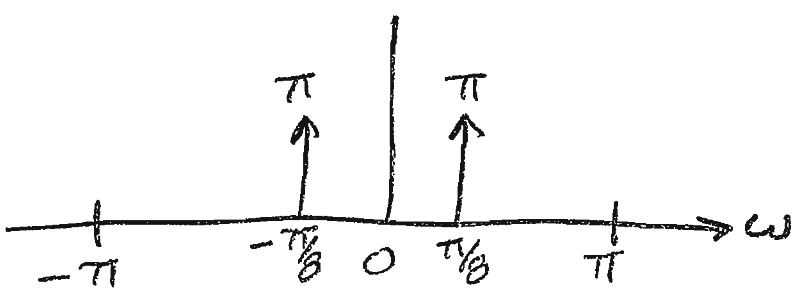
– So $\frac{1}{2}\int_{-\pi}^{\pi} \delta(\omega + \frac{\pi}{8}) e^{j\omega n}\, d\omega = \frac{1}{2} e^{-j\pi/8\, n}$

– And from the last page, we get

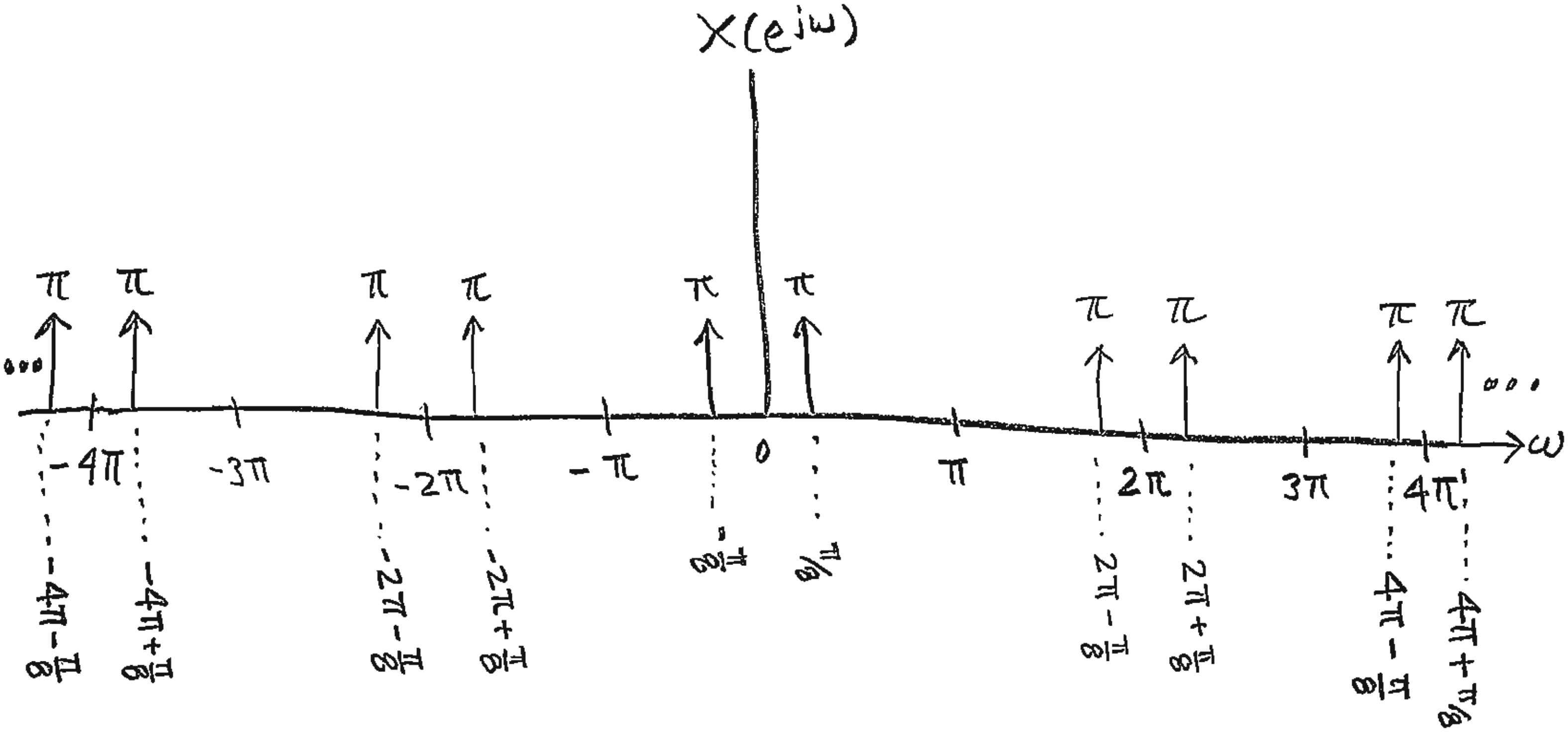
$$x[n] = \frac{1}{2}\int_{-\pi}^{\pi} \delta(\omega - \tfrac{\pi}{8}) e^{j\omega n}\, d\omega + \frac{1}{2}\int_{-\pi}^{\pi} \delta(\omega + \tfrac{\pi}{8}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2} e^{j\frac{\pi}{8}n} + \frac{1}{2} e^{-j\frac{\pi}{8}n} = \cos\frac{\pi}{8}n \checkmark$$

