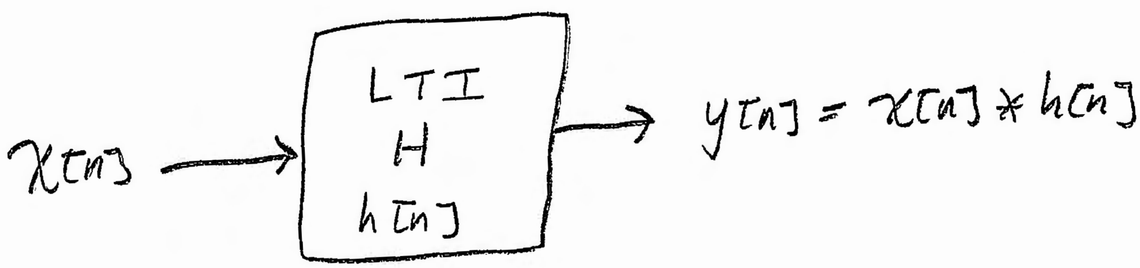
Module 5: LTI Systems in the Frequency Domain

- For working in the time domain, we wrote our signals $x[n]$ as a sum of the shifted kronecker deltas: basis: $\{\delta[n-k]\}_{k\in\mathbb{Z}}$

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$
$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- From this, it followed in about two steps that the output $y[n]$ of a LTI system H is given by the convolution of the input signal $x[n]$ with the system impulse response $h[n]$:



- Then in module 4, we wrote our signals $x[n]$ instead as a sum of the spectral basis signals $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega , \qquad (*)$$

where, for each ω, the number $X(e^{j\omega})$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- When we think of $x[n]$ as being a sum of sinusoidal signals as shown in $(*)$ above, it's called the "frequency domain."

- For an LTI system H, we got that:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \longrightarrow \boxed{\begin{matrix} H \\ \text{LTI} \end{matrix}} \longrightarrow y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n} \, d\omega$$

where: $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

– The function

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

is called the <u>frequency</u> <u>response</u> of the LTI system H.

→ $H(e^{j\omega})$ gives the eigenvalues of the system H that are associated with the eigenfunctions $\{e^{j\omega n}\}_{\omega \in [-\pi, \pi)}$.

→ When we write $x[n]$ as a sum of numbers $X(e^{j\omega})$ times these eigenfunctions:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega,$$

→ The action of the system is simply to multiply each term in the sum times its associated eigenvalue:

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{H(e^{j\omega})\, X(e^{j\omega})}_{Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})}\, e^{j\omega n}\, d\omega$$

- In general, the eigenvalues $H(e^{j\omega})$ are complex numbers.

- So another way to think of what an LTI system does is this:

  - The input signal $x[n]$ can be thought of as a sum of complex sinusoids. If $x[n]$ is real, the imaginary parts all cancel out and $x[n]$ is actually a sum of real-valued cosines & sines (Notes pp. 4.47–4.49).

  - When the signal $x[n]$ goes through a discrete-time LTI system H, each term in the sum gets multiplied by a complex-valued eigenvalue.

  - The eigenvalues are given by the system frequency response $H(e^{j\omega})$.

$\Rightarrow$ Therefore, to understand what the LTI system H really does, it is very important for us to understand what happens when a sinusoidal input term $X(e^{j\omega})e^{j\omega n}$ gets multiplied by a complex number $H(e^{j\omega})$

$$\Bigg\{ \underbrace{X(e^{j\omega})}_{\text{a number}} \underbrace{e^{j\omega n}}_{\cos\omega n + j\sin\omega n}$$

- Suppose that $x[n]$ is input to an LTI system H with frequency response $H(e^{j\omega}) = \text{DTFT}\{h[n]\}$, where $h[n]$ is the system impulse response.

- For some particular frequency $\omega_0 \in [-\pi, \pi)$, the signal $X(e^{j\omega_0})e^{j\omega_0 n}$ is <u>one</u> term in the sum $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$.

  → Since $X(e^{j\omega_0})$ is a complex number, we can write it in polar form as

$$X(e^{j\omega_0}) = \underbrace{|X(e^{j\omega_0})|}_{\text{magnitude}}\, e^{j\underbrace{\arg H(e^{j\omega_0})}_{\text{angle}}}.$$

  → So the term $X(e^{j\omega_0})e^{j\omega_0 n}$ in $x[n]$ has the form

$$|X(e^{j\omega_0})|\, e^{j\arg X(e^{j\omega_0})}\, e^{j\omega_0 n}$$

$$= |X(e^{j\omega_0})|\, e^{j[\omega_0 n + \arg X(e^{j\omega_0})]}$$

$$= |X(e^{j\omega_0})| \cos\left[\omega_0 n + \arg X(e^{j\omega_0})\right]$$

$$+ j\,|X(e^{j\omega_0})| \sin\left[\omega_0 n + \arg X(e^{j\omega_0})\right]$$

→ The real part is a cosine with magnitude $|X(e^{j\omega_0})|$, frequency ω_0, and initial phase offset $\arg H(e^{j\omega_0})$.

→ The imaginary part is a sine with magnitude $|X(e^{j\omega_0})|$, frequency ω_0, and initial phase offset $\arg H(e^{j\omega_0})$.

⇒ When $x[n]$ goes through the system H, this input term $X(e^{j\omega_0})e^{j\omega_0 n}$ gets multiplied by the complex eigenvalue $H(e^{j\omega_0})$.
[because $X(e^{j\omega_0})e^{j\omega_0 n}$ is a number times the eigenfunction $e^{j\omega_0 n}$ ... so $X(e^{j\omega_0})e^{j\omega_0 n}$ is still an eigenfunction of the system H].

→ Since the eigenvalue $H(e^{j\omega_0})$ is a complex number, we can write it in polar form as $H(e^{j\omega_0}) = |H(e^{j\omega_0})| e^{j\arg H(e^{j\omega_0})}$.

→

→ So the term $X(e^{j\omega_0})e^{j\omega_0 n}$ in the input signal $x[n]$ gets turned into a term $H(e^{j\omega_0})X(e^{j\omega_0})e^{j\omega_0 n}$ in the output signal $y[n]$.

→ This has <u>two effects</u> on the input term $X(e^{j\omega_0})e^{j\omega_0 n}$:

1. It scales the <u>magnitude</u> of the term by the magnitude of the eigenvalue, which is $|H(e^{j\omega_0})|$.
2. It shifts the <u>phase</u> of the term by the angle of the eigenvalue, which is $\arg H(e^{j\omega_0})$.

→

- In other words, the term $X(e^{j\omega_0})e^{j\omega_0 n}$ in the input signal $x[n]$ becomes a term in the output signal $y[n]$ that is given by:

$$\underbrace{H(e^{j\omega_0})}_{\text{eigenvalue}}\Big\{\underbrace{|X(e^{j\omega_0})|\, e^{j[\omega_0 n + \arg X(e^{j\omega_0})]}}_{\text{input term}}\Big\}$$

$$= |H(e^{j\omega_0})| e^{j\arg H(e^{j\omega_0})} |X(e^{j\omega_0})| e^{j[\omega_0 n + \arg X(e^{j\omega_0})]}$$

$$= \underbrace{|H(e^{j\omega_0})|}_{\substack{\text{magnitude scaled by} \\ \text{magnitude of} \\ \text{the eigenvalue}}} \cdot |X(e^{j\omega_0})|\, e^{j[\omega_0 n + \arg X(e^{j\omega_0}) + \underbrace{\arg H(e^{j\omega_0})}_{\substack{\text{phase shifted} \\ \text{by the angle} \\ \text{of the eigenvalue}}}]}$$

$$= |H(e^{j\omega_0})|\,|X(e^{j\omega_0})| \cos[\omega_0 n + \arg X(e^{j\omega_0}) + \arg H(e^{j\omega_0})]$$

$$+ j|H(e^{j\omega_0})|\,|X(e^{j\omega_0})| \sin[\omega_0 n + \arg X(e^{j\omega_0}) + \arg H(e^{j\omega_0})]$$

☆☆ And the output signal $y[n]$ is the <u>sum</u> of all of these terms for all the frequencies $\omega \in [-\pi, \pi)$.

- In other words,

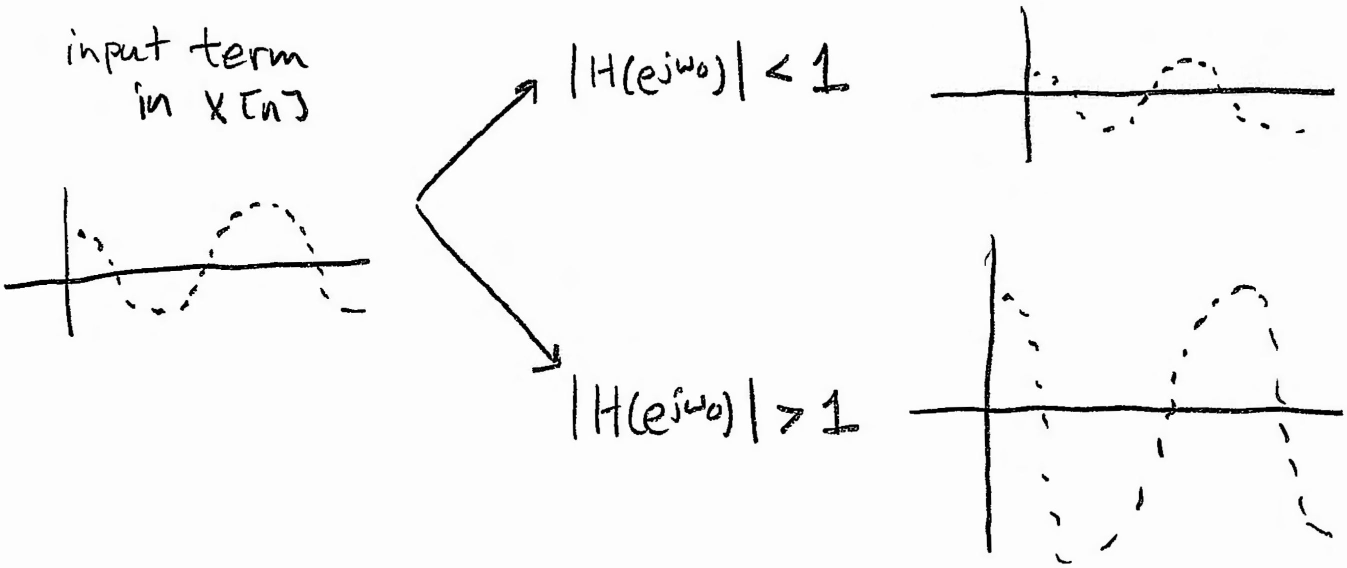
$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{H(e^{j\omega}) X(e^{j\omega})}_{Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})} e^{j\omega n} \, d\omega$$

☆☆☆ Each sinusoidal term in the input signal gets multiplied by a complex eigenvalue, which:

① scales the magnitude of the input term by the magnitude of the eigenvalue

② shifts the phase of the input term by the angle of the eigenvalue.

- It's pretty easy to understand what it means when you scale the magnitude of a sinusoidal input term;



- But how should we think about the phase shift?

☆☆☆ $\Rightarrow$ It is a <u>time</u> <u>shift</u> of the input term.

☆☆☆ <u>FACT</u>: the effect of the phase shift is to <u>delay</u> the input term $X(e^{j\omega_0})e^{j\omega_0 n}$

by $-\dfrac{\arg H(e^{j\omega_0})}{\omega_0}$.

- Here's how to see that:

→ On page 5.8, we got that the input term

$$X(e^{j\omega_0})e^{j\omega_0 n} = |X(e^{j\omega_0})|\cos[\omega_0 n + \arg X(e^{j\omega_0})] + j|X(e^{j\omega_0})|\sin[\omega_0 n + \arg X(e^{j\omega_0})]$$

- makes an output term

$$|H(e^{j\omega_0})||X(e^{j\omega_0})|\cos[\omega_0 n + \arg X(e^{j\omega_0}) + \arg H(e^{j\omega_0})] + j|H(e^{j\omega_0})||X(e^{j\omega_0})|\sin[\omega_0 n + \arg X(e^{j\omega_0}) + \arg H(e^{j\omega_0})]$$

$$= |H(e^{j\omega_0})||X(e^{j\omega_0})|\cos\left[\omega_0 n + \frac{\omega_0 \arg H(e^{j\omega_0})}{\omega_0} + \arg X(e^{j\omega_0})\right] + j|H(e^{j\omega_0})||X(e^{j\omega_0})|\sin\left[\omega_0 n + \frac{\omega_0 \arg H(e^{j\omega_0})}{\omega_0} + \arg X(e^{j\omega_0})\right]$$

$$= |H(e^{j\omega_0})||X(e^{j\omega_0})|\cos\left\{\omega_0\left[n - \underbrace{\frac{-\arg H(e^{j\omega_0})}{\omega_0}}_{\text{Time shift (delay) relative to the input term}}\right] + \arg X(e^{j\omega_0})\right\}$$

$$+ j|H(e^{j\omega_0})||X(e^{j\omega_0})|\sin\left\{\omega_0\left[n - \frac{-\arg H(e^{j\omega_0})}{\omega_0}\right] + \arg X(e^{j\omega_0})\right\}$$

⇒ So both the real and imaginary parts of the sinusoidal input term get <u>time</u> <u>delayed</u>...

