Digital Filter Design

- A digital filter design problem usually consists of a specification for the filter magnitude response $|H(e^{j\omega})|$.
  - The phase response $\arg H(e^{j\omega})$ is often not specified.
    - For certain problems (e.g. digital audio filter), it is understood that the phase must be linear.
    - In many other problems, it is understood that the phase should be as close to linear as possible... i.e., it is desirable for the phase to be approximately linear.
- The digital filter design problem is:

  ☆ Design the frequency response $H(e^{j\omega})$ to meet the specification with the lowest order possible.

→ Recall: The order of the filter is the highest power of $e^{-j\omega}$ that appears in $H(e^{j\omega})$... either upstairs or downstairs.

- Equivalently, the order is:
  → The greatest shift of either $x[n]$ or $y[n]$ that appears in the difference equation.
  → The highest power of z^{-1} that appears in the transfer function $H(z)$.

- Unless otherwise stated, the following requirements are generally assumed:
  - The filter must be <u>causal</u>.
  - The filter must be <u>stable</u>.
  - The impulse response $h[n]$ must be <u>real</u>.
    - → This means that:
      - The numerator and denominator polynomials of $H(e^{j\omega})$ must have <u>real</u> coefficients.
        - These are the same as the numerator and denominator polynomial coefficients of $H(z)$
        - And they are also the same as the coefficients that appear in the difference equation (I/O equation).
        - They must all be <u>real</u>.
    - → This also means that the poles and zeros of $H(z)$ must be <u>real</u> or must occur in <u>complex</u> <u>conjugate</u> <u>pairs</u>.

⇒ Recall from p. 6.101:

- For the digital filter (a discrete-time LTI system) to be <u>both</u> causal and stable,
  - ⇒ All the poles of $H(z)$ must lie strictly <u>inside</u> the unit circle of the z-plane.

- Another consequence of the assumption that the impulse response $h[n]$ is real is the following:

  ☆ The frequency response $H(e^{j\omega})$ must be conjugate symmetric:

  - Real part even
  - Imaginary part odd
  - Magnitude even
  - Phase odd.

- For this reason, the design spec will usually only specify $|H(e^{j\omega})|$ for the non-negative digital frequencies $\omega \geq 0$.

- Recall from pp. 4.31-4.32:

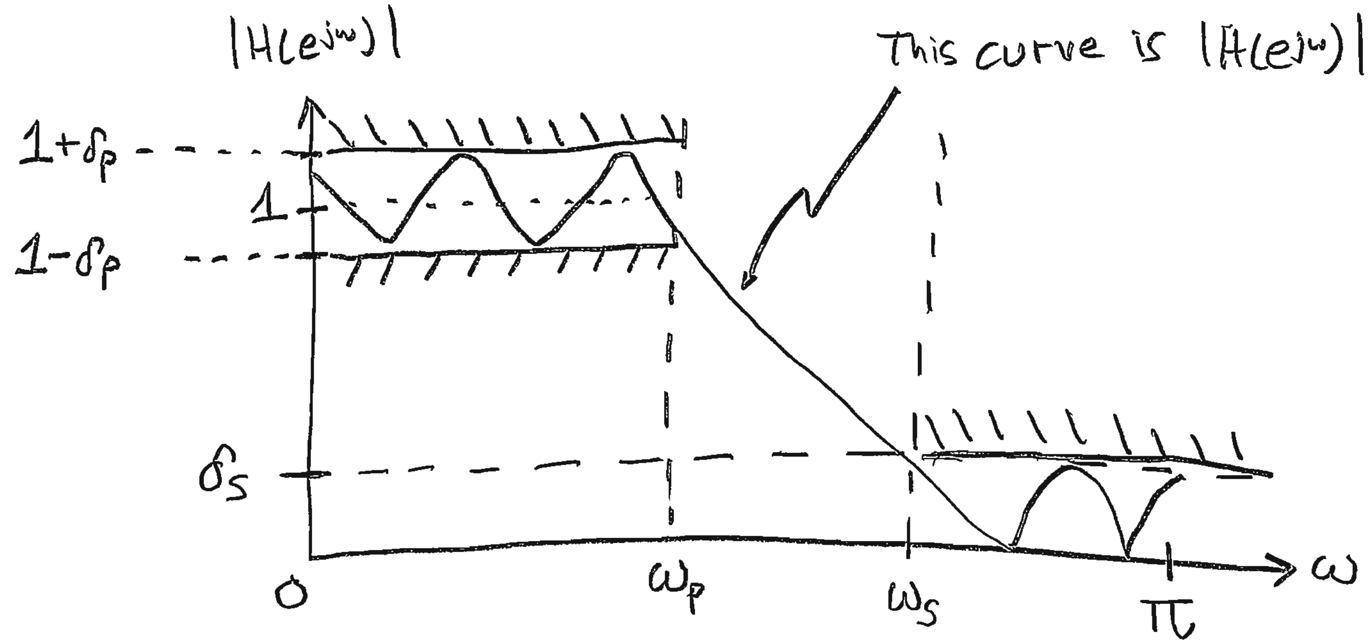
  Any discrete-time Fourier transform $X(e^{j\omega})$ or $H(e^{j\omega})$ is always 2π-periodic in ω.