– So, for $x[n] = \cos\frac{\pi}{8}n$, the graph of $X(e^{j\omega})$ should look like:



- But even though we usually graph $X(e^{j\omega})$ for $\omega = -\pi$ to $\omega = \pi$ only, on p. 4.32 we saw that <u>every</u> DTFT <u>must</u> be 2π-periodic.
- So in reality, the <u>whole</u> graph of $X(e^{j\omega})$ has to look like this:



"2π-periodic"

- If you write the "whole thing" as a function, you get

$$
\begin{aligned}
X(e^{j\omega}) = \; & \cdots + \pi\delta\left(\omega - \left(-4\pi - \tfrac{\pi}{8}\right)\right) + \pi\delta\left(\omega - \left(-4\pi + \tfrac{\pi}{8}\right)\right) \\
& + \pi\delta\left(\omega - \left(-2\pi - \tfrac{\pi}{8}\right)\right) + \pi\delta\left(\omega - \left(-2\pi + \tfrac{\pi}{8}\right)\right) \\
& + \pi\delta\left(\omega - \left(-\pi/8\right)\right) + \pi\delta\left(\omega - \pi/8\right) + \pi\delta\left(\omega - \left(2\pi - \pi/8\right)\right) \\
& + \pi\delta\left(\omega - \left(2\pi + \pi/8\right)\right) + \pi\delta\left(\omega - \left(4\pi - \pi/8\right)\right) \\
& + \pi\delta\left(\omega - \left(4\pi + \pi/8\right)\right) + \cdots
\end{aligned}
$$

$\Rightarrow$ The pattern of the terms in this sum is

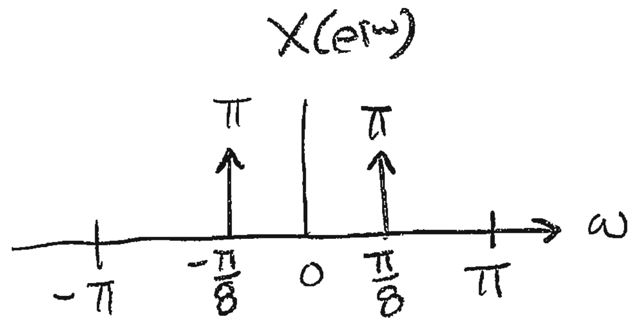
$$\pi\delta\left(\omega - \left(2\pi\ell - \pi/8\right)\right) + \pi\delta\left(\omega - \left(2\pi\ell + \pi/8\right)\right),$$

where ℓ is any integer.

- We can write this with a do loop:

$$X(e^{j\omega}) = \pi \sum_{\ell=-\infty}^{\infty} \left\{ \delta(\omega - \tfrac{\pi}{8} - 2\pi\ell) + \delta(\omega + \tfrac{\pi}{8} - 2\pi\ell) \right\}$$

→ the $\ell = 0$ term of the loop gives us:



⇒ which is the only part that matters for the inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

⇒ because the integration only goes from $\omega = -\pi$ to π.