- And this is the intuitive way to understand the effect of the phase shift.

Putting it All Together for Pages 5.1 – 5.11 :

- A discrete-time LTI system H has a frequency response $H(e^{j\omega})$ that is the DTFT of the impulse response $h[n]$.

- We think of $H(e^{j\omega})$ as a function that gives us the complex-valued eigenvalues of the system H for each frequency $\omega \in [-\pi, \pi)$.

- We think of the input signal $x[n]$ as a sum of sinusoidal terms $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$.

  ⇒ Each term of this sum is an eigenfunction of the LTI system H.

  ⇒ So when $x[n]$ goes through the LTI system H, each term gets multiplied times an eigenvalue.

⇒ So the output signal $y[n]$ is a sum of sinusoidal terms

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) X(e^{j\omega}) e^{j\omega n} \, d\omega$$

⇒ The intuitive effect of multiplying by these eigenvalues is that:

1. each sinusoidal term in the sum gets scaled by the magnitude of its eigenvalue.

2. each sinusoidal term in the sum gets time shifted (delayed) by the angle of its eigenvalue divided by ω... the time delay is $-\frac{\arg H(e^{j\omega})}{\omega}$.

⇒ This is a much easier way to think of what an LTI system does... much easier than trying to intuitively understand the convolution equation.

Let's say this exact same thing one more time in a slightly different way:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad\qquad y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n}\, d\omega$$

complex numbers   sinusoids   complex numbers   sinusoids

- So $x[n]$ is a sum of sinusoids $e^{j\omega n}$ with frequencies $-\pi \le \omega < \pi$. In the sum, these sinusoids are weighted by the numbers $X(e^{j\omega})$.

  $\Rightarrow$ Remember where those numbers come from. For each $\omega \in [-\pi, \pi)$, the <u>weight</u> of the sinusoid $e^{j\omega n}$ in the sum for $x[n]$ is given by the dot product of $x[n]$ with $e^{j\omega n}$:

  $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

  (a number, or weight, for each ω).

- Similarly, $y[n]$ is a sum of those exact same sinusoids $e^{j\omega n}$ with frequencies $-\pi \le \omega < \pi$. In the sum for $y[n]$, the weights of the sinusoids are given by the numbers $Y(e^{j\omega})$.

- The relationship between the weights $Y(e^{j\omega})$ in the output signal $y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n}\, d\omega$ (**)

  - and the weights $X(e^{j\omega})$ in the input signal $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$ (*)

- is this:

Each term $X(e^{j\omega}) e^{j\omega n}$ in the sum (*) for $x[n]$ gets multiplied by a complex eigenvalue $H(e^{j\omega})$ when $x[n]$ goes through the system...

to make an output term

$$Y(e^{j\omega}) e^{j\omega n} = H(e^{j\omega}) X(e^{j\omega}) e^{j\omega n}$$

in the sum (**) for $y[n]$.

- This happens because H is a linear system... so putting in the <u>sum</u> (*) to the system is the same as putting in each term $X(e^{j\omega}) e^{j\omega n}$ separately and then adding up the individual outputs (times $\frac{1}{2\pi}$).

- The eigenvalues are given by the discrete-time Fourier transform of the impulse response:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}.$$

As we saw on pages 4.11, 4.38, and 4.39

- We can write all of this using operator notation like this:

$$y[n] = H\{x[n]\}$$

$$= H\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H\{X(e^{j\omega}) e^{j\omega n}\}\, d\omega \quad \leftarrow \text{Because } H \text{ is linear}$$

(***) $$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) X(e^{j\omega}) e^{j\omega n}\, d\omega \quad \leftarrow$$ Because $X(e^{j\omega})e^{j\omega n}$ is an eigenfunction of the system H with associated eigenvalue $H(e^{j\omega})$

- Comparing (***) above to (**) on PAGE 5.15, we see that $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$.

- Now, if we want to <u>design</u> a digital filter H to process the signal $x[n]$ in a predictable and useful way, the time domain convolution equation

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

is not much help...

- It was good for finding $y[n]$ if we knew $x[n]$ and $h[n]$.
- But it's not much help for <u>designing</u> $h[n]$.

- But the frequency domain equation (***) on PAGE 5.16 is very useful because it gives us an intuitive way to

1. understand what happens to each sinusoid in the sum for $x[n]$ when it goes through the system, and

2. Use this understanding to design the eigenvalues (frequency response) $H(e^{j\omega})$.

$\Rightarrow$ What happens is: each input term $X(e^{j\omega})e^{j\omega n}$ in the input signal $x[n]$ gets multiplied by a complex eigenvalue $H(e^{j\omega})$ to make an output term $H(e^{j\omega})X(e^{j\omega})e^{j\omega n}$ in the output signal $y[n]$.

(Don't forget that the numbers $H(e^{j\omega})$ are generally different for each $\omega \in [-\pi, \pi)$).

$\Rightarrow$ This has two effects on the input term $X(e^{j\omega})e^{j\omega n}$:

→ amplitude gets scaled by $|H(e^{j\omega})|$

→ phase gets shifted by $-\arg H(e^{j\omega})$.

- Once again, here's how to see that:

input term $X(e^{j\omega})e^{j\omega n}$

$\longmapsto$ makes output term

$H(e^{j\omega})X(e^{j\omega})e^{j\omega n}$.

- Writing the number $H(e^{j\omega})$ in polar form as
$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\arg H(e^{j\omega})},$$

- We get that the output term is

$$H(e^{j\omega})X(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|\, X(e^{j\omega})\, e^{j[\omega n + \arg H(e^{j\omega})]}$$

(****) $$= \underbrace{|H(e^{j\omega})|}_{\text{Amplitude scaled by the magnitude of the eigenvalue}} X(e^{j\omega})\, e^{j[\omega n - -\underbrace{\arg H(e^{j\omega})}_{\text{Phase shifted by the angle of the eigenvalue.}}]}$$

- The amplitude scaling part is pretty easy to understand.
  - For example, to make a low pass filter, we design the magnitudes of the eigenvalues to be big for the low frequencies ω and small for the high frequencies ω.

- But to really understand the phase shift part, we need to keep working on it a little more.

(****) on PAGE 5.18 $= |H(e^{j\omega})| X(e^{j\omega}) e^{j[\omega n - - \underbrace{\arg H(e^{j\omega})}_{\text{multiply this by } \frac{\omega}{\omega} = 1}]}$

$$= |H(e^{j\omega})| X(e^{j\omega}) e^{j\left[\omega n - - \frac{\omega \arg H(e^{j\omega})}{\omega}\right]}$$

$$= \underbrace{|H(e^{j\omega})|}_{\text{An amplitude scale}} X(e^{j\omega}) e^{j\omega\left[n - \underbrace{\frac{-\arg H(e^{j\omega})}{\omega}}_{\text{A time shift.}}\right]}$$

☆☆ So each input sinusoidal term $X(e^{j\omega}) e^{j\omega n}$ gets:

- Amplitude scaling by the magnitude of the eigenvalue $|H(e^{j\omega})|$

- Time delayed by an amount $\dfrac{-\arg H(e^{j\omega})}{\omega}$, where

$\arg H(e^{j\omega})$ is the angle of the eigenvalue.

- To <u>design</u> a filter, we design the amplitude scale factors $|H(e^{j\omega})|$ and the time delays $\dfrac{-\arg H(e^{j\omega})}{\omega}$ for all the input sinusoids $e^{j\omega n}$, where $-\pi \le \omega < \pi$

- In other words, we design the eigenvalues for the system H.

- In other words, we design the frequency response $H(e^{j\omega})$.

- The impulse response can then be obtained by taking the inverse DTFT: $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$.

<u>EX</u>: if $x[n]$ is a digital audio signal, then the time/phase alignment between the various input terms $X(e^{j\omega})e^{j\omega n}$ is <u>very</u> <u>important</u>. The exact alignment between the sinusoids $e^{j\omega n}$ is determined by the angles of the numbers $X(e^{j\omega})$.

- If the filter disturbs this alignment, then the output audio signal $y[n]$ will sound <u>terrible</u>.

- So, for a digital audio filter, we need to design the magnitudes of the eigenvalues to achieve the desired processing result... like increase the bass or lower the midrange...

- But it is <u>critical</u> for the spectral phase of the frequency response $\arg H(e^{j\omega})$ to be designed such that <u>ALL INPUT SINUSOIDS GET DELAYED BY THE SAME AMOUNT OF TIME</u>. ☆☆

<u>FACT</u>: This will happen if the spectral phase $\arg H(e^{j\omega})$ is a <u>linear</u> function of ω. (proved on next page)

$\Rightarrow$ In other words, if $\arg H(e^{j\omega}) = C\omega$, where C is a real constant.

$\Rightarrow$ A digital filter H that has $\arg H(e^{j\omega}) = C\omega$ is called a <u>linear phase filter</u>.

→ Linear phase is <u>critical</u> in audio applications

→ Linear phase is <u>highly desirable</u> in many other applications.

- So now let's show that:

  <u>if</u> H is a linear phase filter,
  so that $\arg H(e^{j\omega}) = C\omega$ for some real constant C,
  and $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{jC\omega}$,

  <u>then</u> all input sinusoids get delayed by the same amount of time when they go through the filter.

<u>Proof</u>: let H be a discrete-time LTI filter with frequency response $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{jC\omega}$, where $C \in \mathbb{R}$ is a constant.

Let $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$ be the input signal.

Then each input sinusoid $X(e^{j\omega})\, e^{j\omega n}$ in $x[n]$ becomes a term $H(e^{j\omega})\, X(e^{j\omega})\, e^{j\omega n}$ in the output signal $y[n]$.

This term is given by

$$H(e^{j\omega})\, X(e^{j\omega})\, e^{j\omega n} = |H(e^{j\omega})|\, e^{j\arg H(e^{j\omega})}\, X(e^{j\omega})\, e^{j\omega n}$$

$$= |H(e^{j\omega})|\, e^{jC\omega}\, X(e^{j\omega})\, e^{j\omega n}$$

$$= |H(e^{j\omega})|\, X(e^{j\omega})\, e^{j(\omega n + C\omega)}$$

$$= |H(e^{j\omega})|\, X(e^{j\omega})\, e^{j\omega(n - -C)}$$

$\longrightarrow$

- The time delay of this term is given by $-C$ and does not depend on ω.

- So all sinusoidal input terms $X(e^{j\omega})e^{j\omega n}$ get delayed by the exact same amount when they go through the filter... the delay does not depend on ω.

- So the time alignment between the input sinusoids is preserved at the filter output.

  (we say that "there is no phase distortion introduced by the filter").

QED

TECHNICAL NOTE: the delay amount $-C$ in the above does not have to be an integer. When it is not an integer, the intuition is that when each input sinusoid goes through the filter, it gets "interpolated", which is equivalent to resampling the input sinusoid at a new set of points along its graph.

TECHNICAL NOTE : if the spectral phase of the filter is $\arg H(e^{j\omega}) = C\omega + \phi$, where C and ϕ are real constants and $\phi = \pm\pi$ or $\phi = \pm\pi/2$, then we say that the filter has "generalized linear phase."

- The spectral phase $\arg H(e^{j\omega})$ is still a linear function of ω.
- All input sinusoids will still be time delayed by an amount of $-C$ independent of frequency.
- All input sinusoids will also get an additional phase shift of $e^{\pm j\pi}$ or $e^{\pm j\pi/2}$ that does not depend on frequency and does not correspond to a time delay. Rather, it corresponds to a sign flip and/or to a switching of the real and imaginary parts.
  → Most people still consider this to be a linear phase filter.

Here's how to see it:

If $\arg H(e^{j\omega}) = C\omega + \phi$ and $\phi = -\pi, +\pi, -\pi/2$, or $+\pi/2$, then input term $X(e^{j\omega})e^{j\omega n}$ becomes output term

$$H(e^{j\omega})X(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})| e^{j\overbrace{(C\omega+\phi)}_{\arg H(e^{j\omega})}} X(e^{j\omega})e^{j\omega n}$$

$$= |H(e^{j\omega})| X(e^{j\omega}) e^{j(\omega n - (-C\omega) + \phi)}$$

$$= \underbrace{|H(e^{j\omega})|}_{\text{amplitude scale}} X(e^{j\omega}) e^{j\omega \underbrace{(n - -C)}_{\substack{\text{constant}\\ \text{time delay...}\\ \text{does not depend}\\ \text{on } \omega.}}} \underbrace{e^{j\phi}}_{\substack{\text{extra phase shift}\\ \text{...see below}}}$$

→ if $\phi = \pm\pi$, the extra phase shift is $e^{j\phi} = e^{\pm j\pi} = -1$, which makes the gain negative, but does not depend on ω and does not introduce any additional time delay.