  → We usually graph them from $-\pi$ to π only... or from 0 to 2π only...

  → But they are all 2π-periodic.

- For this reason, the design spec will usually only specify $|H(e^{j\omega})|$ for $0 \leq \omega \leq \pi$.

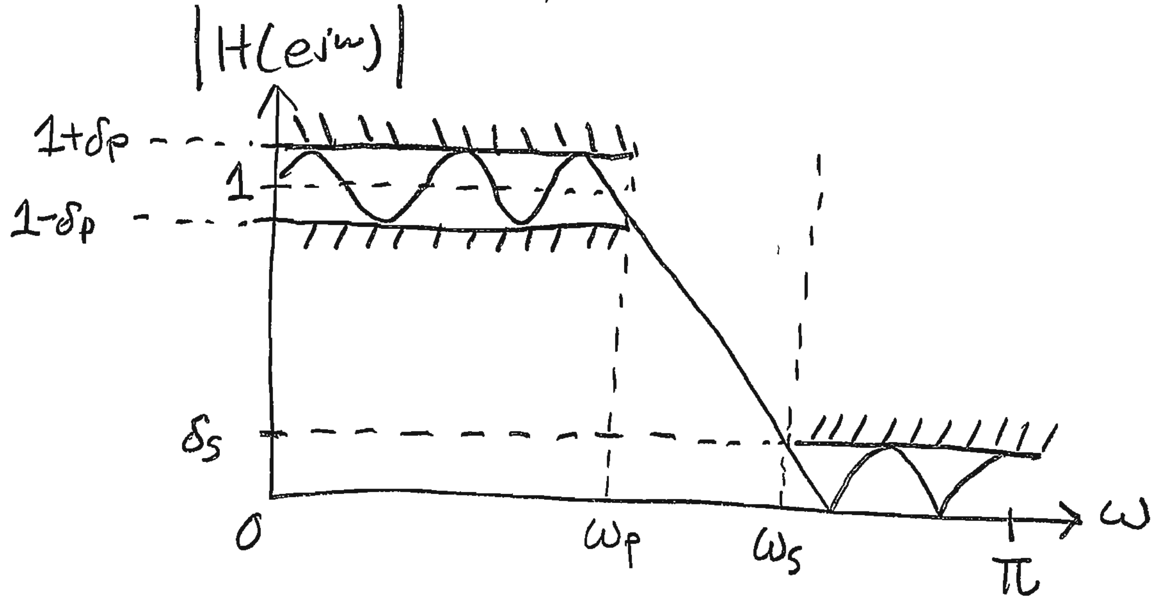
- Here is what a <u>real world</u> digital filter design spec usually looks like:



- This is a low-pass filter
- ω_p is the <u>passband edge frequency</u>.
  - The filter passband goes from $\omega = 0$ (DC) to $\omega = \omega_p$.
  - frequencies in $X(e^{j\omega})$ that lie in the filter passband are <u>not</u> attenuated... they are "passed"... so that they can make it into $Y(e^{j\omega})$.
  - In the passband, $|H(e^{j\omega})| \approx 1 = 0$ dB.
- δ_p is the <u>allowable passband ripple</u>. Some people call it the "passband ripple spec."

⟹ Everywhere in the passband, $|H(e^{j\omega})|$ must be <u>between</u> $1-\delta_p$ and $1+\delta_p$.