⇒ The rest of the terms in the do loop above for $\ell \neq 0$ just "plop down" more copies every 2π to make $X(e^{j\omega})$ 2π-periodic.

→ But they don't enter into the inverse transform when we integrate from $-\pi$ to π.

- Similar reasoning explains the mysterious looking transforms

$$e^{j\omega_0 n} \xleftrightarrow{\text{DTFT}} 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi\ell)$$

$$\cos\omega_0 n \xleftrightarrow{\text{DTFT}} \pi \sum_{\ell=-\infty}^{\infty} \left\{ \delta(\omega - \omega_0 - 2\pi\ell) + \delta(\omega + \omega_0 - 2\pi\ell) \right\}$$

$$\sin\omega_0 n \xleftrightarrow{\text{DTFT}} \frac{\pi}{j} \sum_{\ell=-\infty}^{\infty} \left\{ \delta(\omega - \omega_0 - 2\pi\ell) - \delta(\omega + \omega_0 - 2\pi\ell) \right\}$$

- Even if ω_0 is <u>outside</u> the range $-\pi$ to π,

  - only one term of the sum will enter into the inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega .$$

  - The other terms of the sum are just there to "plop down" more copies every 2π to make sure that $X(e^{j\omega})$ is 2π - periodic.

DTFT Convolution Property

- We are finally ready to explain why we are even talking about the DTFT in the first place.

- Recall: for any $\omega_0 \in \mathbb{R}$, the signal $x[n] = e^{j\omega_0 n}$ is an eigenfunction of <u>any</u> LTI system H.

- The eigenvalues are given by the system frequency response

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n},$$

where $h[n]$ is the impulse response of the system.

- If the input is $x[n] = e^{j\omega_0 n}$, then the output is given by

$$y[n] = \underbrace{H(e^{j\omega_0})}_{\text{a complex number}} \underbrace{e^{j\omega_0 n}}_{\text{the input signal.}}$$

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix} H \\ \text{LTI} \end{matrix}} \longrightarrow y[n] = H(e^{j\omega_0}) e^{j\omega_0 n}$$

- For an <u>arbitrary</u> input signal $x[n]$, we saw that $y[n] = x[n] * h[n]$ (convolution).

- Now, in Module 4, we use the DTFT to write $x[n]$ as a sum of the spectral basis $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$ like this:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega,$$

$$\text{where } X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

- We also write the output signal $y[n]$ as a sum of the spectral basis like this:

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n} \, d\omega,$$

$$\text{where } Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}.$$

- This is called the <u>frequency</u> <u>domain</u>

(It is just a change of basis)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \longrightarrow \boxed{\begin{matrix} H \\ \text{LTI} \end{matrix}} \longrightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}\,d\omega$$

- For frequencies ω_1, ω_2, ω_3, etc...,

  $x[n]$ is a sum of terms $\frac{1}{2\pi}X(e^{j\omega_1})e^{j\omega_1 n}$, $\frac{1}{2\pi}X(e^{j\omega_2})e^{j\omega_2 n}$, $\frac{1}{2\pi}X(e^{j\omega_3})e^{j\omega_3 n}$, and lots of others...

- Every one of them is an eigenfunction of the LTI system H.

- So every term in the sum simply gets multiplied times an eigenvalue when it goes through the system.

Input term in $x[n]$	makes	Output term in $y[n]$
$\vdots$		$\vdots$
$\frac{1}{2\pi}X(e^{j\omega_1})e^{j\omega_1 n}$	$\longmapsto$	$\frac{1}{2\pi}X(e^{j\omega_1})H(e^{j\omega_1})e^{j\omega_1 n}$
$\frac{1}{2\pi}X(e^{j\omega_2})e^{j\omega_2 n}$	$\longmapsto$	$\frac{1}{2\pi}X(e^{j\omega_2})H(e^{j\omega_2})e^{j\omega_2 n}$
$\frac{1}{2\pi}X(e^{j\omega_3})e^{j\omega_3 n}$	$\longmapsto$	$\frac{1}{2\pi}X(e^{j\omega_3})H(e^{j\omega_3})e^{j\omega_3 n}$
$\vdots$		$\vdots$