→ if $\phi = \pm\pi/2$, the extra phase shift is $e^{j\phi} = e^{\pm j\pi/2} = \pm j$, which switches the real and imaginary parts and may make the gain negative, but does not depend on ω and does not introduce any additional time delay.

- Whew! All of that stuff on PAGES 5.4 – 5.24 is very important for intuitively understanding what an LTI filter does and for designing LTI filters.

- Now we are going to develop one more similar result for the case when $h[n]$ is real... then we'll get on with the main part of the story.

FACT : if $x[n]$ is a real-valued signal, then the discrete-time Fourier transform $X(e^{j\omega})$ is a conjugate symmetric function of ω.

- This means that :

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad \text{[conjugate symmetric]}$$

$$\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\} \quad \text{[real part is even]}$$

$$\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\} \quad \text{[imaginary part is odd]}$$

$$|X(e^{j\omega})| = |X(e^{-j\omega})| \quad \text{[magnitude is even]}$$

$$\arg X(e^{j\omega}) = -\arg X(e^{-j\omega}) \quad \text{[phase is odd]}$$

→ These relations are shown on page 6 of the formula sheet for Test 2.

→ They are not hard to prove, but we will save that for ECE 3793.

- Now suppose that H is a discrete-time LTI system with an impulse response $h[n]$ that is real.
  - This means that the frequency response $H(e^{j\omega})$ is conjugate symmetric: $H(e^{j\omega}) = H^*(e^{-j\omega})$

    → The spectral magnitude is even:

    $$|H(e^{j\omega})| = |H(e^{-j\omega})|.$$

    → The spectral phase is odd:

    $$\arg H(e^{j\omega}) = -\arg H(e^{-j\omega}).$$

- If the input is a real cosine $x[n] = \cos \omega_0 n$ for some fixed $\omega_0 \in \mathbb{R}$,
  - Then the output is given by

    $$y[n] = |H(e^{j\omega_0})| \cos\left[\omega_0 n + \arg H(e^{j\omega_0})\right]$$

    ⇒ The output is a real cosine of the same frequency.
    - The amplitude is scaled by $|H(e^{j\omega_0})|$
    - The phase is shifted by $\arg H(e^{j\omega_0})$.

Proof : $x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$.

Since $x[n]$ is a sum of two eigen-functions, the output is the sum of the same two eigenfunctions multiplied by their respective eigenvalues.

$$y[n] = \frac{1}{2}H(e^{j\omega_0})e^{j\omega_0 n} + \frac{1}{2}H(e^{-j\omega_0})e^{-j\omega_0 n}$$

$$= \frac{1}{2}|H(e^{j\omega_0})| e^{j\arg H(e^{j\omega_0})} e^{j\omega_0 n}$$

$$+ \frac{1}{2}\underbrace{|H(e^{-j\omega_0})|}_{\substack{=|H(e^{j\omega_0})| \\ \text{because magnitude} \\ \text{is even}}} e^{j\overbrace{\arg H(e^{-j\omega})}^{\substack{= -\arg H(e^{j\omega_0}) \\ \text{because phase} \\ \text{is odd}}}} e^{-j\omega_0 n}$$

$$= \frac{1}{2}|H(e^{j\omega_0})| e^{j[\omega_0 n + \arg H(e^{j\omega_0})]}$$

$$+ \frac{1}{2}|H(e^{j\omega_0})| e^{-j[\omega_0 n + \arg H(e^{j\omega_0})]}$$

$$= |H(e^{j\omega_0})| \left\{ \frac{1}{2}e^{j[\omega_0 n + \arg H(e^{j\omega_0})]} + \frac{1}{2}e^{-j[\omega_0 n + \arg H(e^{j\omega_0})]} \right\}$$

$$= |H(e^{j\omega_0})| \cos[\omega_0 n + \arg H(e^{j\omega_0})]$$

QED //

- A similar result holds for sine.
  - If $h[n]$ is <u>real</u> and the input is $x[n] = \sin \omega_0 n$, then the output is $y[n] = |H(e^{j\omega_0})| \sin[\omega_0 n + \arg H(e^{j\omega_0})]$.

    → Amplitude scaled by $|H(e^{j\omega_0})|$

    → Phase shifted by $\arg H(e^{j\omega_0})$.

  - The proof is almost identical to the one we just did for cosine.

Now Back to the Main Story

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- The last equation is the discrete-time Fourier transform "convolution property."
  - We spent a lot of time deriving it intuitively
  - Now here is a proof based purely on math:

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\left(x[n]*h[n]\right)e^{-j\omega n}$$

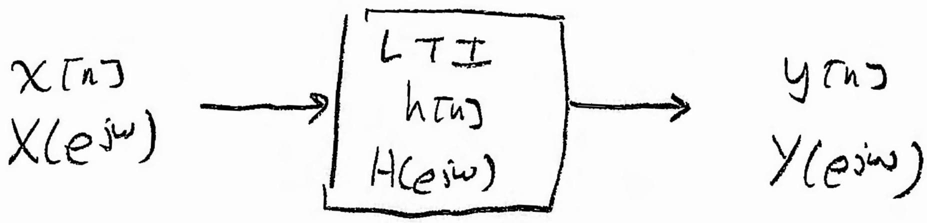
$$= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\underbrace{\sum_{n=-\infty}^{\infty} x[n-k]e^{-j\omega n}}_{e^{-j\omega k}X(e^{j\omega}) \text{ by the time shift property}}$$

$$Y(e^{jw}) = \sum_{k=\infty}^{\infty} h[k]\, e^{-jwk}\, X(e^{jw})$$

$$= X(e^{jw}) \sum_{k=-\infty}^{\infty} h[k]\, e^{-jwk} = \underline{\underline{X(e^{jw})\, H(e^{jw})}}$$

- From now on, you should always try to think of an LTI system simultaneously in the time domain and frequency domain



$$y[n] = x[n] * h[n]$$

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

- The last equation can also be solved for $X(e^{jw})$ or $H(e^{jw})$ as follows:

$$X(e^{jw}) = \frac{Y(e^{jw})}{H(e^{jw})}$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} \leftarrow \underline{\text{used often}}$$

- This leads to three basic types of problems that are all easy to solve:

① "Analysis" or "Convolution":

given [$x[n]$ or $X(e^{j\omega})$] AND [$h[n]$ or $H(e^{j\omega})$],

→ find $y[n]$.

- use $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$ unless specifically instructed to use time domain convolution.

② "Deconvolution":

given [$y[n]$ or $Y(e^{j\omega})$] AND [$h[n]$ or $H(e^{j\omega})$],

→ find $x[n]$

- use $X(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{H(e^{j\omega})}$

③ "System Identification"

given [$x[n]$ or $X(e^{j\omega})$] AND [$y[n]$ or $Y(e^{j\omega})$],

→ Find $H(e^{j\omega})$ or $h[n]$

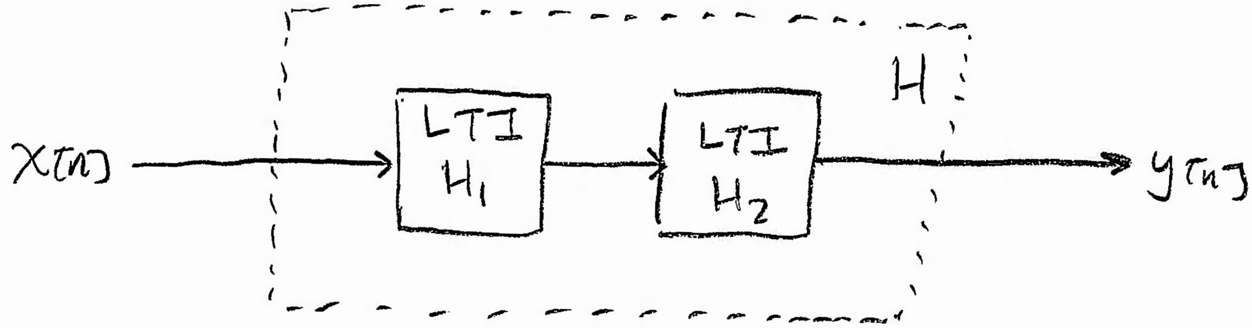
- use $H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$

- We already saw an example of the first type of problem on pages 4.60–4.66.

- They all work pretty much the same way:
  - use Tables and or properties to find the transforms of the two given signals.
  - Use the $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$ equation to find the transform of the unknown signal.
  - Do a PFE.
  - Use Tables and or properties to invert each term of the PFE and find the unknown signal.

LTI System Interconnections

- "Series" or "Cascade" Connection.



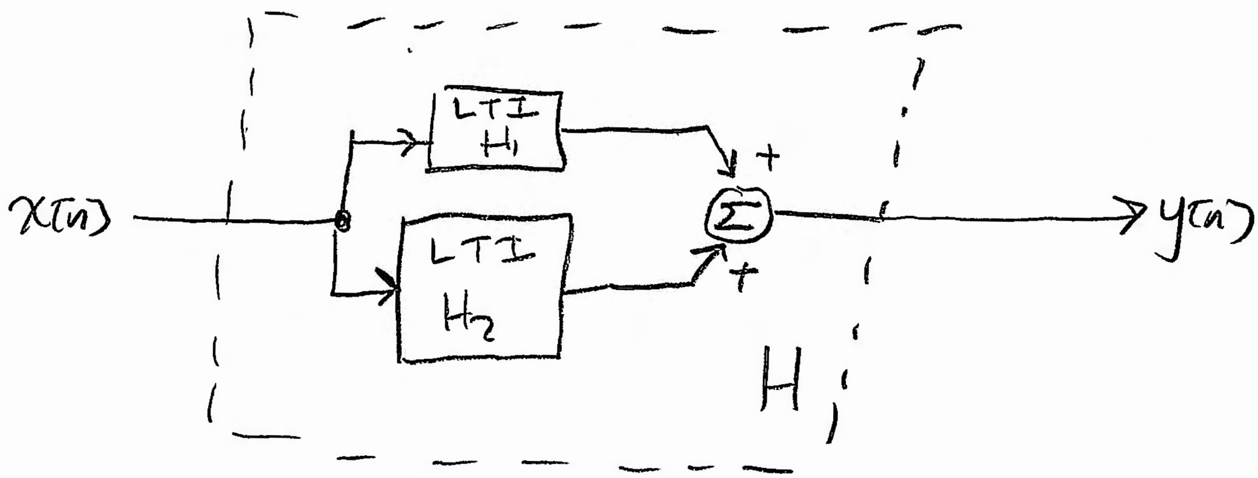
- If H_1 and H_2 are LTI, then so is the overall system H.
  - The impulse response is given by the convolution of the individual impulse responses:

$$h[n] = h_1[n] * h_2[n].$$

  - The frequency response is given by the product of the individual frequency responses:

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

- parallel connection:



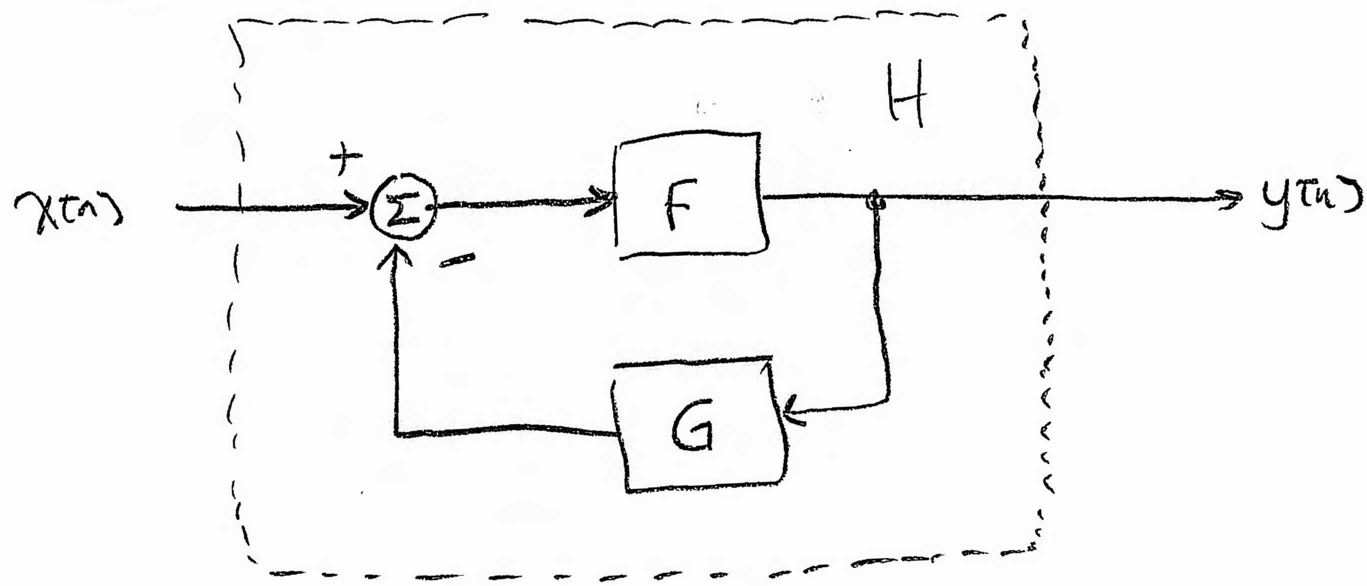
- If H_1 and H_2 are LTI, then so is the overall system H.
  - The impulse response is given by the sum of the individual impulse responses:

$$h[n] = h_1[n] + h_2[n]$$

  - The frequency response is the sum of the individual frequency responses:

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

- Negative Feedback Connection:



- If F and G are both LTI, then so is the overall system H.