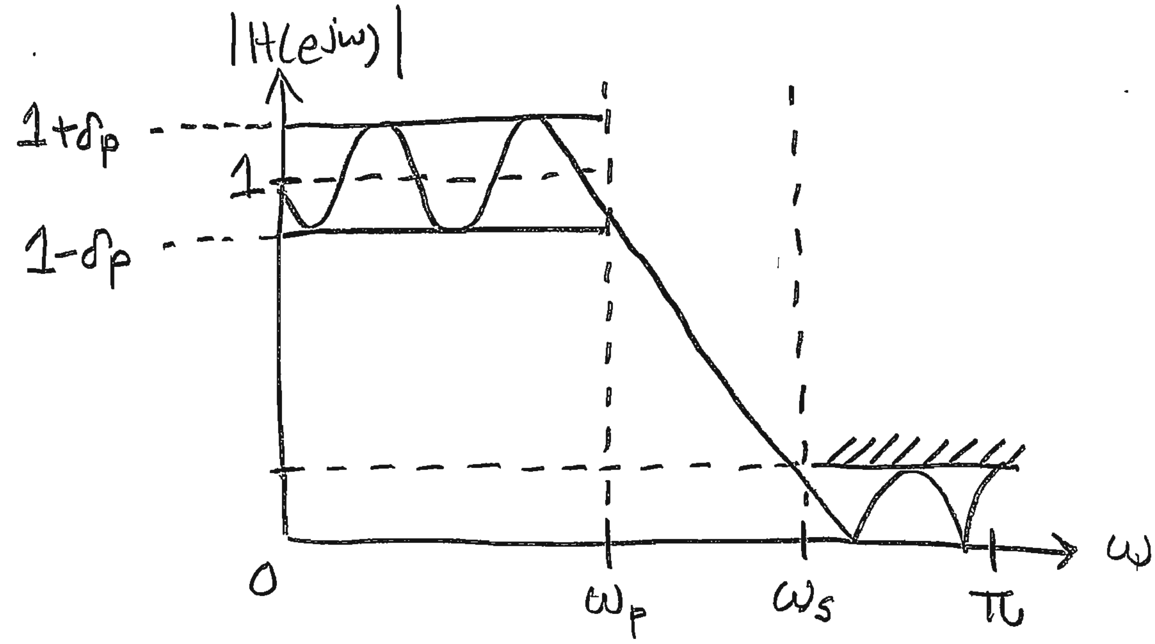
→ In other words, $|H(e^{j\omega})|$ must be <u>one</u> to within $\pm\delta_p$.



- ω_s is the stopband edge frequency.
  - The filter stopband goes from $\omega = \omega_s$ to $\omega = \pi$.
  - frequencies in $X(e^{j\omega})$ that lie in the filter stopband are greatly attenuated ... they are "stopped" from making it into $Y(e^{j\omega})$.
  - In the stopband, $|H(e^{j\omega})| \approx 0$.
- δ_s is the allowable stopband ripple.
  - It is often expressed in deciBells as $\alpha_s = -20\log_{10}\delta_s$ dB. (since $0 < \delta_s < 1$, α_s is a positive number).
  - $\alpha_s = -20\log_{10}\delta_s$ dB is called the minimum stopband attenuation.

$\Rightarrow$ Everywhere in the stopband, $|H(e^{j\omega})|$ must be below δ_s ...

$\rightarrow$ In other words, the filter must provide at least α_s dB of attenuation everywhere in the stopband.



- The region from ω_p to ω_s is called the filter <u>transition band</u>.

  - For the digital filter design problem, the shape of $|H(e^{j\omega})|$ in the transition band is "don't care."

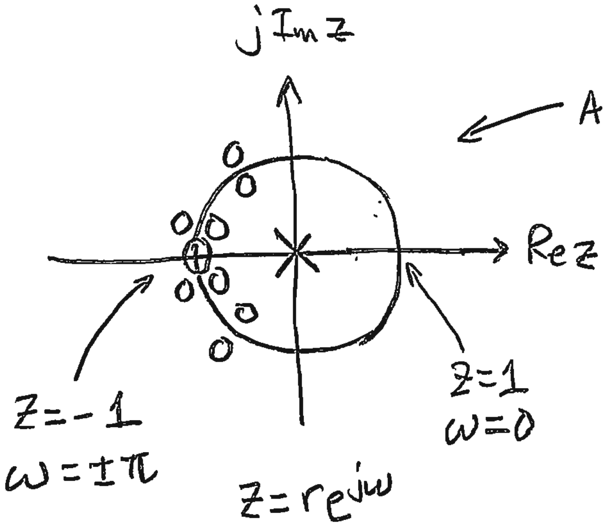
    - But:

      - $|H(e^{j\omega})|$ must be <u>at or above</u> $1-\delta_p$ at the passband edge frequency ω_p.

