ECE 3793
Test 1B

Wednesday, November 8, 2000
7:00 PM - 9:00 PM

Fall 2000 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. You have 120 minutes to complete the test. All work must be your own.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

Formulas from Trigonometry:

$\sin^2 A + \cos^2 A = 1$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$\cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan \frac{A}{2} = \frac{\sin A}{1+\cos A}$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\cos A - \cos B = 2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(B-A)$

$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

$\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$

Differentiation Formulas:

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \left(-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, (0 < \cos^{-1} u < \pi)$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}, \left(-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}e^u = e^u \frac{du}{dx}$

$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}, \ a \neq 0, 1$

Integration Formulas:

Integration by parts: $\int u\,dv = uv - \int v\,du$

$\int \frac{du}{u} = \ln|u|$

$\int e^u\,du = e^u$

$\int a^u\,du = \frac{a^u}{\ln a}, \quad a > 0, a \neq 1$

$\int \sin u\,du = -\cos u$

$\int \cos u\,du = \sin u$

$\int \tan u\,du = -\ln\cos u$

$\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

$\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

$\int \tan^2 u\,du = \tan u - u$

$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)$

$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$

$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$

$\int x \sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$

$\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$

$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$\int x \cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$

$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$

$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$

$\int x e^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$

$\int \ln x\,dx = x\ln x - x$

$\int x \ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$

Summation Formulas:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \alpha \neq 1 \qquad \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \quad |a| < 1 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1$$

$$\sum_{k=0}^{n} k a^k = \frac{a\{1-(n+1)a^n + n a^{n+1}\}}{(1-a)^2}$$

Signals:

$$\delta(t) = \tfrac{d}{dt}u(t) \qquad \delta[n] = u[n] - u[n-1]$$

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt \qquad \langle x_1[n], x_2[n]\rangle = \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$$

$$\mathcal{E}v\{x(t)\} = \tfrac{1}{2}\{x(t) + x(-t)\} \qquad \mathcal{O}d\{x(t)\} = \tfrac{1}{2}\{x(t) - x(-t)\}$$

Complex Exponential Signals:

e^{jw_0t}	e^{jw_0n}
Distinct signals for distinct w_0	Identical signals for values of w_0 separated by multiples of 2π
Periodic for any choice of w_0	Periodic only if $w_0/(2\pi) = m/N \in \mathbb{Q}$
Fundamental frequency w_0	Fundamental frequency w_0/m
Fundamental period: $w_0 = 0$: undefined $w_0 \neq 0$: $2\pi/w_0$	Fundamental period: $w_0 = 0$: one $w_0 \neq 0$: $2\pi m/w_0$

Systems:

System H is linear if $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$.

System H is time invariant if $H\{x(t-t_0)\} = y(t-t_0)$.

System H is memoryless if the current output depends only on the current input.

System H is invertible if distinct inputs produce distinct outputs.

System H is invertible if an inverse system G exists which "undoes" the action of H.

System H is causal if the current output depends only on the past and present inputs.

LTI system H is causal iff $h(t) = 0 \; \forall \; t < 0$.

Bounded: $x(t)$ is bounded if $\exists \; B \in \mathbb{R}$, $B > 0$, such that $|x(t)| \leq B \; \forall \; t \in \mathbb{R}$.

System H is BIBO stable if every bounded input produces a bounded output.

LTI system H is BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \qquad h(t) = \tfrac{d}{dt}s(t)$$

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad h[n] = s[n] - s[n-1]$$

1. **25 pts.** Consider a continuous-time system H with input $x(t)$ and output $y(t)$ related by

$$y(t) = \begin{cases} 0, & t < 0, \\ x(t) + x(t-2), & t \geq 0. \end{cases}$$

(a) **5 pts.** Is the system H memoryless? Justify your answer.

When $t=4$, $y(4) = x(2) + x(4)$.

Since $y(4)$ depends on $x(2)$, the system is <u>not</u> <u>memoryless</u>.