- In this case, there is no general solution for the impulse response $h[n]$.
  - If you are given specific impulse responses $f[n]$ and $g[n]$, then you <u>can</u> find $h[n]$.
  - But there is no general closed form way to write $h[n]$ in terms of $f[n]$ and $g[n]$ unless you actually know what $f[n]$ and $g[n]$ are.

- But there is a closed form solution for $H(e^{j\omega})$ in terms of $F(e^{j\omega})$ and $G(e^{j\omega})$.

- It is given by:

$$H(e^{j\omega}) = \frac{F(e^{j\omega})}{1 + F(e^{j\omega})G(e^{j\omega})}.$$

- $F(e^{j\omega})$ is called the "forward path gain"
- $G(e^{j\omega})$ is called the "reverse path gain"
- Here is a mnemonic that I use to remember this formula:

"Forward path gain over one plus forward times reverse"!

- Although this formula is given on the course formula sheet for Test 2.

- Here's how to derive this formula:

- In the frequency domain, the output of LTI system F is connected directly to the output of the overall system H, which is $Y(e^{j\omega})$.

- This is connected also to the input of LTI system G.

- So, the output of system G is given by $Y(e^{j\omega})G(e^{j\omega})$ (by the discrete-time Fourier transform convolution property).

- This signal enters the negative input of the summing junction where it gets added to $X(e^{j\omega})$ to make the input of LTI system F.

  - So, the input to F is given by $X(e^{j\omega}) - Y(e^{j\omega})G(e^{j\omega})$

- The output of F, which is $Y(e^{j\omega})$, must be equal to the input of F times the frequency response of F. In other words, at the terminals of F we get:

$$\text{output} = F(e^{j\omega}) \cdot \text{Input}$$

$$Y(e^{j\omega}) = F(e^{j\omega})\left[X(e^{j\omega}) - Y(e^{j\omega})G(e^{j\omega})\right]$$

$\longrightarrow$

- Multiplying out, we get

$$Y(e^{j\omega}) = F(e^{j\omega})X(e^{j\omega}) - F(e^{j\omega})G(e^{j\omega})Y(e^{j\omega})$$

$$Y(e^{j\omega}) + F(e^{j\omega})G(e^{j\omega})Y(e^{j\omega}) = F(e^{j\omega})X(e^{j\omega})$$

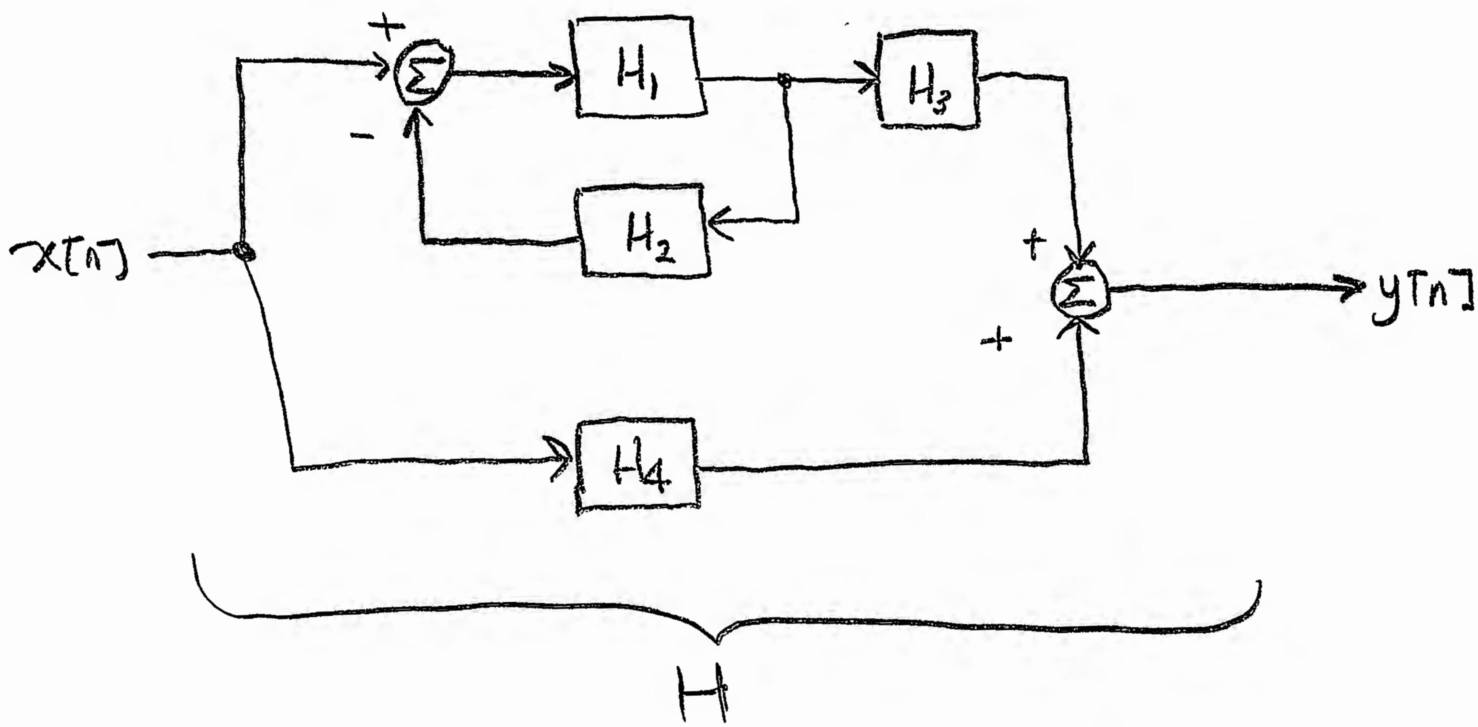
$$Y(e^{j\omega})\left[1 + F(e^{j\omega})G(e^{j\omega})\right] = X(e^{j\omega})F(e^{j\omega})$$

- Now solve for $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{F(e^{j\omega})}{1 + F(e^{j\omega})G(e^{j\omega})}$$

EXAMPLE : Block Diagram Reduction:

- An LTI system H is formed by connecting four LTI systems $H_1 - H_4$ as shown in the diagram below:



- You are given the following:

$$H_1(e^{j\omega}) = \frac{1}{1-2e^{-j\omega}} \qquad h_3[n] = 4\left(\tfrac{1}{3}\right)^n u[n]$$

$$h_2[n] = 3\delta[n] \qquad h_4[n] = \left(-\tfrac{1}{6}\right)^n u[n]$$

$\Rightarrow$ Find the overall system frequency response $H(e^{j\omega})$ and impulse response $h[n]$.

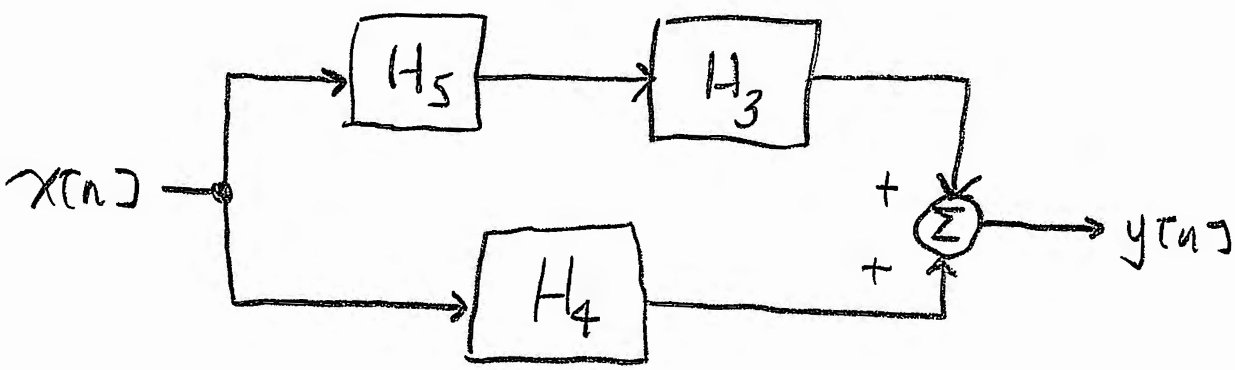
Solution:

Table: $H_2(e^{j\omega}) = 3$

$$H_3(e^{j\omega}) = \frac{4}{1-\frac{1}{3}e^{-j\omega}}$$

$$H_4(e^{j\omega}) = \frac{1}{1+\frac{1}{6}e^{-j\omega}}$$

- Combine H_1 & H_2 (negative feedback connection) into a single new block... call it H_5.
- The new picture will be



where

$$H_5(e^{j\omega}) = \frac{H_1(e^{j\omega})}{1+H_1(e^{j\omega})H_2(e^{j\omega})} = \longrightarrow$$

$$\ldots H_5(e^{j\omega}) = \frac{\frac{1}{1-2e^{-j\omega}}}{1+\frac{3}{1-2e^{-j\omega}}} \cdot \left[\frac{1-2e^{-j\omega}}{1-2e^{-j\omega}}\right]$$

This is 1... in a sneaky way that will clear the denominators upstairs and downstairs

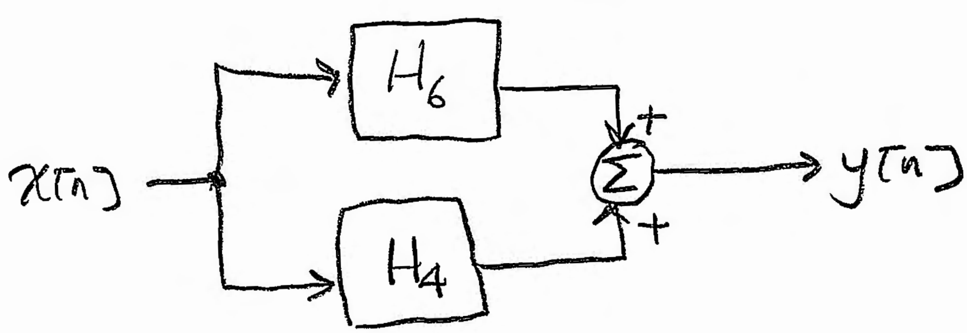
$$= \frac{1}{1-2e^{-j\omega}+3}$$

$$= \frac{1}{4-2e^{-j\omega}} \cdot \frac{1/4}{1/4}$$

This will get $H_5(e^{j\omega})$ into the form that's in our tables

$$H_5(e^{j\omega}) = \frac{1/4}{1-\frac{1}{2}e^{-j\omega}}$$

- Now combine H_5 and H_3 in series and call the new block H_6. The new picture will be



where ... →

$$H_6(e^{j\omega}) = H_5(e^{j\omega})H_3(e^{j\omega})$$

$$= \frac{1/4}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{4}{1-\frac{1}{3}e^{-j\omega}}$$

$$= \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} \quad //$$

- Now combine H_6 and H_4 in parallel to find the overall frequency response $H(e^{j\omega})$:

$$H(e^{j\omega}) = H_6(e^{j\omega}) + H_4(e^{j\omega})$$

$$= \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} + \frac{1}{1+\frac{1}{6}e^{-j\omega}}$$

→ To add these, we must get a common denominator.

→ Multiply $H_6(e^{j\omega})$ by $1 = \dfrac{1+\frac{1}{6}e^{-j\omega}}{1+\frac{1}{6}e^{-j\omega}}$

→ Multiply $H_4(e^{j\omega})$ by $1 = \dfrac{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$

$$H(e^{j\omega}) = \frac{1+\frac{1}{6}e^{-j\omega} + (1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{6}e^{-j\omega})}$$

$$= \frac{1+\frac{1}{6}e^{-j\omega} + 1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{6}e^{-j\omega})}$$

$$\boxed{H(e^{j\omega}) = \frac{2-\frac{2}{3}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1+\frac{1}{6}e^{-j\omega})}}$$

- By applying the quadratic formula, I can see that the roots of the numerator are complex. So it won't make this look any better if we try to factor the numerator.