- And $y[n]$ is exactly the sum of all these output terms... because H is linear:

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}\,d\omega$$

- But we also have by definition that

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\,e^{j\omega n}\,d\omega$$

- Comparing these two equations, we see that

~~thus~~ $$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$\Rightarrow$ In other words, the numbers $Y(e^{j\omega})$ that we need to write $y[n]$ as a sum of the spectral basis are given by the product

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$\rightarrow$ Our change of basis turns convolution into multiplication.

$\rightarrow$ This is called the convolution property of the DTFT.

- Formal statement of the DTFT convolution property:

if $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ and $h[n] \xleftrightarrow{DTFT} H(e^{j\omega})$

and $y[n] = x[n] * h[n]$,

then $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$.

<u>EX</u>: on problem 3 of Homework 4, we had an LTI system H with input $x[n] = (\frac{1}{4})^n u[n]$ and impulse response $h[n] = (\frac{1}{3})^n u[n]$. We used time domain convolution to show that

$$y[n] = x[n] * h[n] = \begin{cases} 0 & , n<0 \\ 4(\frac{1}{3})^n - 3(\frac{1}{4})^n & , n \geq 0 \end{cases}$$

$$= \left[4(\tfrac{1}{3})^n - 3(\tfrac{1}{4})^n\right] u[n].$$

- Use the DTFT to solve this problem:

Table: $X(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{4}e^{-j\omega}}$

Table $H(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

- To get this into terms that are in our DTFT table, we have to do a partial fraction expansion (PFE) like this:

$$\frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} = \frac{A}{(1-\frac{1}{4}e^{-j\omega})} + \frac{B}{(1-\frac{1}{3}e^{-j\omega})}$$

$\Rightarrow$ A and B are constants.

- We will use the "Heaviside cover up method" to find them.

- First, to avoid having to do complex arithmetic when computing the PFE, we replace $e^{-j\omega}$ with "θ".

- This is one of the main reasons that we write the DTFT as $X(e^{j\omega})$ instead of $X(\omega)$.

$\rightarrow$ $e^{j\omega}$ is called the <u>character</u>

- Replacing e^{-jw} with θ, we get

$$\frac{1}{(1-\frac{1}{4}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{(1-\frac{1}{4}\theta)} + \frac{B}{(1-\frac{1}{3}\theta)}$$

- To find A, we multiply both sides by the "thing" thats under A and then evaluate both sides at the value of θ that makes the "thing" under A zero....

→ In other words, we multiply both sides by $(1-\frac{1}{4}\theta)$ and evaluate at $\theta = 4$.

- multiply both sides by $(1-\frac{1}{4}\theta)$:

$$\frac{(1-\frac{1}{4}\theta)}{(1-\frac{1}{4}\theta)(1-\frac{1}{3}\theta)} = \frac{A(1-\frac{1}{4}\theta)}{(1-\frac{1}{4}\theta)} + \frac{B(1-\frac{1}{4}\theta)}{(1-\frac{1}{3}\theta)}$$

$$\frac{1}{1-\frac{1}{3}\theta} = A + \frac{B(1-\frac{1}{4}\theta)}{1-\frac{1}{3}\theta}$$

- Now evaluate at $\theta = 4$:

$$\frac{1}{1-\frac{1}{3}\cdot 4} = A + \cancelto{0}{\frac{B(1-1)}{1-\frac{1}{3}\cdot 4}}$$

$$A = \frac{1}{1-4/3} = \frac{1}{-1/3} = -3$$

- This always works the same way.
- When you mutiply both sides by $(1-\frac{1}{4}\theta)$ and evaluate at $\theta=4$, you get "$=A$" on the right side.
  - On the left side, you get the left-hand side with $(1-\frac{1}{4}\theta)$ removed from the denominator... evaluated at $\theta=4$.
- You can use these facts to perform both steps at the same time... like this:

$$\frac{1}{(1-\frac{1}{4}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1-\frac{1}{4}\theta} + \frac{B}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1}{1-\frac{1}{3}\theta}\right|_{\theta=4} = \frac{1}{1-4/3} = \frac{1}{-1/3} = -3$$

$$B = \left.\frac{1}{1-\frac{1}{4}\theta}\right|_{\theta=3} = \frac{1}{1-3/4} = \frac{1}{1/4} = 4$$