(b) **5 pts.** Is the system H causal? Justify your answer.

For $t<0$, the output does not depend on the input.

For $t \geq 0$, the output $y(t)$ depends on $x(t)$ {current input} and $x(t-2)$ {past input}.

$\Rightarrow$ The output never depends on a <u>future</u> input,

$\Rightarrow$ The system <u>is</u> <u>causal</u>.

(c) **5 pts.** Is the system H BIBO stable? Justify your answer.

Suppose $x(t)$ is a bounded input. Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|x(t)| \leq B \ \forall t \in \mathbb{R}$.

when $t<0$, $y(t)=0$, so $|y(t)| = 0 < 2B$.

when $t \geq 0$,
$$|y(t)| = |x(t) + x(t-2)| \leq |x(t)| + |x(t-2)| \leq B + B = 2B.$$

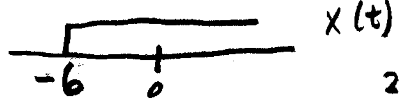
So $|y(t)| \leq 2B \ \forall t \in \mathbb{R}$. $\Rightarrow$ $y(t)$ is bounded.

Since $y(t)$ is bounded, the system <u>is</u> stable.

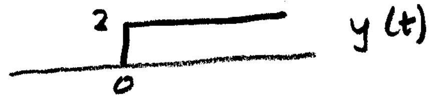
Problem 1, cont...

(d) **5 pts.** Is the system H time invariant? Justify your answer.

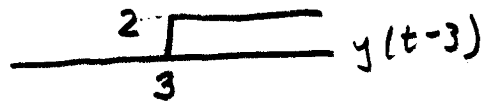
Let $x(t) = u(t+6)$.



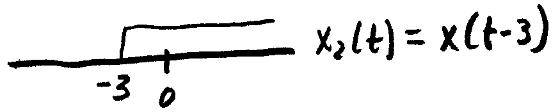
Then $y(t) = \begin{cases} 0, & t<0 \\ 2, & t \geq 0 \end{cases} = 2u(t)$.



Then $y(t-3) = 2u(t-3)$.



Now let $x_2(t) = x(t-3) = u(t+3)$



Then $y_2(t) = \begin{cases} 0, & t<0 \\ 2, & t \geq 0 \end{cases} = 2u(t) = y(t) \neq y(t-3)$.

Since $y_2(t) \neq y(t-3)$, the system is <u>not</u> time invariant

(e) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = \begin{cases} 0, & t<0 \\ x_1(t) + x_1(t-2), & t \geq 0 \end{cases} = [x_1(t) + x_1(t-2)]u(t)$.

Let $y_2(t) = H\{x_2(t)\} = \begin{cases} 0, & t<0 \\ x_2(t) + x_2(t-2), & t \geq 0 \end{cases} = [x_2(t) + x_2(t-2)]u(t)$.

Let a, b be constants and let $x_3(t) = ax_1(t) + bx_2(t)$.

Then $y_3(t) = \begin{cases} 0, & t<0 \\ x_3(t) + x_3(t-2), & t \geq 0 \end{cases} = [x_3(t) + x_3(t-2)]u(t)$

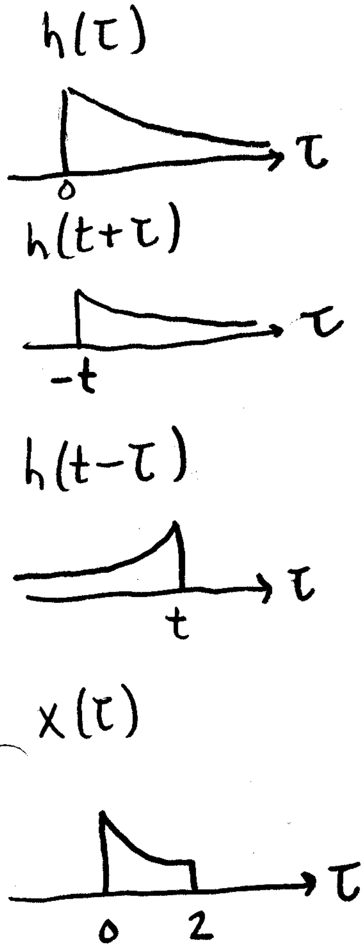
$= [ax_1(t) + bx_2(t) + ax_1(t-2) + bx_2(t-2)]u(t)$