- To find $h[n]$, we need to use partial fractions.

  → I can see that the numerator is <u>quadratic</u> in $e^{-j\omega}$, while the denominator is <u>cubic</u> in $e^{-j\omega}$

→

- So we can apply PFE directly to $H(e^{j\omega})$ without needing to use long division first.

<u>NOTE:</u> From looking at the figure on PAGE 5.41, I can see that H_6 and H_4 are in parallel.

- So $h[n] = h_6[n] + h_4[n]$.
- So the quickest way to find $h[n]$ would be to do a PFE on $H_6(e^{j\omega})$ to find $h_6[n]$, then add it to the given $h_4[n]$.

- But, for the sake of illustration, I'm going to go ahead and do the PFE on $H(e^{j\omega})$ even though it will take more work.

- To do the PFE, write θ for $e^{-j\omega}$:

$$\frac{2-\frac{2}{3}\theta+\frac{1}{6}\theta^2}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)(1+\frac{1}{6}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{3}\theta} + \frac{C}{1+\frac{1}{6}\theta}$$

$$A = \left.\frac{2-\frac{2}{3}\theta+\frac{1}{6}\theta^2}{(1-\frac{1}{3}\theta)(1+\frac{1}{6}\theta)}\right|_{\theta=2} = \frac{2-\frac{4}{3}+\frac{2}{3}}{(1-\frac{2}{3})(1+\frac{1}{3})} = \frac{2-\frac{2}{3}}{(\frac{1}{3})(\frac{4}{3})}$$

$$= \frac{4/3}{1/3 \cdot 4/3} = 1/\tfrac{1}{3} = 3$$

$$B = \left.\frac{2-\frac{2}{3}\theta+\frac{1}{6}\theta^2}{(1-\frac{1}{2}\theta)(1+\frac{1}{6}\theta)}\right|_{\theta=3} = \frac{2-2+\frac{3}{2}}{(1-\frac{3}{2})(1+\frac{1}{2})} = \frac{3/2}{(-\frac{1}{2})(\frac{3}{2})}$$

$$= \frac{1}{-1/2} = -2$$

$$C = \left.\frac{2-\frac{2}{3}\theta+\frac{1}{6}\theta^2}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)}\right|_{\theta=-6} = \frac{2+4+6}{(1+3)(1+2)} = \frac{12}{4\cdot 3} = \underline{1}$$

So

$$H(e^{j\omega}) = \frac{3}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{1+\frac{1}{6}e^{-j\omega}}$$

Table: $$\boxed{h[n] = 3\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[n] + \left(-\tfrac{1}{6}\right)^n u[n]}$$

Relationship Between the Frequency Response AND The I/O Equation

- The input-output equation (a.k.a. I/O-relation) of a discrete-time LTI system is a linear constant coefficients difference equation.
  - This means that:
    - a linear combination of the shifts of the input signal $x[n]$
  - is equal to
    - a linear combination of the shifts of the output signal $y[n]$.

- Here is an example:

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{1}{4} x[n-1]$$

- We will find the frequency response $H(e^{j\omega})$
  - To do this, we will need the time shift property of the DTFT (recall from PAGE 4.81):

    if $x[n] \leftrightarrow X(e^{j\omega})$ then $x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$.

  - We don't know what $x[n]$ is... it could be anything.
    - $\rightarrow$ But whatever it is, it has a DTFT $X(e^{j\omega})$ (assuming the transform exists).
  - And whatever $y[n]$ is, it has a transform $Y(e^{j\omega})$ (assuming the transform exists).
  - And, whatever $x[n]$ and $y[n]$ are, we know from the time shift property that:

$$x[n-1] \leftrightarrow e^{-j\omega} X(e^{j\omega})$$

$$y[n-1] \leftrightarrow e^{-j\omega} Y(e^{j\omega})$$

$$y[n-2] \leftrightarrow e^{-j2\omega} Y(e^{j\omega})$$

  - Finally, recall from PAGE 5.30,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

- Now, take the discrete-time Fourier transform (DTFT) on both sides of the I/O equation:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1]$$

$$\text{DTFT}\left\{y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2]\right\} = \text{DTFT}\left\{x[n] - \frac{1}{4}x[n-1]\right\}$$

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega}) - \frac{1}{4}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 - \frac{1}{4}e^{-j\omega}\right]$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}}$$

→ Notice that $H(e^{j\omega})$ is a ratio of two polynomials in the "character" $e^{-j\omega}$.

→ we say that "$H(e^{j\omega})$ is rational in $e^{-j\omega}$"

⇒ This will <u>always</u> be the case when the I/O equation is a linear constant coefficients difference equation.

Note: there do exist theoretical discrete-time LTI systems for which the I/O equation is not a constant coefficients difference equation.

- But they are highly theoretical.
- They are difficult to even construct.
- You will not see them in ECE 2713.
- In fact, you will not see them in any undergraduate class.

- For our purposes in this class, a discrete-time LTI system will always have an I/O equation that is a linear constant coefficient difference equation.

- The frequency response will always be rational in $e^{-j\omega}$, just like we saw on page 5.48.

- Now let's find the impulse response of the LTI system H from page 5.48.

- we had:

I/O equation: $y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1]$

frequency response: $$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}$$

- To find the impulse response, we must factor the denominator of $H(e^{j\omega})$ and perform a PFE.
- To factor the denominator of $H(e^{j\omega})$, we use the "foil" rule. We write:

$$1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega} = (1 - ae^{-j\omega})(1 - be^{-j\omega})$$
$$= 1 - (a+b)e^{-j\omega} + abe^{-j2\omega}$$

- we need numbers a and b such that

$$\underbrace{ab = \frac{1}{6}}_{\text{possible choices:}} \qquad a+b = \frac{5}{6}$$

$a=1 \quad b=\frac{1}{6} \longmapsto a+b = \frac{7}{6}$ X

$a=\frac{1}{6} \quad b=1 \longmapsto a+b = \frac{7}{6}$ X

$a=\frac{1}{2} \quad b=\frac{1}{3} \longmapsto a+b = \frac{5}{6}$ ✓

etc.

- So, by the foil rule, we get $a = \frac{1}{2}$ and $b = \frac{1}{3}$:

$$\text{denom} = 1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega} = \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right)$$

- And we have:

$$H(e^{j\omega}) = \underbrace{\frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}}_{\text{"sum form" of the denominator}} = \underbrace{\frac{1 - \frac{1}{4}e^{-j\omega}}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{-j\omega}\right)}}_{\text{"product form" of the denominator}}$$

★★★ SUPER IMPORTANT: you need to be able to go back and forth between these two forms.

- To go from the sum form to the product form, use the foil rule.
- To go from the product form to the sum form, just multiply out.

- In this example, the denominator was quadratic in $e^{-j\omega}$ and the numerator was linear in $e^{-j\omega}$.

  - So we didn't need to go back and forth between the sum and product forms for the numerator.

    - But in cases where the numerator is a polynomial in $e^{-j\omega}$ of order 2 or greater, you must also be able to factor and/or multiply out the numerator.

      $\Rightarrow$ You do it the same way we just did for the denominator.

- So now let's find the impulse response.

- Up to now, we've got:

I/O equation: $y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1]$

frequency response: $$H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-j2\omega}}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$\longrightarrow$

- Performing a PFE on $H(e^{j\omega})$, we get (write θ for $e^{-j\omega}$):

$$\frac{1-\frac{1}{4}\theta}{(1-\frac{1}{2}\theta)(1-\frac{1}{3}\theta)} = \frac{A}{1-\frac{1}{2}\theta} + \frac{B}{1-\frac{1}{3}\theta}$$

$$A = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{1-\frac{1}{2}}{1-\frac{2}{3}} = \frac{1/2}{1/3} = \frac{3}{2}$$

$$B = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{1-\frac{3}{4}}{1-\frac{3}{2}} = \frac{1/4}{-1/2} = -\frac{2}{4} = -\frac{1}{2}$$

$$H(e^{j\omega}) = \frac{3/2}{1-\frac{1}{2}e^{-j\omega}} - \frac{1/2}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $\boxed{h[n] = \frac{3}{2}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^n u[n]}$

- Now let's do an example going the other way:

<u>EX</u>: H is a discrete-time LTI system with impulse response

$$h[n] = \frac{4}{9}\left(-\frac{1}{2}\right)^n u[n] + \frac{5}{9}\left(\frac{1}{4}\right)^n u[n]$$

Find the system I/O equation.

Solution:

Table: $H(e^{j\omega}) = \text{DTFT}\{h[n]\}$

$$= \frac{4/9}{1+\frac{1}{2}e^{-j\omega}} + \frac{5/9}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{4/9\left(1-\frac{1}{4}e^{-j\omega}\right) + 5/9\left(1+\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{4/9 - 1/9\, e^{-j\omega} + 5/9 + 5/18\, e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega} - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$$

$$= \frac{\frac{9}{9} + 3/18\, e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$$

$$= \frac{1+\frac{1}{6}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$\longrightarrow$

- Now cross multiply the last equation on the bottom of page 5.54:

$$Y(e^{j\omega})\left[1+\tfrac{1}{4}e^{-j\omega}-\tfrac{1}{8}e^{-j2\omega}\right] = X(e^{j\omega})\left[1+\tfrac{1}{6}e^{-j\omega}\right]$$

$$Y(e^{j\omega})+\tfrac{1}{4}e^{-j\omega}Y(e^{j\omega})-\tfrac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})+\tfrac{1}{6}e^{-j\omega}X(e^{j\omega})$$

- Finally, take the inverse DTFT on both sides:

$$\boxed{y[n]+\tfrac{1}{4}y[n-1]-\tfrac{1}{8}y[n-2] = x[n]+\tfrac{1}{6}x[n-1]}$$

- From this example and the last one on pages 5.47–5.53, we see that there is a <u>very close</u> relationship between the I/O equation and the frequency response.

→

- In the first example, we had

$$\begin{cases} y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1] \\ H(e^{j\omega}) = \dfrac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}} \end{cases}$$

- In the second example, we had

$$\begin{cases} y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + \frac{1}{6}x[n-1] \\ H(e^{j\omega}) = \dfrac{1 + \frac{1}{6}e^{-j\omega}}{1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}} \end{cases}$$

- We see that the <u>numerator</u> of $H(e^{j\omega})$ comes from the "x-side" of the I/O equation

- The <u>denominator</u> of $H(e^{j\omega})$ comes from the "y-side" of the I/O equation.

$\Rightarrow$ This is <u>very</u> <u>very</u> important!!

- Once you see how this works and do it a couple of times, you will be able to just:
  - write down the frequency response from the I/O equation
  - write down the I/O equation from the frequency response.

EX:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{4}x[n-1]$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}}$$

EX:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + \frac{1}{6}x[n-1]$$

$$H(e^{j\omega}) = \frac{1 + \frac{1}{6}e^{-j\omega}}{1 + \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$$

$\Rightarrow$ All of this really follows straight from the time shift property of the DTFT... it's very important.

General Form of the I/O Equation

- The general form of the I/O equation for a discrete-time LTI system is the general form of a linear constant coefficients difference equation.
  - This means that a linear combination of the shifts of the output signal $y[n]$ is equal to a linear combination of the shifts of the input signal $x[n]$.
- Here is what it looks like:

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \dots + b_M x[n-M].$$

- N is the order of the "y-side" of the equation
- a_0 through a_N are the coefficients of the y-side of the equation.
  - → The denominator of $H(e^{j\omega})$ will be an N^{th} order polynomial in $e^{-j\omega}$ with coefficients $a_0, a_1, \dots, a_N$.

- M is the order of the "x-side" of the equation.
- b_0 through b_M are the coefficients of the x-side of the equation

→ The numerator of $H(e^{j\omega})$ will be an M^{th}-order polynomial in $e^{-j\omega}$ with coefficients $b_0, b_1, \ldots, b_M$.

<u>Note</u>: if $a_0 \neq 1$, then we usually divide both sides of the difference equation a_0 so that the coefficient of $y[n]$ is <u>one</u>.

⇒ You can see in the examples on page 5.57 that the coefficient of $y[n]$ was "1" in both examples

- Now, to <u>save writing</u>, let's re-write the general difference equation using "capital-Σ do loops"...

⟶

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

- Take the DTFT on both sides:

$$\mathrm{DTFT}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathrm{DTFT}\left\{\sum_{\ell=0}^{M} b_\ell x[n-\ell]\right\}$$

$$\sum_{k=0}^{N} a_k \,\mathrm{DTFT}\{y[n-k]\} = \sum_{\ell=0}^{M} b_\ell \,\mathrm{DTFT}\{x[n-\ell]\}$$

$$\sum_{k=0}^{N} a_k e^{-j\omega k}\, Y(e^{j\omega}) = \sum_{\ell=0}^{M} b_\ell e^{-j\omega \ell}\, X(e^{j\omega})$$

$$Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{\ell=0}^{M} b_\ell e^{-j\omega \ell}$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{\ell=0}^{M} b_\ell e^{-j\omega \ell}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}}$$ ☆☆☆

- This probably seems a little bit crazy and complicated at first.