      - $|H(e^{j\omega})|$ must be <u>at or below</u> δ_s at the stopband edge frequency ω_s.

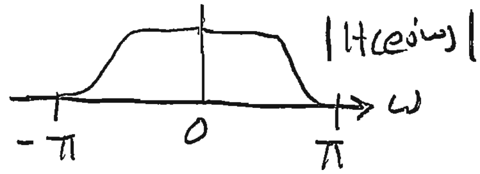
$\Rightarrow$ So in practice, the transition band is where $|H(e^{j\omega})|$ "drops" or "transitions" from $1-\delta_p$ to δ_s.

- A digital filter with a sharp transition band is called "high performance."
  - Designing a high performance filter usually requires a high filter order... meaning higher cost, increased complexity, and greater delay.
  - Recall from p. 6.140:
    - An FIR filter has <u>all</u> of the poles of $H(z)$ at $z=0$.
    - The designer can place the zeros anywhere in the z-plane that she or he wants,
      - but the poles must all be at $z=0$.
    - To achieve high performance in an FIR filter, the designer will generally need to place many zeros close together and close to the unit circle... to pull the surface $H(z)$ abruptly down towards zero in the transition band



A high-performance FIR filter pole-zero plot.

Recall: $H(e^{j\omega})$ is given by $H(z)$ above the unit circle.

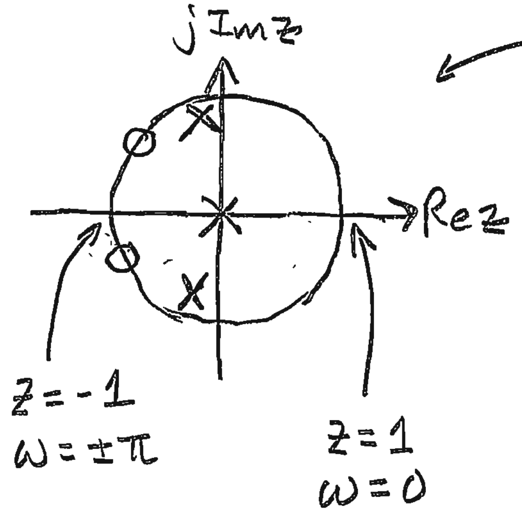


- For an IIR filter design, the designer can also place poles in the z-plane at locations other than $z=0$

  → But for a causal, stable filter, they must always be <u>strictly</u> <u>inside</u> the unit circle.

- This means that a sharp transition band can be achieved with a <u>lower</u> order using an IIR design (as compared to an FIR design).

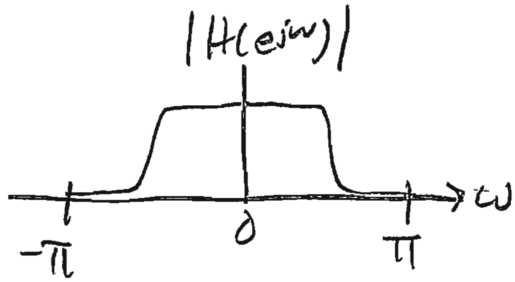
  - Because with an IIR design you can place poles and zeros close together (and close to the unit circle) to achieve a sharp transition.



← A high-performance IIR filter.

- The poles pull the surface $H(z)$ up towards ∞
- The zeros pull $H(z)$ down to zero
- Above the unit circle where $H(z)\big|_{|z|=1} = H(e^{j\omega})$, this makes a sharp transition from passband to stopband.

$z = re^{j\omega}$



- But FIR filters have at least two desirable advantages:

① <u>Always</u> <u>stable</u>. Because $h[n]$ has a finite length with an FIR filter, it is <u>always</u> true that $\sum_{n=-\infty}^{\infty} |h[n]| < \infty \rightarrow$ <u>always</u> <u>stable</u>.