- Now plug A & B back into the PFE expression for $Y(e^{j\omega})$:

$$Y(e^{j\omega}) = \frac{-3}{1-\frac{1}{4}e^{-j\omega}} + \frac{4}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $y[n] = 4\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{4}\right)^n u[n]$ ✓

→ Agrees with the answer we got by convolution on HW 4.

- The Heaviside cover up method works the same way even when the left side has more "roots" downstairs and a nontrivial numerator.

- For Example, suppose you got

$$Y(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})\underbrace{(1+\frac{1}{2}e^{-j\omega})}_{a=-\frac{1}{2}}(1-\frac{1}{3}e^{-j\omega})}$$

- You would do it like this:

$$\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)(1+\frac{1}{2}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1+\frac{1}{2}\theta} + \frac{C}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1-\frac{1}{4}\theta}{(1+\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=2} = \frac{1-\frac{1}{2}}{(1+1)(1-2/3)} = \frac{1/2}{2\cdot 1/3}$$

$$= \frac{\frac{1}{2}}{2/3} = \frac{3}{2}\cdot\frac{1}{4} = 3/4$$

$$B = \left.\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=-2} = \frac{1+\frac{1}{2}}{(1+1)(1+2/3)} = \frac{3/2}{2\cdot 5/3}$$

$$= \frac{3/2}{10/3} = \frac{3}{10}\cdot\frac{3}{2} = \frac{9}{20}$$

$$C = \left.\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)(1+\frac{1}{2}\theta)}\right|_{\theta=3} = \frac{1-\frac{3}{4}}{(1-\frac{3}{2})(1+3/2)} = \frac{1/4}{(-1/2)(5/2)}$$

$$= \frac{1/4}{-5/4} = -\frac{4}{5}\cdot\frac{1}{4} = -\frac{1}{5}$$

$\longrightarrow$

$$Y(e^{j\omega}) = \frac{3/4}{1-\frac{1}{2}e^{-j\omega}} + \frac{9/20}{1+\frac{1}{2}e^{-j\omega}} - \frac{1/5}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $y[n] = \frac{3}{4}\left(\frac{1}{2}\right)^n u[n] + \frac{9}{20}\left(-\frac{1}{2}\right)^n u[n]$

$$- \frac{1}{5}\left(\frac{1}{3}\right)^n u[n]$$

NOTE: if you multiply out the numerator and denominator of a DTFT, you will see that they are polynomials in the "chararter" $e^{-j\omega}$.

- In the example we just did, we could multiply out to get:

$$Y(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$\longrightarrow$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}-\frac{1}{2}e^{-j\omega}-\frac{1}{4}e^{-j2\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{\left(1-\frac{1}{4}e^{-j2\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}-\frac{1}{4}e^{-j2\omega}+\frac{1}{12}e^{-j3\omega}}$$

→ numerator is a first-order polynomial in $e^{-j\omega}$

→ denominator is a third-order polynomial in $e^{-j\omega}$.

→ if you write "x" instead of $e^{-j\omega}$, it would be

$$\frac{1-\frac{1}{4}x}{1-\frac{1}{3}x-\frac{1}{4}x^2+\frac{1}{12}x^3}$$

which looks more like the polynomials you saw in high school.

Recall from high school: An n'th order polynomial has n roots.

→ The roots are the numbers x that, when plugged in, make the polynomial zero.

⇒ If the denominator polynomial of a DTFT has any <u>repeated roots</u>, then the Heaviside cover up method requires <u>more steps</u>.

⇒ In this case, the <u>denominator</u> of the DTFT will contain one or more terms of the form

$(1-ae^{-j\omega})^N$, where $N > 1$.