- But it is <u>extremely</u> <u>useful</u> because it is the <u>general</u> <u>form</u> for the difference equation (I/O equation) and the frequency response.

- Compare ~~that~~ on page 5.60 to the first example on page 5.57:

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{1}{4} x[n-1]$$

$a_0 = 1 \quad a_1 = -\frac{5}{6} \quad a_2 = \frac{1}{6} \quad b_0 = 1 \quad b_1 = -\frac{1}{4}$

$N = 2 \qquad M = 1$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4} e^{-j\omega}}{1 - \frac{5}{6} e^{-j\omega} + \frac{1}{6} e^{-j2\omega}}$$

$$H(e^{j\omega}) = \frac{\sum_{\ell=0}^{M} b_\ell e^{-j\omega \ell}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- Again notice that the numerater of $H(e^{j\omega})$ comes directly from the x-side of the difference equation.

- The denominater of $H(e^{j\omega})$ comes directly from the y-side of the difference equation.

- If $M < N$, then $H(e^{j\omega})$ is a strictly proper fraction in $e^{-j\omega}$. In this case, partial fractions can be applied directly to $H(e^{j\omega})$ by factoring the denominater to get the product form of the denominator.

- If $M \geq N$, then $H(e^{j\omega})$ is an Improper fraction.
  - partial fractions can not be applied directly.

This is Super Important.

★★ <u>DO</u> <u>NOT</u> attempt to apply a PFE to an improper $H(e^{j\omega})$.

- The PFE will <u>seem</u> like it works
- But the answer you get will be total garbage!
- Worse yet, there will be <u>no warning</u> that the obtained PFE is completely wrong... it will seem like it worked.

- If you get an $H(e^{j\omega})$ with $M \geq N$, so that $H(e^{j\omega})$ is an improper fraction,
  - Then you need to do long division to get a quotient plus a remainder that is a proper fraction.
    - Then apply PFE to the remainder.

- Here's an example of that:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] + \frac{1}{6}x[n-1] - \frac{1}{6}x[n-2]$$

$$Y(e^{j\omega})\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

$M = N = 2$: $H(e^{j\omega})$ is an improper fraction.

$$\begin{array}{r|l}
 & -8/6 \quad \leftarrow \text{Quotient} \\
\hline
1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega} & 1 + \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega} \quad \leftarrow \text{dividend} \\
\nearrow & -8/6 + e^{-j\omega} - 1/6\, e^{-j2\omega} \\
\text{divisor} & \overline{\frac{7}{3} - \frac{5}{6}e^{-j\omega}} \quad \leftarrow \text{Remainder}
\end{array}$$

$$H(e^{j\omega}) = \frac{-8}{6} + \underbrace{\frac{\frac{7}{3} - \frac{5}{6}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}}_{\substack{\text{Do a PFE on the}\\ \text{remainder... it is Now}\\ \text{a proper fraction.}}}$$

DEF: The highest power of $e^{-j\omega}$ that appears in the numerator or denominator of the frequency response $H(e^{j\omega})$ is called the <u>order</u> of the LTI system H.

EX:
$$H(e^{j\omega}) = \frac{1 + 2e^{-j\omega}}{1 - 2e^{-j\omega} + 3e^{-j2\omega}}$$

Highest Power = 2

$\Rightarrow$ 2nd order system

EX:
$$H(e^{j\omega}) = \frac{1}{2} - \frac{1}{4}e^{-j\omega} + e^{-j2\omega} - \frac{1}{4}e^{-j3\omega}$$

Highest power = 3

$\Rightarrow$ 3rd order system

Note: in the second example above, the denominator is just "1", which is a <u>zeroth</u> order polynomial...

because
$$H(e^{j\omega}) = \frac{\frac{1}{2} - \frac{1}{4}e^{-j\omega} + e^{-j2\omega} - \frac{1}{4}e^{-j3\omega}}{1}$$

FIR and IIR Filters

- If H is a discrete-time LTI system and the impulse response $h[n]$ has a finite length, then H is called a finite impulse response filter or FIR filter. EX: $h[n] = \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$.

- Similarly, a discrete-time LTI system H with an impulse response $h[n]$ that has infinite length is called an infinite impulse response filter or IIR filter. EX: $h[n] = (\frac{1}{2})^n u[n]$.

FIR FILTERS

- Suppose H is an FIR filter and the length of $h[n]$ is N, so that for some numbers

$a_0, a_1, a_2, \dots a_{N-1}$, we have

$$h[n] = a_0\delta[n] + a_1\delta[n-1] + a_2\delta[n-1] + \dots + a_{N-1}\delta[n-(N-1)].$$

This can be written with a Σ do loop:

$$h[n] = \sum_{k=0}^{N-1} a_k \delta[n-k]$$

$\longrightarrow$

Then the filter output is:

$$y[n] = x[n] * h[n]$$

$$= x[n] * \left(a_0\delta[n] + a_1\delta[n-1] + \dots + a_{N-1}\,\delta[n-(N-1)]\right)$$

or:

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + \dots + a_{N-1}\, x[n-(N-1)]$$

⇒ This is the system input/output equation.

⇒ Note that it has <u>no</u> <u>shifts</u> of $y[n]$.

☆ The difference equation for an FIR filter has no shifts of the output signal $y[n]$.

Now, still keeping $h[n]$ the same, i.e.

$$h[n] = a_0\delta[n] + a_1\delta[n-1] + \dots + a_{N-1}\,\delta[n-(N-1)],$$

we see that the frequency response is

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\}$$

$$= a_0 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + \dots + a_{N-1}\, e^{-j(N-1)\omega}$$

⇒ The order of the filter is $\underline{\underline{N-1}}$.

(one less than the length of $h[n]$)

- We can also find $H(e^{j\omega})$ and the order from the I/O equation (difference equation) like this:

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + \dots + a_{N-1} x[n-(N-1)]$$

- Take DTFT on both sides:

$$Y(e^{j\omega}) = a_0 X(e^{j\omega}) + a_1 e^{-j\omega} X(e^{j\omega}) + a_2 e^{-j2\omega} X(e^{j\omega}) + \dots + a_{N-1} e^{-j(N-1)\omega} X(e^{j\omega})$$

$$= X(e^{j\omega}) \left[a_0 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + \dots + a_{N-1} e^{-j(N-1)\omega} \right]$$

- Divide both sides by $X(e^{j\omega})$:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{a_0 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + \dots + a_{N-1} e^{-j(N-1)\omega}}{1}$$

$\Rightarrow$ order $= N-1$

EX: $h[n] = \delta[n] - \frac{5}{6}\delta[n-1] + \frac{1}{6}\delta[n-2]$

$\Rightarrow$ FIR because $h[n]$ has finite length

length of $h[n] = 3$

$\Rightarrow$ order $= 2$

$$H(e^{j\omega}) = 1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}$$

$\Rightarrow$ denominator of $H(e^{j\omega}) = 1$ ✓

$\Rightarrow$ order $=2$ ✓

I/O relation $\equiv$ "I/O equation" $\equiv$ "difference equation":

$$y[n] = x[n] * h[n] = x[n] - \frac{5}{6}x[n-1] + \frac{1}{6}x[n-2]$$

$\Rightarrow$ No shifts of $y[n]$

$\Rightarrow$ FIR ✓

IIR FILTERS

- For an IIR filter H:

  - The difference equation <u>does</u> have shifts of the output signal $y[n]$.

  - The denominator of $H(e^{j\omega})$ is <u>not</u> 1 ... it is a "nontrivial" polynomial.

  - The length of $h[n]$ is infinite

{These three statements ↑ are <u>equivalent</u> ....

if one of them is true, then all of them are true}

⟹ And they are all three equivalent to saying that H is an IIR filter.

EX:

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{1}{4} x[n-1]$$

$\Rightarrow$ I/O equation has shifts of $y[n]$

$\Rightarrow$ This is an IIR filter.

- To find the frequency response $H(e^{j\omega})$, take DTFT on both sides:

$$Y(e^{j\omega}) - \frac{5}{6} e^{-j\omega} Y(e^{j\omega}) + \frac{1}{6} e^{-j2\omega} Y(e^{j\omega})$$
$$= X(e^{j\omega}) - \frac{1}{4} e^{-j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \frac{5}{6} e^{-j\omega} + \frac{1}{6} e^{-j2\omega}\right] = X(e^{j\omega})\left[1 - \frac{1}{4} e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{4} e^{-j\omega}}{1 - \frac{5}{6} e^{-j\omega} + \frac{1}{6} e^{-j2\omega}}$$

$\Rightarrow$ Denominator is a nontrivial polynomial... i.e., the denominator is <u>not</u> 1

$\Rightarrow$ It's an IIR filter

$\Rightarrow$ The highest power of $e^{-j\omega}$ is 2... (from the denominator]

$\Rightarrow$ order = 2

- To find the impulse response, take inverse DTFT of $H(e^{j\omega})$:

$\Rightarrow$ $H(e^{j\omega})$ is a proper fraction

$\Rightarrow$ So we can do a PFE

$$H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-j2\omega}}$$

$$= \frac{1-\frac{1}{4}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{3}e^{-j\omega}}$$

$$A = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{3}\theta}\right|_{\theta=2} = \frac{1-\frac{1}{2}}{1-\frac{2}{3}} = \frac{1/2}{1/3} = \frac{3}{2}$$

$$B = \left.\frac{1-\frac{1}{4}\theta}{1-\frac{1}{2}\theta}\right|_{\theta=3} = \frac{1-\frac{3}{4}}{1-\frac{3}{2}} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{2}{4} = -\frac{1}{2}$$

$\longrightarrow$

$$H(e^{j\omega}) = \frac{3/2}{1-\frac{1}{2}e^{-j\omega}} - \frac{1/2}{1-\frac{1}{3}e^{-j\omega}}$$

Table: $h[n] = \frac{3}{2}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{3}\right)^n u[n]$

$\Rightarrow$ $h[n]$ has infinite length

$\Rightarrow$ it's an IIR filter.

Phase Response of A Digital LTI FILTER

- For both FIR filters and IIR filters, the frequency response $H(e^{j\omega})$ is a complex-valued function of ω in general.

- We can write it in polar form like this:

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j \arg H(e^{j\omega})}$$

where $|H(e^{j\omega})| = \left\{ \mathrm{Re}[H(e^{j\omega})]^2 + \mathrm{Im}[H(e^{j\omega})]^2 \right\}^{1/2}$

$\arg H(e^{j\omega}) = \arctan\left[\mathrm{Im}[H(e^{j\omega})] / \mathrm{Re}[H(e^{j\omega})] \right]$

$|H(e^{j\omega})|$ is called the magnitude response of the filter.

- Sometimes also called the "spectral magnitude"

$\arg H(e^{j\omega})$ is called the phase response of the filter.

- Sometimes also called the "spectral phase".

$\Rightarrow$ We have seen on pages 5.21 – 5.24 that linear phase is often desirable because it means that all of the sinusoids in the input signal will be delayed by the same amount of time in going through the filter.

$\Rightarrow$ Very important for audio filters.

- There are <u>no</u> <u>known</u> techniques for designing IIR filters to have linear phase.
    - If linear phase is needed, it can only be approximated by an IIR filter
- However, there <u>are</u> known techniques for designing FIR filters to have linear phase

  <u>EX</u>: Parks-McClellan algorithm - a numerical optimization algorithm for designing linear phase FIR filters.

    - Covered in ECE 4213

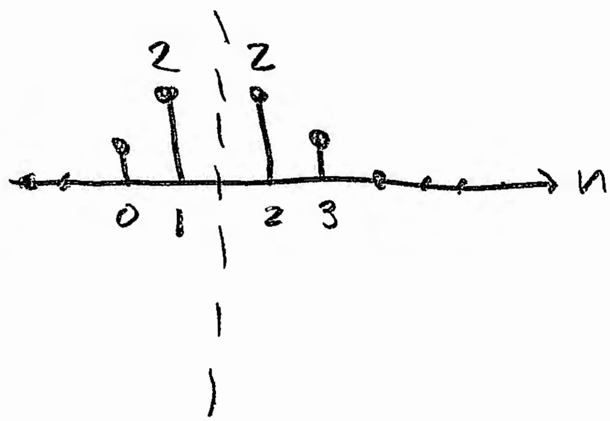
<u>FACT</u>: For a linear phase FIR filter, the impulse response has symmetry about the midpoint.

