ECE 3793
Test 1

Tuesday, October 10, 2000
7:15 PM - 10:15 PM

Fall 2000 — Name: SOLUTION

Dr. Havlicek — Student Num:________________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **five** problems. Work any **four** of them. Only **four** problems will be graded. Below, you must circle the numbers of the **four** problems you wish to have graded.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. **2.** **3.** **4.** **5.**

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

Formulas from Trigonometry:

$\sin^2 A + \cos^2 A = 1$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin 2A = 2 \sin A \cos A$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$\cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\cos 2A = \cos^2 A - \sin^2 A$

$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$\tan \frac{A}{2} = \frac{\sin A}{1+\cos A}$

$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

$\cos A - \cos B = 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A)$

$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

Differentiation Formulas:

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \left(-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}, \left(-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$

$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, (0 < \cos^{-1} u < \pi)$

$\frac{d}{dx}e^u = e^u \frac{du}{dx}$

$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}, \; a \neq 0, 1$

Integration Formulas:

Integration by parts: $\int u\,dv = uv - \int v\,du$

$\int \frac{du}{u} = \ln|u|$

$\int a^u\,du = \frac{a^u}{\ln a}, \quad a > 0, \, a \neq 1$

$\int \cos u\,du = \sin u$

$\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

$\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

$\int \tan^2 u\,du = \tan u - u$

$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)$

$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$

$\int x \sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$

$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$

$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$

$\int \ln x\,dx = x\ln x - x$

$\int e^u\,du = e^u$

$\int \sin u\,du = -\cos u$

$\int \tan u\,du = -\ln \cos u$

$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$

$\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$

$\int x \cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$

$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

$\int x e^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$

$\int x \ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$

Summation Formulas:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \alpha \neq 1 \qquad \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \quad |a| < 1 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1$$

$$\sum_{k=0}^{n} k a^k = \frac{a\{1-(n+1)a^n + na^{n+1}\}}{(1-a)^2}$$

Signals:

$$\delta(t) = \tfrac{d}{dt}u(t) \qquad \delta[n] = u[n] - u[n-1]$$

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt \qquad \langle x_1[n], x_2[n]\rangle = \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$$

$$\mathcal{E}v\{x(t)\} = \tfrac{1}{2}\{x(t) + x(-t)\} \qquad \mathcal{O}d\{x(t)\} = \tfrac{1}{2}\{x(t) - x(-t)\}$$

Complex Exponential Signals:

e^{jw_0t}	e^{jw_0n}
Distinct signals for distinct w_0	Identical signals for values of w_0 separated by multiples of 2π
Periodic for any choice of w_0	Periodic only if $w_0/(2\pi) = m/N \in \mathbb{Q}$
Fundamental frequency w_0	Fundamental frequency w_0/m
Fundamental period: $w_0 = 0$: undefined $w_0 \neq 0$: $2\pi/w_0$	Fundamental period: $w_0 = 0$: one $w_0 \neq 0$: $2\pi m/w_0$

Systems:

System H is linear if $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$.

System H is time invariant if $H\{x(t-t_0)\} = y(t-t_0)$.

System H is memoryless if the current output depends only on the current input.

System H is invertible if distinct inputs produce distinct outputs.

System H is invertible if an inverse system G exists which "undoes" the action of H.

System H is causal if the current output depends only on the past and present inputs.

LTI system H is causal iff $h(t) = 0 \;\forall\; t < 0$.

Bounded: $x(t)$ is bounded if $\exists\; B \in \mathbb{R}$, $B > 0$, such that $|x(t)| \leq B \;\forall\; t \in \mathbb{R}$.

System H is BIBO stable if every bounded input produces a bounded output.

LTI system H is BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \qquad h(t) = \tfrac{d}{dt}s(t)$$

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad h[n] = s[n] - s[n-1]$$

1. **25 pts.** Consider a continuous-time system H with input $x(t)$ and output $y(t)$ related by

$$y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau + 5.$$

(a) **5 pts.** Is the system H memoryless? Justify your answer.