→ In this case, the PFE must have terms $\frac{A_1}{(1-ae^{-j\omega})} + \frac{A_2}{(1-ae^{-j\omega})^2} + \cdots + \frac{A_N}{(1-ae^{-j\omega})^N}$

- in other words, it has to have $\frac{1}{(1-ae^{-j\omega})^K}$ for all powers K from $K=1$ up to the highest power that appears in the denominator... which is called the <u>multiplicity</u> of the repeated root.

- Here's an example:

$$Y(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{\underbrace{(1-\frac{1}{2}e^{-j\omega})^2}_{\substack{\text{repeated root} \\ \text{term with} \\ \text{multiplicity 2.}}}(1-\frac{1}{3}e^{-j\omega})}$$

- For the PFE, we would write

$$\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)^2(1-\frac{1}{3}\theta)} = \frac{A_1}{1-\frac{1}{2}\theta} + \frac{A_2}{(1-\frac{1}{2}\theta)^2} + \frac{B}{1-\frac{1}{3}\theta}$$

- You can find A_2 and B using the basic Heaviside cover up method:

$$A_2 = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{1-\frac{1}{2}}{1-2/3}$$

$$= \frac{1/2}{1/3} = \frac{3}{2}$$

$$B = \left.\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)^2}\right|_{\theta=3} = \frac{1-\frac{3}{4}}{(1-3/2)^2} = \frac{1/4}{(-1/2)^2}$$

$$= \frac{1/4}{1/4} = 1$$

- But this will fail for A_1. If we multiply both sides by $(1-\frac{1}{2}\theta)$ [the thing that's under A_1] and evaluate at $\theta=2$, we get $\infty = \infty$, which doesn't help us find A_1.

- To find A_1, you multiply both sides by what's under A_2 (the highest power of the repeated root).

- Then you have to differentiate both sides with respect to θ... i.e., take $\frac{d}{d\theta}$ on both sides.

- Then evaluate both sides at $\theta = 2$ (the value that makes the repeated root term zero).

- This is usually cumbersome because you have to differentiate quotients... requiring the quotient rule (or product rule), and terms $(1 - a\theta)^N$, requiring the chain rule.

- To find A_1 in our current example, you multiply both sides by $(1-\frac{1}{2}\theta)^2$, take $\frac{d}{d\theta}$ on both sides, then plug in $\theta=2$:

$$\frac{1-\frac{1}{4}\theta}{1-\frac{1}{3}\theta} = A_1(1-\tfrac{1}{2}\theta) + A_2 + \frac{B(1-\frac{1}{2}\theta)^2}{(1-\frac{1}{3}\theta)}$$

- To avoid using the quotient rule... and use the product rule instead, I will write this as:

$$(1-\tfrac{1}{4}\theta)(1-\tfrac{1}{3}\theta)^{-1} = A_1(1-\tfrac{1}{2}\theta) + A_2 + B(1-\tfrac{1}{2}\theta)^2(1-\tfrac{1}{3}\theta)^{-1}$$

- Now I have to take $\frac{d}{d\theta}$ on both sides (and my heart sinks, because it will be a pain)

$$\frac{d}{d\theta}\left[\text{left side}\right] = \frac{d}{d\theta}\left[(1-\tfrac{1}{4}\theta)(1-\tfrac{1}{3}\theta)^{-1}\right]$$

$$= \left[\frac{d}{d\theta}(1-\tfrac{1}{4}\theta)\right](1-\tfrac{1}{3}\theta)^{-1} + (1-\tfrac{1}{4}\theta)\left[\frac{d}{d\theta}(1-\tfrac{1}{3}\theta)^{-1}\right]$$

$$= -\tfrac{1}{4}(1-\tfrac{1}{3}\theta)^{-1} + (1-\tfrac{1}{4}\theta)(-1)(1-\tfrac{1}{3}\theta)^{-2}(-\tfrac{1}{3})$$

$$= \frac{-1/4}{(1-\tfrac{1}{3}\theta)} + \frac{\tfrac{1}{3}(1-\tfrac{1}{4}\theta)}{(1-\tfrac{1}{3}\theta)^2}$$