- This can be an issue with an IIR design.
  - For a high performance IIR design, you usually need to place poles near the unit circle.
  - Cost and profit considerations will often require the hardware or software to be implemented in fixed point arithmetic.
  - The resulting roundoff error can cause a pole that was <u>designed</u> to be inside the unit circle (but close) to actually jump <u>outside</u> the unit circle when the filter is implemented with finite precision arithmetic...

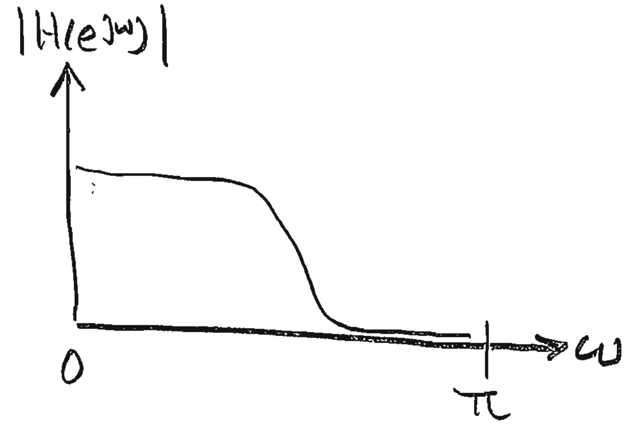
$\Rightarrow$ Making the <u>implemented</u> filter <u>unstable</u>.

② Can be designed for <u>linear</u> <u>phase</u>... by making $h[n]$ have symmetry about the midpoint

(Recall: pp. 5.73 - 5.87)

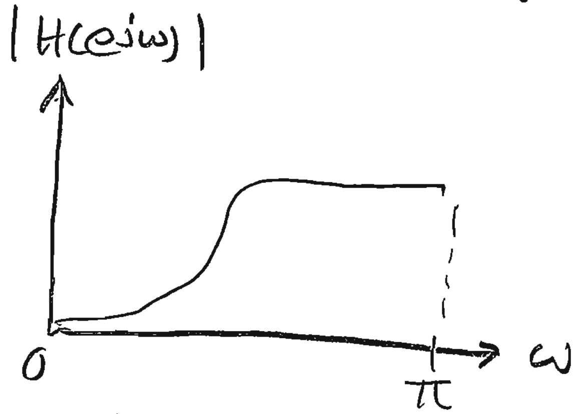
- Here are the main types of digital filters:

Low Pass:



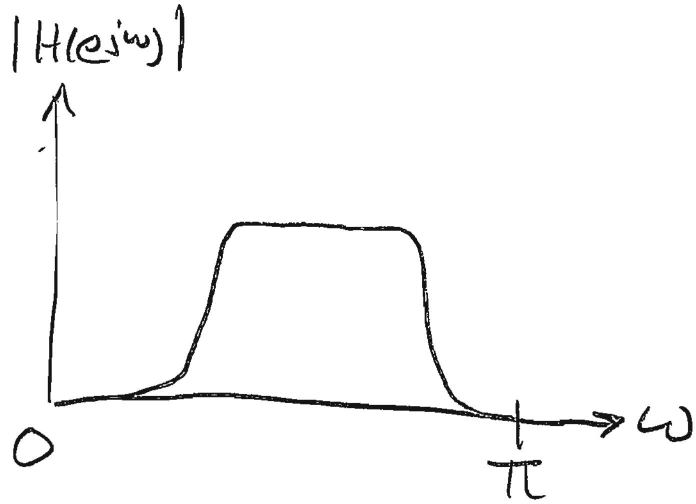
- Passes the low frequencies
- Stops or "blocks" the high frequencies