H is NOT memoryless because

$y(1) = \int_{-\infty}^{2} x(\tau)\, d\tau + 5$ which depends on $x(-1)$.

(b) **5 pts.** Is the system H causal? Justify your answer.

H is NOT causal because

$y(2) = \int_{-\infty}^{4} x(\tau)\, d\tau + 5$ which depends on the future input $x(3)$.

(c) **5 pts.** Is the system H BIBO stable? Justify your answer.

Suppose $x(t) = u(-t)$. Then $|x(t)| \leq 1 \ \forall t$, so $x(t)$ is a bounded input.

But $|y(0)| = \left| \int_{-\infty}^{0} u(-\tau)\, d\tau + 5 \right|$

$= 5 + \int_{0}^{\infty} u(\tau)\, d\tau \to \infty$, so $y(t)$ is not bounded.

The system is unstable because a bounded input produced an unbounded output.

(d) **5 pts.** Is the system H time invariant? Justify your answer.

Let $y_1(t)$ be the output when $x_1(t)$ is the input.

Then $y_1(t) = \int_{-\infty}^{2t} x_1(\tau)d\tau + 5$.

Then $y_1(t-t_0) = \int_{-\infty}^{2(t-t_0)} x_1(\tau)d\tau + 5 = \int_{-\infty}^{2t-2t_0} x_1(\tau)d\tau + 5$.

Let $x_2(t) = x_1(t-t_0)$.

Then $y_2(t) = \int_{-\infty}^{2t} x_2(\tau)d\tau + 5 = \int_{-\infty} x_1(\tau - t_0)d\tau + 5$ $\left\{\begin{array}{l}\theta = \tau - t_0 \\ d\theta = d\tau \\ \tau = \theta + t_0\end{array}\right.$

$= \int_{-\infty}^{2t-t_0} x_1(\theta)d\theta + 5 = \int_{-\infty}^{2t-t_0} x_1(\tau)d\tau + 5 \neq y_1(t-t_0)$.

The system is <u>NOT</u> time invariant because $y_2(t) \neq y_1(t-t_0)$.

(e) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = \int_{-\infty}^{2t} x_1(\tau)d\tau + 5$.

Let $y_2(t) = H\{x_2(t)\} = \int_{-\infty}^{2t} x_2(\tau)d\tau + 5$.

Let a and b be constants.

Then $ay_1(t) + by_2(t) = a\int_{-\infty}^{2t} x_1(\tau)d\tau + b\int_{-\infty}^{2t} x_2(\tau) + 5(a+b)$.

Let $x_3(t) = ax_1(t) + bx_2(t)$.

Then $y_3(t) = H\{x_3(t)\} = \int_{-\infty}^{2t} x_3(\tau)d\tau + 5$

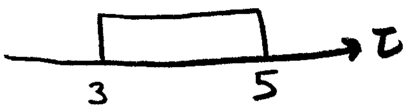
$= \int_{-\infty}^{2t} ax_1(\tau) + bx_2(\tau)\, d\tau + 5$

$= a\int_{-\infty}^{2t} x_1(\tau)d\tau + b\int_{-\infty}^{2t} x_2(\tau)d\tau + 5 \neq ay_1(t) + by_2(t)$.

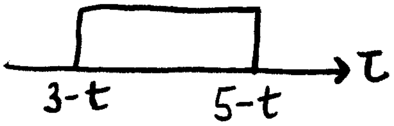
Therefore the system is <u>NOT</u> linear.

2. **25 pts.** Consider a continuous-time LTI system H with impulse response $h(t) = e^{-3t}u(t)$. The system input is given by $x(t) = u(t-3) - u(t-5)$. Find the system output $y(t)$.