$$\frac{d}{d\theta}\left[\text{right side}\right] = \frac{d}{d\theta}\left[A_1(1-\tfrac{1}{2}\theta) + A_2 + B(1-\tfrac{1}{2}\theta)^2(1-\tfrac{1}{3}\theta)^{-1}\right]$$

$$= -\tfrac{1}{2}A_1 + 0 + B\left\{\left[\frac{d}{d\theta}(1-\tfrac{1}{2}\theta)^2\right](1-\tfrac{1}{3}\theta)^{-1} + (1-\tfrac{1}{2}\theta)^2\left[\frac{d}{d\theta}(1-\tfrac{1}{3}\theta)^{-1}\right]\right\}$$

$$= -\tfrac{1}{2}A_1 + B\left\{2(1-\tfrac{1}{2}\theta)(-\tfrac{1}{2})(1-\tfrac{1}{3}\theta)^{-1} + (1-\tfrac{1}{2}\theta)^2(-1)(1-\tfrac{1}{3}\theta)^{-2}(-\tfrac{1}{3})\right\}$$

$$= -\tfrac{1}{2}A_1 + B\left\{\frac{-(1-\tfrac{1}{2}\theta)}{1-\tfrac{1}{3}\theta} + \frac{\tfrac{1}{3}(1-\tfrac{1}{2}\theta)^2}{(1-\tfrac{1}{3}\theta)^2}\right\}$$

- So, after taking $\frac{d}{d\theta}$ on both sides, I've got

$$\frac{-1/4}{(1-\frac{1}{3}\theta)} + \frac{\frac{1}{3}(1-\frac{1}{4}\theta)}{(1-\frac{1}{3}\theta)^2} = -\frac{1}{2}A_1 + B\left\{\frac{-(1-\frac{1}{2}\theta)}{1-\frac{1}{3}\theta} + \frac{\frac{1}{3}(1-\frac{1}{2}\theta)^2}{(1-\frac{1}{3}\theta)^2}\right\}$$

- Now we evaluate <u>both</u> sides of this at $\theta = 2$:

$$\frac{-\frac{1}{4}}{1-\frac{2}{3}} + \frac{\frac{1}{3}(1-\frac{1}{2})}{(1-\frac{2}{3})^2} = -\frac{1}{2}A_1 + B\left\{\frac{0}{1-2/3} + \frac{\frac{1}{3}\cdot 0^2}{(1-\frac{2}{3})^2}\right\}$$

$$\frac{-\frac{1}{4}}{1/3} + \frac{\frac{1}{3}(\frac{1}{2})}{(1/3)^2} = -\frac{1}{2}A_1$$

$$-\frac{3}{4} + \frac{1/6}{1/9} = -\frac{1}{2}A_1$$

$$-\frac{3}{4} + \frac{9}{6} = -\frac{1}{2}A_1$$

$$-\frac{3}{4} + \frac{3}{2} = -\frac{1}{2}A_1$$

$$\frac{-3+6}{4} = -\frac{1}{2}A_1 \longmapsto \frac{3}{4} = -\frac{1}{2}A_1 \longmapsto A_1 = \frac{-6}{4} = -\frac{3}{2}$$

Whew!!

- For a third-order root you have to <u>also</u> differentiate <u>twice</u>.

- For an N^{th}-order root, you have terms

$$\frac{A_1}{1-ae^{-j\omega}} + \frac{A_2}{(1-ae^{-j\omega})^2} + \cdots + \frac{A_N}{(1-ae^{-j\omega})^N}$$

→ You don't have to differentiate to find A_N

→ $\frac{d}{d\theta}$ to find A_{N-1}

→ $\frac{d^2}{d\theta^2}$ to find A_{N-2}

→ $\frac{d^3}{d\theta^3}$ to find A_{N-3}

$\vdots$

→ $\frac{d^{N-1}}{d\theta^{N-1}}$ to find A_1

⇒ It is <u>miserable</u>, but it <u>works</u>.