$\longrightarrow$

- This can happen in 4 different ways:

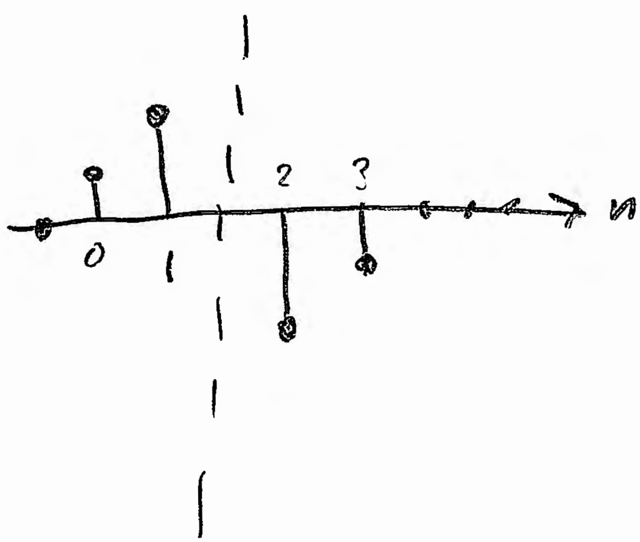
① length even, even symmetric about midpoint:

EX: $h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]$



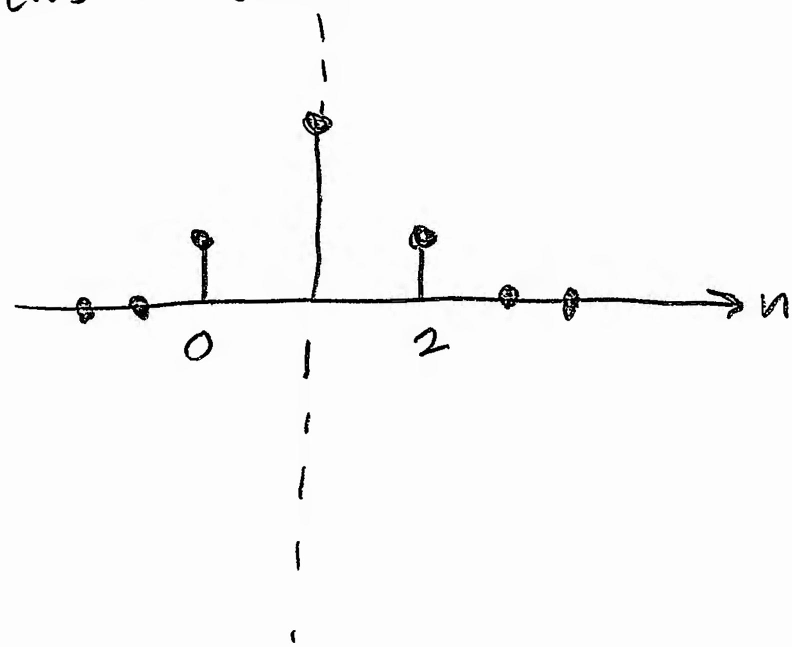
② length even, odd symmetric about midpoint:

EX: $h[n] = \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3]$



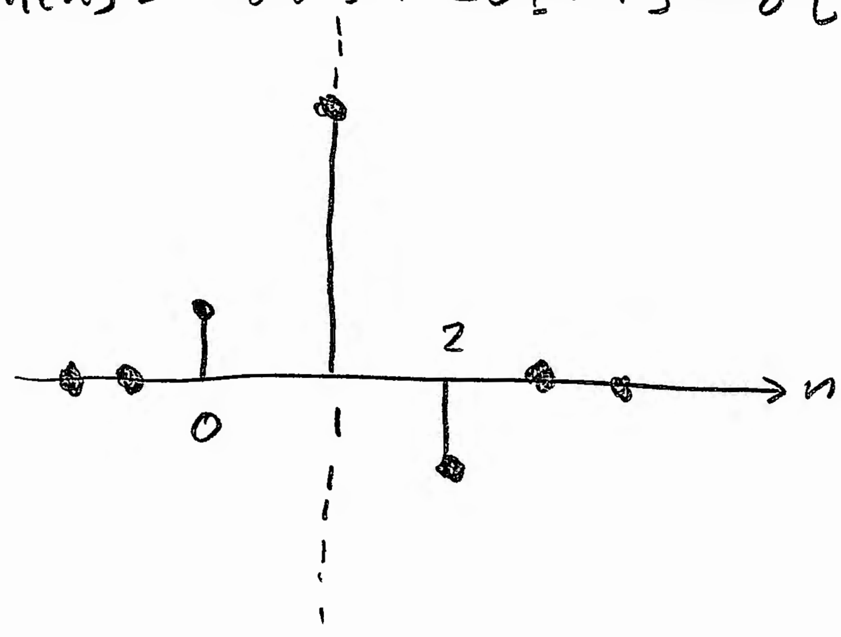
③ length odd, even symmetric about the middle sample :

EX : $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$



④ length odd, odd symmetric about the middle sample :

EX : $h[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$



- When you have an FIR digital filter,
  - and you see that there is symmetry about the midpoint of the impulse response
  - then:
    1. you know that it is a <u>linear phase</u> FIR filter, and
    2. there is a "trick" (a "shortcut") for writing the frequency response in polar form.

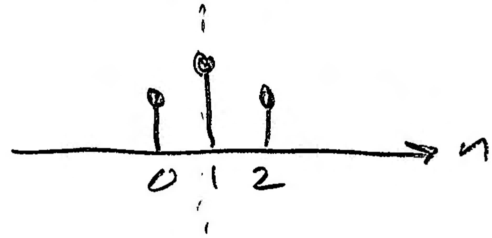
Here is the trick/shortcut:

- Take the DTFT of the impulse response to get the frequency response.
- Factor out <u>half</u> the highest power of $e^{-j\omega}$ that appears.
- Use Euler's formula on what's left to get cosines or sines.

- If the symmetry was **even** about the midpoint of $h[n]$, then you get cosines.

- If the symmetry was **odd**, then you get sines.

EX: $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$



"even symmetry" about midpoint.

Table: $H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$

→ highest power of $e^{-j\omega}$: 2

→ **half** the highest power: 1

⟹ Factor out $e^{-j\omega}$

⟶

<u>EX</u>...

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

½ the highest power

$$= \left[e^{j\omega} + 2 + e^{-j\omega}\right] e^{-j\omega}$$

$$= \underbrace{\left[2 + 2\cos\omega\right]}_{|H(e^{j\omega})|} \underbrace{e^{-j\omega}}_{e^{j \arg H(e^{j\omega})}}$$

$\Rightarrow$ The spectral phase (i.e. the "phase response" of the filter) is:

$$\arg H(e^{j\omega}) = \underline{\underline{-\omega}}$$

$\Longrightarrow$ Linear Phase ✓

<u>Note:</u> half the highest power might be a fraction. If the highest power is $e^{-j3\omega}$ ... then you have to factor out $e^{-j\frac{3}{2}\omega}$.

Note: if the symmetry is odd about the midpoint of $h[n]$, then use Euler's formula for sine instead. The phase term will then be $e^{-j(\beta\omega \pm \frac{\pi}{2})}$, where

$\beta =$ half the highest power...

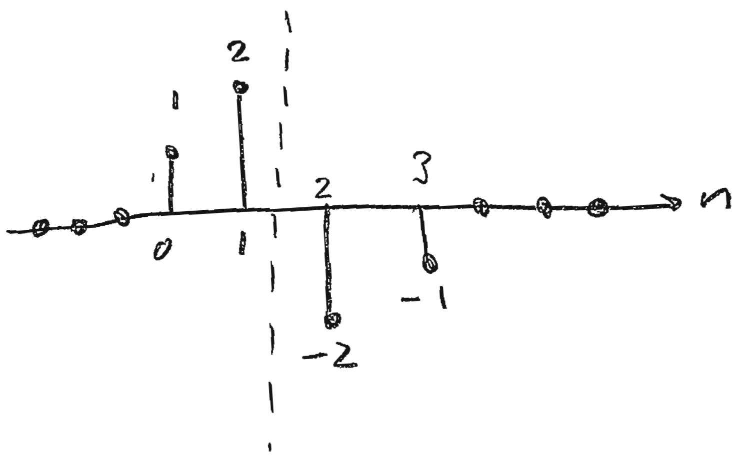
giving a phase response of

$$\arg H(e^{j\omega}) = -\beta\omega \pm \frac{\pi}{2}.$$

EX: using the "linear phase trick" on an FIR filter when the impulse response has odd symmetry about the midpoint:

Given: H is a discrete-time LTI system with impulse response

$$h[n] = \delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3]$$



Note: $h[n]$ has finite length

$\Rightarrow$ It's an FIR filter ✓

Note: $h[n]$ has odd symmetry about the midpoint

$\Rightarrow$ It's a linear phase FIR filter ✓

Problem: find the frequency response $H(e^{j\omega})$ and write it in polar form... in other words, find the magnitude response $|H(e^{j\omega})|$ and the phase response $\arg H(e^{j\omega})$

$\longrightarrow$

Solution:

① use the DTFT to find $H(e^{j\omega})$:

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \text{DTFT}\{\delta[n] + 2\delta[n-1] - 2\delta[n-2] - \delta[n-3]\}$$

Table (plus time shift property):

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} - 2e^{-j2\omega} - e^{-j3\omega}$$

② use the "linear phase" trick:

- Highest power of $e^{-j\omega}$ $\Rightarrow$ $e^{-j3\omega}$
- Half the highest power $\Rightarrow$ $e^{-j\frac{3}{2}\omega}$

$\Rightarrow$ Factor out $e^{-j\frac{3}{2}\omega}$:

$$H(e^{j\omega}) = \underbrace{e^{j\frac{3}{2}\omega}e^{-j\frac{3}{2}\omega}}_{1} + 2\underbrace{e^{j\frac{1}{2}\omega}e^{-j\frac{3}{2}\omega}}_{e^{-j\omega}} - 2\underbrace{e^{-j\frac{1}{2}\omega}e^{-j\frac{3}{2}\omega}}_{e^{-j2\omega}} - \underbrace{e^{-j\frac{3}{2}\omega}e^{-j\frac{3}{2}\omega}}_{e^{-j3\omega}}$$

$$= \left[e^{j\frac{3}{2}\omega} + 2e^{j\frac{1}{2}\omega} - 2e^{-j\frac{1}{2}\omega} - e^{-j\frac{3}{2}\omega}\right]e^{-j\frac{3}{2}\omega}$$

$$= \left[\left(e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}\right) + 2\left(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}\right)\right]e^{-j\frac{3}{2}\omega}$$

$\longrightarrow$

$\Rightarrow$ Now use the <u>sine</u> form of Euler's equation:

$$H(e^{j\omega}) = \left[2j\left(\frac{e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega}}{2j}\right) + 2j2\left(\frac{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}}{2j}\right)\right]e^{-j\frac{3}{2}\omega}$$

$$= \left[2j\left(\sin\tfrac{3}{2}\omega\right) + 4j\left(\sin\tfrac{1}{2}\omega\right)\right]e^{-j\frac{3}{2}\omega}$$

$\rightarrow$ factor out "j".

$$= \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right] j e^{-j\frac{3}{2}\omega}$$

$\rightarrow$ Note: $j = e^{j\pi/2}$

$$= \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right] e^{j\pi/2} e^{-j\frac{3}{2}\omega}$$

$$\Rightarrow H(e^{j\omega}) = \underbrace{\left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)\right]}_{|H(e^{j\omega})|} \underbrace{e^{j\left(-\frac{3}{2}\omega + \frac{\pi}{2}\right)}}_{e^{j\arg H(e^{j\omega})}}$$

$$\arg H(e^{j\omega}) = -\tfrac{3}{2}\omega + \tfrac{\pi}{2}$$

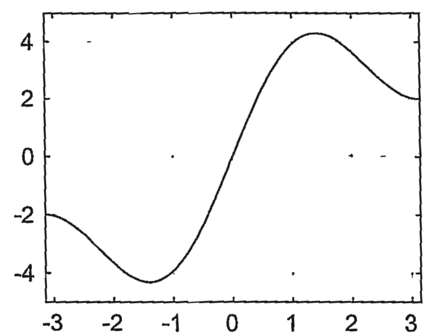
$\Rightarrow$ <u>Linear</u> <u>Phase</u> ✓

TECHNICAL NOTE about the last example on pages 5.82 - 5.84:

- By applying the "linear phase" trick to the FIR filter H, we got that:

$$|H(e^{j\omega})| = 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right)$$

→ If you plot this from $\omega = -\pi$ to $\omega = +\pi$, it looks like this:



→ It is negative for $-\pi \leq \omega < 0$!!

→ How can this be the magnitude $|H(e^{j\omega})|$??

→ I thought magnitudes had to be ≥ 0 ??

ANSWER: technically, that is true. Magnitudes do have to be ≥ 0.

⇒ But in a situation like this, with a linear phase FIR filter, DSP engineers often allow the magnitude to go negative for some ω.