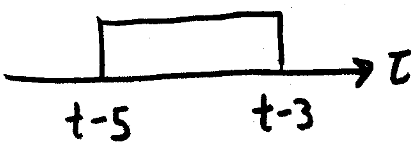
$x(\tau)$



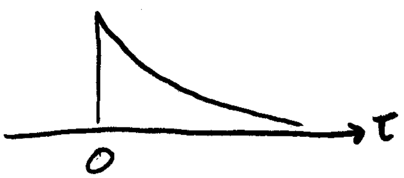
$x(t+\tau)$



$x(t-\tau)$



$h(\tau)$



(I) when $t-3<0 \Rightarrow t<3$: $y(t)=0$

(II) when $t-3 \geq 0$ and $t-5<0$

$\Rightarrow 3 \leq t < 5$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^{t-3} e^{-3\tau}\, d\tau = -\frac{1}{3} e^{-3\tau} \Big|_{\tau=0}^{t-3}$$

$$= -\frac{1}{3}\left[e^{-3(t-3)} - 1\right] = \frac{1}{3} - \frac{1}{3} e^{-3(t-3)}$$

(III) when $t-5 \geq 0 \Rightarrow t \geq 5$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{t-5}^{t-3} e^{-3\tau}\, d\tau = -\frac{1}{3} e^{-3\tau} \Big|_{\tau=t-5}^{t-3}$$

$$= -\frac{1}{3}\left[e^{-3(t-3)} - e^{-3(t-5)}\right] = -\frac{1}{3} e^{-3(t-3)}\left[1 - e^{6}\right]$$

All Together:
$$y(t) = \begin{cases} 0, & t<3 \\ \frac{1}{3} - \frac{1}{3} e^{-3(t-3)}, & 3 \leq t < 5 \\ \frac{1}{3}\left[e^{6}-1\right] e^{-3(t-3)}, & t \geq 5 \end{cases}$$

(Problem 2.11a), given on HW3)

3. **25 pts**. Consider an **invertible** discrete-time LTI system H with impulse response

$$h[n] = (-1)^n \, u[n+1].$$

Call the inverse system G.

(a) **5 pts**. Is the system H causal? Justify your answer.

H is <u>not</u> causal because

$$h(-1) = (-1)^{-1} u[0] = \frac{1}{-1} = -1 \neq 0.$$

(system H is causal iff $h[n]=0 \ \forall \, n<0$).

(b) **5 pts**. Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |(-1)^n u[n+1]| = \sum_{n=-1}^{\infty} |(-1)^n|$$

$$= \sum_{n=-1}^{\infty} 1 \longrightarrow \infty.$$

So H is <u>unstable</u> because the impulse response is not absolutely summable.

(c) **5 pts**. Find the impulse response $g[n]$ of the inverse system.

Hint: 1) Use the convolution sum to write the input/output relation of H. Put the "$n-k$" on the input $x[n]$ of H and the "k" on the impulse response $h[n]$ of H. 2) Write out the sum to get an expression for the output $y[n]$ of H in terms of the input $x[n]$ using "$\cdots$" instead of a "Σ". 3) Use this to write similar expressions for the shifted outputs $y[n-1]$ and $y[n-2]$ of H that can be combined to solve for the input $x[n]$.

For H: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-1}^{\infty} (-1)^k x[n-k]$

$$= (-1)^{-1}x[n+1] + (-1)^0 x[n] + (-1)^1 x[n-1] + (-1)^2 x[n-2] - \cdots$$

$$= -x[n+1] + x[n] - x[n-1] + x[n-2] - \cdots$$

So $y[n-1] = -x[n] + x[n-1] - x[n-2] + x[n-3] - \cdots$

$y[n-2] = -x[n-1] + x[n-2] - x[n-3] + x[n-4] - \cdots$

Then $y[n-1] + y[n-2] = -x[n]$,

or $x[n] = -y[n-1] - y[n-2]$.

The input/output relationship for the inverse system G is therefore

$$y[n] = -x[n-1] - x[n-2]$$

$$= x[n] * (-\delta[n-1] - \delta[n-2])$$

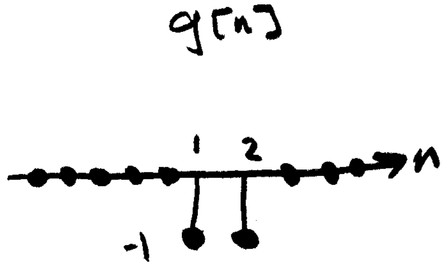
So $\underline{g[n] = -\delta[n-1] - \delta[n-2]}$.