$= a[x_1(t) + x_1(t-2)]u(t) + b[x_2(t) + x_2(t-2)]u(t)$

$= ay_1(t) + by_2(t)$ ✓

Therefore the system <u>is</u> linear.

2. **25 pts.** Consider a continuous-time LTI system H with impulse response $h(t) = e^{-t}u(t)$. The system input is given by $x(t) = e^{-3t}[u(t) - u(t-2)]$. Find the system output $y(t)$.



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

I.) $t<0$: $y(t)=0$.

II.) $t\geq 0$ and $t<2$: $0\leq t<2$

$$y(t) = \int_0^t e^{-3\tau}e^{-(t-\tau)}\,d\tau$$

$$= \int_0^t e^{-3\tau}e^{-t}e^{\tau}\,d\tau = e^{-t}\int_0^t e^{-2\tau}\,d\tau$$

$$= \frac{e^{-t}}{-2}\left[e^{-2\tau}\right]_0^t = \frac{-e^{-t}}{2}\left[e^{-2t}-1\right]$$

$$= \frac{1}{2}\left[e^{-t}-e^{-3t}\right] = \frac{1}{2}e^{-t}\left[1-e^{-2t}\right]$$

III.) $t\geq 2$;

$$y(t) = \int_0^2 e^{-3\tau}e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^2 e^{-2\tau}\,d\tau$$

$$= \frac{-e^{-t}}{2}\left[e^{-2\tau}\right]_0^2 = \frac{e^{-t}}{2}\left[1-e^{-4}\right]$$

All Together:

$$y(t) = \begin{cases} 0, & t<0 \\ \frac{1}{2}e^{-t}[1-e^{-2t}], & 0\leq t<2 \\ \frac{1}{2}e^{-t}[1-e^{-4}], & t\geq 2 \end{cases}$$

3. **25 pts**. Consider an **invertible** discrete-time LTI system H with impulse response

$$h[n] = (-1)^n u[n+1].$$

Call the inverse system G.

(a) **5 pts**. Is the system H causal? Justify your answer.

H is not causal because

$$h[-1] = (-1)^{-1} u[0] = \frac{1}{-1} \cdot 1 = -1 \neq 0.$$

(b) **5 pts**. Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-1}^{\infty} |(-1)^n| = \sum_{n=-1}^{\infty} 1 \rightarrow \infty.$$

H is not stable because the system impulse response is not absolutely summable.

Problem 3, cont...

(c) **5 pts.** Find the impulse response $g[n]$ of the inverse system.

Hint: 1) Use the convolution sum to write the input/output relation of H. Put the "$n-k$" on the input $x[n]$ of H and the "k" on the impulse response $h[n]$ of H. 2) Write out the sum to get an expression for the output $y[n]$ of H in terms of the input $x[n]$ using "$\cdots$" instead of a "Σ". 3) Use this to write similar expressions for the shifted outputs $y[n-1]$ and $y[n-2]$ of H that can be combined to solve for the input $x[n]$.

For H: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-1}^{\infty} (-1)^k x[n-k]$

$= (-1)^{-1} x[n+1] + (-1)^0 x[n] + (-1)^1 x[n-1] + \cdots$

$= -x[n+1] + x[n] - x[n-1] + x[n-2] - x[n-3] + \cdots$

So $y[n-1] = -x[n] + x[n-1] - x[n-2] + x[n-3] - x[n-4] + \cdots$

So $y[n-2] = -x[n-1] + x[n-2] - x[n-3] + x[n-4] - x[n-5] + \cdots$

So $y[n-1] + y[n-2] = -x[n]$

or $x[n] = -y[n-1] - y[n-2]$

$\Rightarrow$ The input/output relation of system G is then

$y[n] = -x[n-1] - x[n-2]$

$= x[n] * (-\delta[n-1] - \delta[n-2])$

So the impulse response of system G is

$\underline{g[n] = -\delta[n-1] - \delta[n-2]}.$

Problem 3, cont. . .