High Pass



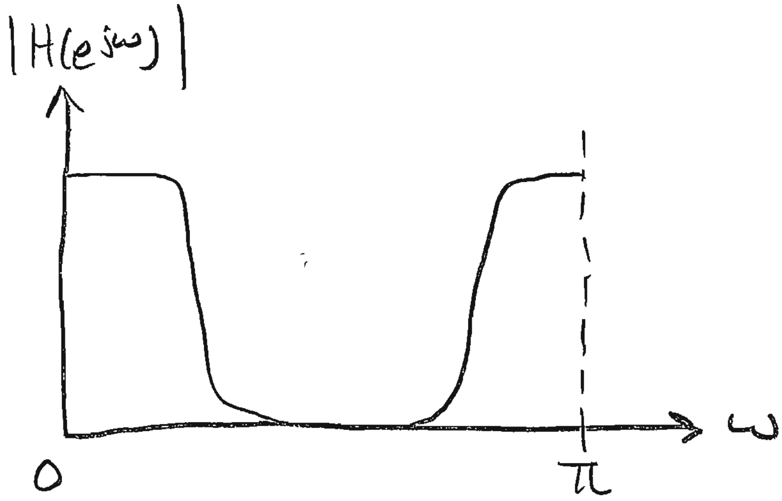
- passes the high frequencies
- Stops or "blocks" the low frequencies

Band Pass



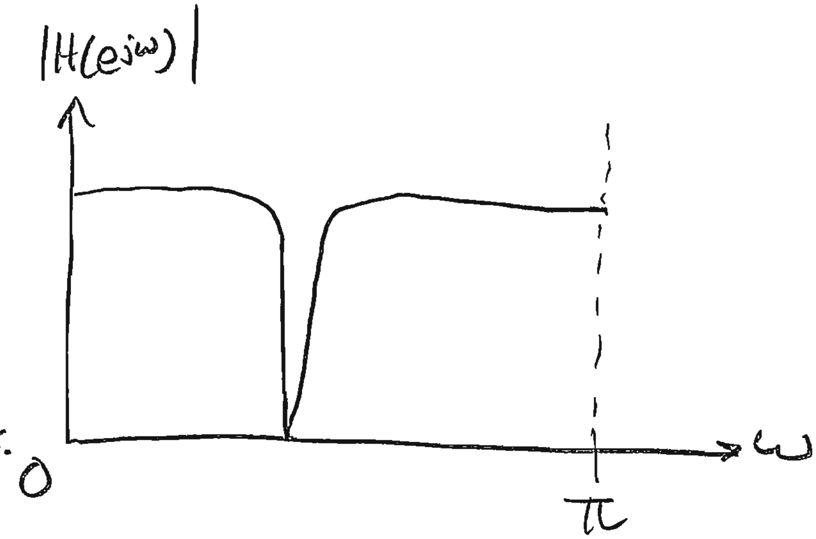
- passes only a midrange band of frequencies

Band Stop



- Stops or "blocks" a midrange band of frequencies.

Notch Filter



- Removes only a <u>very</u> <u>narrow</u> band of frequencies.

→ used to suppress feedback in high-power audio systems.

→ Used to remove a sinusoidal noise or "buzz"... like "AC line noise" at 60 Hz analog freq.

- When doing a design by hand, you will almost always design a low pass filter.

  - There are transformations for:

    - converting a high pass filter into a low pass filter

    - converting a band pass filter into a low pass filter.

  etc...

  - These transformations are beyond the scope of ECE 2713, but they are taught in ECE 4213 (DSP).

- So, for example, if you are asked to design a high pass filter, the usual procedure is:

  1. Use a transformation to convert the high pass spec into an equivalent low pass spec.

  2. Design the low pass filter.

  3. Use another transformation to convert the designed low pass filter into the required high pass filter.

- In ECE 2713, we will learn one technique for designing low pass FIR digital filters:

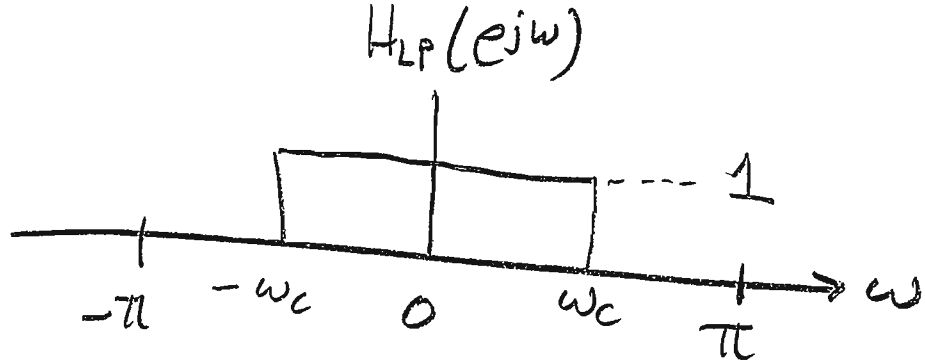
  - It is called the "windowed" FIR design method.
  - It is widely used in real-world practice.
  - This method does not allow you to control the passband ripple directly. It is controlled indirectly by assuming that $\delta_p = \delta_s$ and then by designing for the transition bandwidth.