- In ECE 2713, I will not test you on repeated roots,

→ **Except** for possibly the two cases that are actually in your DTFT table:

$$(n+1)a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{(1-ae^{-j\omega})^2}$$

$$|a|<1$$

$$\frac{(n+r-1)!}{n(r-1)!} a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{(1-ae^{-j\omega})^r}$$

$$|a|<1$$

- A ratio of two polynomials is called a "rational function".

- Most DTFT's that you will see in LTI systems turn out to be rational functions of $e^{-j\omega}$.

  - Numerator is a polynomial in $e^{-j\omega}$

  - Denominator is a polynomial in $e^{-j\omega}$.

- To compute the PFE, you need to convert the denominator from a "sum form" to a "product form".

- In our first example back on pages 4.60 - 4.64, we got

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} = \frac{4}{1-\frac{1}{3}e^{-j\omega}} - \frac{3}{1-\frac{1}{4}e^{-j\omega}}$$

Multiplying out the denominator on the left side, we get

$$\frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} = \frac{1}{1-\frac{1}{3}e^{-j\omega}-\frac{1}{4}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}}$$

$$= \frac{1}{1-\frac{4}{12}e^{-j\omega}-\frac{3}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}}$$

$$= \frac{1}{1-\frac{7}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}}$$

So:

$$\frac{1}{\underbrace{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}_{\text{denom is a product}}} = \frac{1}{\underbrace{1-\frac{7}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}}_{\text{denom is a sum}}}$$

$\Rightarrow$ You <u>must</u> be able to go back and forth between these two forms.

- To go from the product form to the sum form, you simply multiply out as we just saw:

$$(1-\tfrac{1}{4}\theta)(1-\tfrac{1}{3}\theta) = 1-\tfrac{1}{3}\theta-\tfrac{1}{4}\theta+\tfrac{1}{12}\theta^2$$

$$= 1-\tfrac{7}{12}\theta+\tfrac{1}{12}\theta^2.$$

- To go the other way, <u>use the foil rule</u>.

- write: $1-\frac{7}{12}\theta+\frac{1}{12}\theta^2 = (1+a\theta)(1+b\theta)$

→ we need numbers a and b such that $a+b = -\frac{7}{12}$

$$ab = \tfrac{1}{12}$$

- So a and b need to both be negative... so that the sum is negative but the product is positive.

- Since the product $ab = \frac{1}{12}$ has a "12" downstairs, our choices for a, b

are 1 and $\frac{1}{12}$, -1 and $-\frac{1}{12}$, $\frac{1}{3}$ and $\frac{1}{4}$, or $-\frac{1}{3}$ and $-\frac{1}{4}$.

→ Only $-\frac{1}{3}$ and $-\frac{1}{4}$ can give us the sum $a+b = -\frac{7}{12}$.

* So, by <u>foil</u> <u>rule</u>,

$$\text{denom} = 1 - \tfrac{7}{12}\theta + \tfrac{1}{12}\theta^2 = (1-\tfrac{1}{4}\theta)(1-\tfrac{1}{3}\theta)$$

$\Rightarrow$ And:
$$\frac{1}{1-\frac{7}{12}e^{-j\omega}+\frac{1}{12}e^{-j2\omega}} = \frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

- Sometimes the roots of the denominator turn out to be <u>complex</u> ... and you have to use the quadratic formula to factor the denominator.
  - We will try to avoid this in ECE 2713.

Time Shift Property of DTFT

- This is the <u>second</u> most important property.

- If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$, then:

$$x[n-1] \longleftrightarrow e^{-j\omega} X(e^{j\omega})$$

$$x[n-2] \longleftrightarrow e^{-j2\omega} X(e^{j\omega})$$

$$x[n-3] \longleftrightarrow e^{-j3\omega} X(e^{j\omega})$$

$$\vdots$$

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$\Rightarrow$ This also works when n_0 is <u>negative</u>:

$$x[n- -1] = x[n+1] \Leftrightarrow e^{j\omega} X(e^{j\omega})$$

$$x[n- -2] = x[n+2] \Leftrightarrow e^{j2\omega} X(e^{j\omega})$$

$$\vdots$$

etc.