(d) **5 pts**. Is the inverse system G causal? Justify your answer.

Since $g[n] = 0 \ \forall n < 0$, system G <u>is</u> <u>causal</u>.

(e) **5 pts**. Is the inverse system G BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |g[n]| = \sum_{n=-\infty}^{\infty} \left| -\delta[n-1] - \delta[n-2] \right|$$

$$= \underbrace{|-1|}_{n=1} + \underbrace{|-1|}_{n=2} = 2 < \infty.$$

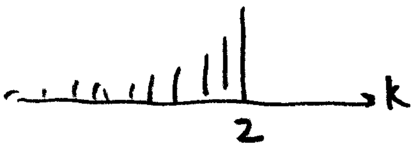
Therefore, system G <u>is</u> <u>BIBO</u> <u>stable</u> because the impulse response is absolutely summable.

4. **25 pts.** Consider a discrete-time LTI system H with impulse response

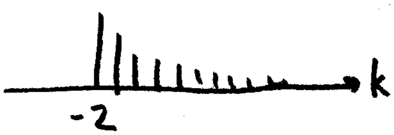
$$h[n] = \left(\frac{1}{4}\right)^{-n} u[-n+2].$$

The system input is given by $x[n] = \left(\frac{1}{2}\right)^n u[n+2]$. Find the system output $y[n]$.

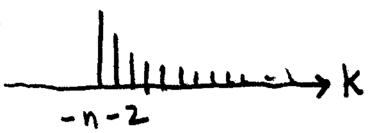
$h[k]$



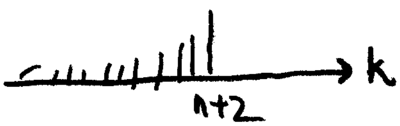
$x[k]$



$x[n+k]$



$x[n-k]$



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

I) $n+2 < 2$: $n < 0$:

$$y[n] = \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{8}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-n-2}^{\infty} \left(\tfrac{1}{8}\right)^{k}$$

$$= \left(\tfrac{1}{2}\right)^n \frac{\left(\tfrac{1}{8}\right)^{-n-2} - \left(\tfrac{1}{8}\right)^{\infty}}{1 - \tfrac{1}{8}} = \left(\tfrac{1}{2}\right)^n \frac{\left(\tfrac{1}{8}\right)^{-n} \left(\tfrac{1}{8}\right)^{-2}}{7\left(\tfrac{1}{8}\right)}$$

$$= \left(\tfrac{1}{7}\right)(8)(8)^n(8)^2\left(\tfrac{1}{2}\right)^n = \frac{8^3}{7}(2)^{3n}(2)^{-n}$$

$$= \frac{8^3}{7}(2)^{2n} = \frac{8^3}{7} 4^n$$

II) $n+2 \geq 2$: $n \geq 0$:

$$y[n] = \sum_{k=-\infty}^{2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{\infty} \left(\tfrac{1}{8}\right)^{k}$$

$$= \left(\tfrac{1}{2}\right)^n \frac{\left(\tfrac{1}{8}\right)^{-2} - \left(\tfrac{1}{8}\right)^{\infty}}{1 - 1/8} = \left(\tfrac{1}{2}\right)^n \left(\tfrac{8}{7}\right)(8)^2 = \frac{8^3}{7}\left(\tfrac{1}{2}\right)^n.$$

<u>All Together</u>

$$y[n] = \begin{cases} \frac{8^3}{7} 4^n, & n < 0 \\ \frac{8^3}{7}\left(\frac{1}{2}\right)^n, & n \geq 0. \end{cases}$$