⇒ It makes it easier to see the linear phase characteristic. ⟶

TECHNICAL NOTE: ...

- Technically, the magnitude response of the filter is

$$|H(e^{j\omega})| = \left| 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \right|$$

- But note that: (in polar form)

$$+1 = e^{j0}$$ 

$$-1 = e^{j\pi}$$ 

- So if we define a function $\gamma(\omega)$ according to

$$\gamma(\omega) = \begin{cases} 0, & 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \geq 0 \\ \pi, & 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) < 0 \end{cases}$$

- Then

$$H(e^{j\omega}) = \left[2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \right] e^{j\left(-\frac{3}{2}\omega + \frac{\pi}{2}\right)}$$

$$= \underbrace{\left| 2\sin\left(\tfrac{3}{2}\omega\right) + 4\sin\left(\tfrac{1}{2}\omega\right) \right|}_{\text{True magnitude response, } \geq 0} \; \underbrace{e^{j\gamma(\omega)}} \; e^{j\underbrace{\left(-\frac{3}{2}\omega + \frac{\pi}{2}\right)}_{\text{linear phase response}}}$$

This term just accounts for the sign. At all ω,
$e^{j\gamma(\omega)} = +1$ or
$e^{j\gamma(\omega)} = -1$

→

TECHNICAL NOTE ...

- Technically, this is called "generalized linear phase."
- The term $e^{j\gamma(\omega)}$ is always equal to either $+1$ (at some values of ω) or -1 (at other values of ω).
  - → This term does not actually add any delay to the filter... rather, it just accounts for sign.
  - → Intuitively, it says that certain frequencies will get "flipped" (i.e., multiplied by -1) when they go through the filter.

⇒ In such cases, DSP engineers will often say that:

- The magnitude response of the filter is
$$|H(e^{j\omega})| = 2\sin(\tfrac{3}{2}\omega) + 4\sin(\tfrac{1}{2}\omega)$$
even though this function is negative for some ω,
- The phase response of the filter is
$$\arg H(e^{j\omega}) = -\tfrac{3}{2}\omega + \tfrac{\pi}{2}$$
- The filter has "generalized linear phase."

Two More Adjectives to describible discrete-time LTI systems:

- In other words, two more "system properties"
- These two important additional properties are:
  - Causality
  - Stability

CAUSALITY

DEF: A discrete-time LTI system H is called causal if the current value of the output signal $y[n]$ NEVER depends on any future value of the input signal $x[n]$.

EX: $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$$y[n] = x[n] * h[n]$$
$$= x[n] * (\delta[n] + \delta[n-1] + \delta[n-2])$$
$$= x[n] + x[n-1] + x[n-2]$$

current value of the input signal

value of the input signal one time ago

value of the input signal two times ago.

EX... for example, plugging in $n=7$, we get:

$$y[7] = x[7] + x[6] + x[5]$$

- At time $n=7$, the system output is equal to the number $y[7]$.
    - This depends on the number $x[7]$ (the current value of the input signal,
    - And on $x[6]$ and $x[5]$... past values of the input signal,
    - But <u>never</u> on any <u>future</u> value of the input signal (like $x[8]$, for example).

$\Rightarrow$ The system is <u>causal</u> because the <u>current</u> value of the output signal never depends on <u>future</u> values of the input signal.

$\longrightarrow$

EX : $h[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

$$y[n] = x[n] * h[n]$$
$$= x[n] * \left(\delta[n+1] + \delta[n] + \delta[n-1]\right)$$
$$= \underbrace{x[n+1]}_{\text{Future value of the input signal}} + \underbrace{x[n]}_{\text{Current value of the input signal}} + \underbrace{x[n-1]}_{\text{Past value of the input signal.}}$$

$\Rightarrow$ NOT CAUSAL because the current value of the output signal depends on a future value of the input signal.

$\rightarrow$ For example, at time $n=7$, we get

$$y[7] = x[8] + x[7] + x[6],$$

which depends on the future value of the input signal $x[8]$...

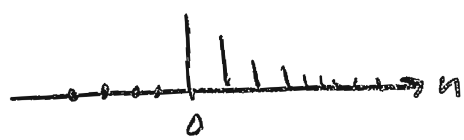
$\Rightarrow$ NOT CAUSAL.

FACT : A discrete-time LTI system H is causal if and only if the impulse response $h[n] = 0 \ \forall \ n < 0$. ☆☆☆

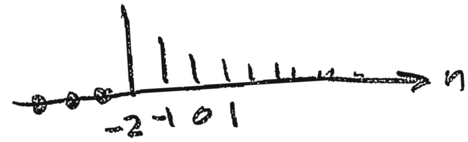
- in other words, if and only if $h[n] = 0$ for all the negative n's.
- You will prove this fact in ECE 3793.

- Non-causal systems can not be implemented "online" or in "real-time."
  - They can only be used on recorded data... when the whole input signal has been previously recorded and stored in memory.

EX : $h[n] = \left(\frac{1}{2}\right)^n u[n]$



→ Causal because $h[n] = 0 \ \forall \ n < 0$.



EX : $h[n] = \left(\frac{1}{2}\right)^{n+2} u[n+2]$

→ NOT causal because there are some $n < 0$ for which $h[n] \neq 0$ (like $n = -2$, for example).

STABILITY

- In order to talk about stability, we must first say what it means for a signal to be **bounded**.

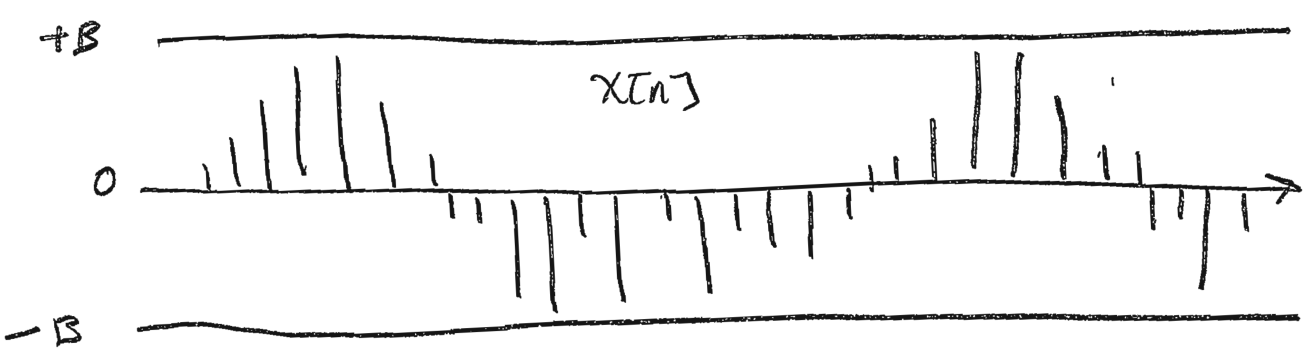
DEF: A discrete-time signal $x[n]$ is **bounded** if $\exists$ a real number $B > 0$ such that $|x[n]| \leq B$ $\forall n \in \mathbb{Z}$.

EX: $x[n] = \cos(\frac{\pi}{17} n)$

$\longrightarrow$ **Bounded** by $B = 1$ because

$$|x[n]| = |\cos(\tfrac{\pi}{17} n)| \leq 1 \quad \forall n \in \mathbb{Z}.$$

- Intuitively, **bounded** means that there exists a positive real number B such that the graph of $x[n]$ lives entirely within a tunnel of radius B:



EX: $x[n] = 100\left(\frac{1}{2}\right)^n u[n]$

$\rightarrow$ Bounded by $B = 100$, because $|x[n]|$ is <u>always</u> ≤ 100. In other words, $|x[n]| \leq 100 \;\; \forall n \in \mathbb{Z}$.

EX: $x[n] = 2^n u[n]$

$\rightarrow$ <u>NOT</u> <u>bounded</u> ... i.e., "unbounded".

- Because $\lim_{n \to \infty} x[n] = \infty$, there is no tunnel big enough to hold the graph of $x[n]$.

<u>Note</u>: if a signal <u>is</u> bounded, then the bound is <u>not</u> unique.

- For example, $x[n] = 100\left(\frac{1}{2}\right)^n u[n]$ is bounded by $B = 100$ (as we said above), and also by $B = 101$ and $B = 200$ because

$$|x[n]| \leq 100 \quad \forall n \in \mathbb{Z}$$
$$|x[n]| \leq 101 \quad \forall n \in \mathbb{Z}$$
$$|x[n]| \leq 200 \quad \forall n \in \mathbb{Z}$$

DEF: A discrete-time system H is <u>stable</u> if every bounded input signal produces a bounded output signal.

Note: this definition does not say <u>anything</u> about what happens if the input signal is unbounded.

→ A stable system can produce an <u>unbounded</u> output signal when given an <u>unbounded</u> input signal.

→ But a stable system can <u>never</u> produce an unbounded output signal when the input signal is <u>bounded</u>.

⇒ Intuitively, this means that the system <u>itself</u> will never do anything to make a signal "blow up."

→ For a <u>stable</u> system, the output signal can't blow up unless the input signal blows up.

→ given a <u>bounded</u> input signal, the output signal will always be bounded.

<u>Note</u>: the bound on the output signal does <u>not</u> have to be the same as the bound on the input signal.

<u>EX</u>: H is an ideal digital amplifier with a gain of 100... so that $h[n] = 100\,\delta[n]$:

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n] = 100\,x[n]$$

- if $x[n]$ is bounded by $B = 2$ (say $x[n] = 2\cos\left(\frac{\pi}{17}n\right)$ for example)
- Then $y[n]$ is bounded by $B = 200$ ($y[n] = 200\cos\left(\frac{\pi}{17}n\right)$ in this case, so $|y[n]| \leq 200 \;\; \forall n \in \mathbb{Z}$)

- The definition of <u>stable</u> just requires that there is <u>some</u> bound on $y[n]$ any time $x[n]$ is bounded.

- A system that is not stable is called <u>unstable</u>.

<u>FACT</u>: A discrete-time LTI system is stable if and only if the impulse response $h[n]$ is absolutely summable, ... in other words, if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty .$$

- This fact is more difficult to prove than others we have seen this semester. You will learn how to prove it in ECE 3793.

<u>EX</u>: H is a discrete-time LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$

→ STABLE, because

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 < \infty$$

formula sheet

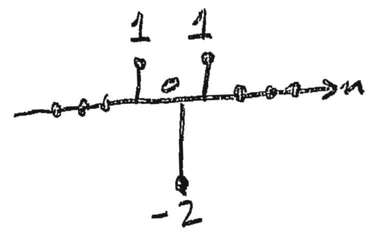
EX: H is a discrete-time LTI system with impulse response $h[n] = 2^n u[n]$.

$\rightarrow$ **UNSTABLE**, because

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 2^n \longrightarrow \infty$$

Note: **ALL** FIR digital filters are stable.

- For example, suppose H is an FIR filter with impulse response $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$



Then $\sum_{n=-\infty}^{\infty} |h[n]| = |1| + |-2| + |1|$

$= 1 + 2 + 1$

$= 4 < \infty$ ✓

- Here is one more important fact to conclude Chapter 5:

<u>FACT</u>: if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then the

DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ <u>exists</u>.

- In other words, if $x[n]$ is absolutely summable, then the DTFT sum <u>converges</u>.

<u>EX</u>: $x[n] = \left(\frac{1}{2}\right)^n u[n]$

→ Absolutely summable because

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

↑ formula sheet

→ Table: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

- If $x[n]$ is not absolutely summable, then the DTFT might or might not exist.

→ If the DTFT does exist, then it will generally involve distributions in the frequency domain.

EX: $x[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} 1 \rightarrow \infty$$ NOT ABSOLUTELY SUMMABLE.

But, from the formula sheet, we have

$$\text{DTFT}\{x[n]\} = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k)$$

→ So even though $u[n]$ is not absolutely summable, the DTFT $U(e^{j\omega})$ exists.

→

EX: $x[n] = 2^n u[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} 2^n \longrightarrow \infty$$

NOT ABSOLUTELY SUMMABLE

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (2e^{-j\omega})^n = \lim_{A\to\infty} \sum_{n=0}^{A} (2e^{-j\omega})^n$$

$$= \lim_{A\to\infty} \frac{(2e^{-j\omega})^0 - (2e^{-j\omega})^{A+1}}{1-2e^{-j\omega}}$$

$$= \lim_{A\to\infty} \frac{1 - 2^A e^{-j(A+1)\omega}}{1-2e^{-j\omega}} \longrightarrow \text{DIVERGENT SUM}$$

$\Rightarrow$ So $X(e^{j\omega})$ does not exist in this case.

Note: the fact that absolute summability of the signal is sufficient to guarantee the existence of the DTFT means that every stable discrete-time LTI system has a frequency response:

- If the system is stable, then the impulse response $h[n]$ is absolutely summable. This is sufficient to guarantee existence of the frequency response $H(e^{j\omega})$.