- We will learn to do this in a way that will always produce an FIR impulse response $h[n]$ that is symmetric about the midpoint.

$\Rightarrow$ So we will always get an FIR filter with <u>linear</u> <u>phase</u>.

Windowed FIR filter design:

- An "ideal" low pass filter with cutoff frequency ω_c has frequency response:



- The impulse response is (table):

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$$

- This filter is not realizable... it cannot be built...

  - because it is <u>unstable</u> and it is <u>not</u> <u>causal</u>.

- However, we can make it <u>stable</u> if we multiply $h_{LP}[n]$ times a finite-length window to "chop" it off.

- Then we can shift the finite length version to make it causal.

- This is called "windowed FIR filter design"

Common Window Functions for FIR filter design:

Rectangular: $$w[n] = 1, \qquad -M \le n \le M,$$

Hann: $$w[n] = \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{M}\right)\right], \qquad -M \le n \le M,$$

Hamming: $$w[n] = 0.54 + 0.46\cos\left(\frac{\pi n}{M}\right), \qquad -M \le n \le M,$$

Blackman: $$w[n] = 0.42 + 0.5\cos\left(\frac{\pi n}{M}\right) + 0.08\cos\left(\frac{2\pi n}{M}\right), \qquad -M \le n \le M.$$

Main Properties of the Window Functions:

Type of Window	Main Lobe Width Δ_{ML}	Relative Sidelobe Level $A_{s\ell}$	Minimum Stopband Attenuation	Transition Bandwidth $\Delta\omega$
Rectangular	$4\pi/(2M+1)$	13.3 dB	20.9 dB	$0.92\pi/M$
Hann	$8\pi/(2M+1)$	31.5 dB	43.9 dB	$3.11\pi/M$
Hamming	$8\pi/(2M+1)$	42.7 dB	54.5 dB	$3.32\pi/M$
Blackman	$12\pi/(2M+1)$	58.1 dB	75.3 dB	$5.56\pi/M$

Design Steps:

1. Convert the minimum stopband attenuation spec δ_s to dB using the formula $\alpha_s = -20\log_{10}\delta_s$.
2. Look in column 4 (Minimum Stopband Attenuation) of the table above to determine which window functions can provide at least α_s dB of stopband attenuation.
3. Let $\Delta\omega = \omega_s - \omega_p$. Use the last column of the table to figure out which window function $w[n]$ can meet the stopband spec with the smallest value M.
   - To do this, set $\Delta\omega$ equal to the formula in the last column of the table and solve for M. M must be an integer and you must always round up. For example, 2.001 means $M = 3$.
   - The order of your filter will be $2M$.
   - The length of $h[n]$ will be $2M+1$.
4. Let $\omega_c = \dfrac{\omega_p + \omega_s}{2}$.
5. Let $h_{LP}[n] = \dfrac{\sin(\omega_c n)}{\pi n}$.
6. For the window function $w[n]$ that meets the stopband spec with the smallest M, compute the "centered" impulse response $h_1[n] = w[n]h_{LP}[n]$, $-M \le n \le M$.
7. Shift it right by M to make it causal: $h[n] = h_1[n-M] = w[n-M]h_{LP}[n-M]$, $0 \le n \le 2M$.

Here is an example problem for Windowed FIR filter design. I am going to follow the steps on page 6 of our ECE 2713 Final Exam formula sheet.

Use the window design method to design a causal lowpass FIR digital filter that meets the following specifications:

Passband Edge Freq.	$\omega_p = 0.3\pi$ rad/sample
Stopband Edge Freq.	$\omega_s = 0.35\pi$ rad/sample
Max. Passband Ripple	$\delta_p = 0.01$
Max. Stopband Ripple	$\delta_s = 0.01$

Give the filter impulse response $h[n]$.

① $\alpha_s = -20\log_{10}\delta_s = -20\log_{10} 0.01 = 40\text{ dB}$

② The Hann, Hamming, and Blackman windows can all provide more than 40 dB of stopband attenuation.
→ I know that Hann will give the smallest M because it is highest in the table, but let's go ahead and compute the M's for each window anyway.

③ $\Delta\omega = \omega_s - \omega_p = 0.35\pi - 0.3\pi = 0.05\pi$.

Rectangular: can not provide 40 dB of stopband attenuation

Hann: $\Delta\omega = 0.05\pi = \frac{3.11\pi}{M} \rightarrow M = \left\lceil \frac{3.11}{0.05} \right\rceil = \lceil 62.2 \rceil = 63$

Hamming: $\Delta\omega = 0.05\pi = \frac{3.32\pi}{M} \rightarrow M = \left\lceil \frac{3.32}{0.05} \right\rceil = \lceil 66.4 \rceil = 67$

Blackman: $\Delta\omega = 0.05\pi = \frac{5.56\pi}{M} \rightarrow M = \left\lceil \frac{5.56}{0.05} \right\rceil = \lceil 111.2 \rceil = 112$

⟶ Hann meets the stopband spec with the smallest M... which means lowest order.

(4) $\omega_c = \dfrac{\omega_p + \omega_s}{2} = \dfrac{0.3\pi + 0.35\pi}{2} = 0.325\pi$

(5) $$h_{LP}[n] = \frac{\sin(0.325\pi n)}{\pi n}$$

(6) $$w[n]\, h_{LP}[n] = \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{63}\right)\right] \frac{\sin(0.325\pi n)}{\pi n}, \quad -63 \leq n \leq 63$$

(7) Shift to make causal:

$$h[n] = \frac{1}{2}\left[1 + \cos\left(\frac{\pi(n-63)}{63}\right)\right] \frac{\sin[0.325\pi(n-63)]}{\pi(n-63)},$$

$$0 \leq n \leq 126$$

4. 25/20 pts. Use the window design method with an appropriate fixed window from the Table to design a causal lowpass FIR digital filter that meets the following specifications:

Passband Edge Freq.	$\omega_p = 0.2\pi$ rad/sample
Stopband Edge Freq.	$\omega_s = 0.9\pi$ rad/sample
Max. Passband Ripple	$\delta_p = 0.005$
Max. Stopband Ripple	$\delta_s = 0.005$

Give the filter impulse response $h[n]$.

$$\alpha_s = -20 \log_{10} \delta_s = -20 \log_{10} 0.005 = 46.0206 \text{ dB.}$$

Table : Hamming and Blackman can meet the spec.

$$\Delta\omega = \omega_s - \omega_p = 0.9\pi - 0.2\pi = 0.7\pi$$

Hamming: $\Delta\omega = \frac{3.32\pi}{M}$; $M = \left\lceil \frac{3.32\pi}{\Delta\omega} \right\rceil = \left\lceil \frac{3.32\pi}{0.7\pi} \right\rceil = \lceil 4.74 \rceil = 5$

Blackman; $\Delta\omega = \frac{5.56\pi}{M}$; $M = \left\lceil \frac{5.56\pi}{\Delta\omega} \right\rceil = \left\lceil \frac{5.56\pi}{0.7\pi} \right\rceil = \lceil 7.94 \rceil = 8$

$\Rightarrow$ Hamming meets the spec with a lower order

$M = 5$. Order $= N = 2M = 10$. Length $= 2M+1 = 11$.

$$\omega_c = \frac{\omega_p + \omega_z}{2} = \frac{0.2\pi + 0.9\pi}{2} = \frac{1.1\pi}{2} = 0.55\pi.$$

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\sin 0.55\pi n}{\pi n}$$

$$w[n] = 0.54 + 0.46 \cos\left(\frac{\pi}{M} n\right) = 0.54 + 0.46 \cos\left(\frac{\pi}{5} n\right)$$

$$-5 \le n \le 5.$$

$$w[n] h_{LP}[n] = \left\{0.54 + 0.46 \cos\left(\frac{\pi}{5} n\right)\right\} \frac{\sin 0.55\pi n}{\pi n}, \quad -5 \le n \le 5.$$

shift to make causal:

$$h[n] = \frac{\sin 0.55\pi(n-5)}{\pi(n-5)} \left\{0.54 + 0.46 \cos\frac{\pi}{5}(n-5)\right\}, \quad 0 \le n \le 10$$