Check: $g[n] * h[n] = -h[n-1] - h[n-2] = -(-1)^{n-1}u[n] - (-1)^{n-2}u[n-1]$
$= (-1)^{n-2}\{u[n] - u[n-1]\} = (-1)^{n-2}\delta[n] = \delta[n]$ ✓

Problem 3, cont...

(d) **5 pts**. Is the inverse system G causal? Justify your answer.

The system G <u>is</u> causal because $g[n] = 0 \ \forall n < 0$.



(e) **5 pts**. Is the inverse system G BIBO stable? Justify your answer.

The system G <u>is</u> BIBO stable since

$$\sum_{n=-\infty}^{\infty} |g[n]| = \sum_{n=-\infty}^{\infty} \left| -\delta[n-1] - \delta[n-2] \right|$$

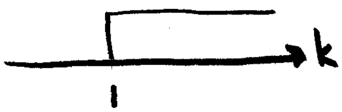
$$= \underbrace{|-1|}_{\substack{n=1 \\ \text{term}}} + \underbrace{|-1|}_{\substack{n=2 \\ \text{term}}} = 2 < \infty.$$

4. **25 pts.** Consider a discrete-time LTI system H with impulse response

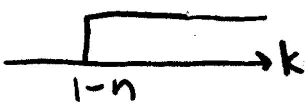
$$h[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1].$$

The system input is given by $x[n] = u[n-1]$. Find the system output $y[n]$.

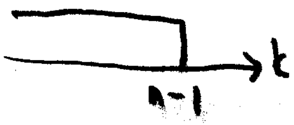
$x[k]$



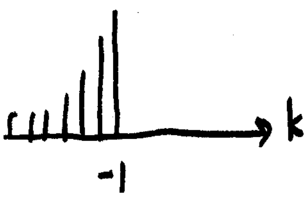
$x[n+k]$



$x[n-k]$



$h[k]$



I $\quad n-1 < -1 \Rightarrow n < 0$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{n-1} \left(\tfrac{1}{3}\right)^{-k}$$

$$= \lim_{A\to\infty} \sum_{k=-A}^{n-1} 3^k = \lim_{A\to\infty} \frac{3^{-A} - 3^n}{1-3} = \frac{-3^n}{-2} = \frac{3^n}{2}$$

II $\quad n-1 \geq -1 \Rightarrow n \geq 0$;

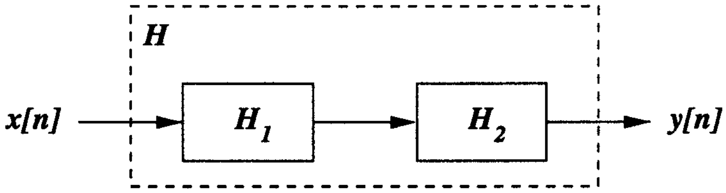
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{-1} \left(\tfrac{1}{3}\right)^{-k}$$

$$= \lim_{A\to\infty} \sum_{k=-A}^{-1} 3^k = \lim_{A\to-\infty} \frac{3^{-A} - 3^0}{1-3} = \frac{-1}{-2} = \frac{1}{2}$$

All Together: $\quad y[n] = \begin{cases} \dfrac{3^n}{2}, & n < 0 \\[2mm] \dfrac{1}{2}, & n \geq 0 \end{cases}$

(problem 2.6, given on HW 3)

5. **25 pts**. Consider a discrete-time system H formed by connecting two LTI systems H_1 and H_2 in series as shown in the figure below.



The impulse response of LTI system H_1 is given by $h_1[n] = \cos\left(\frac{\pi}{4}n\right)$. The impulse response of LTI system H_2 is given by $h_2[n] = \delta[n] - \delta[n+8]$. Find the output of the overall system H when the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

$$h[n] = h_1[n] * h_2[n]$$

$$= \cos\left(\tfrac{\pi}{4}n\right) * \left\{\delta[n] - \delta[n+8]\right\}$$

$$= \cos\left(\tfrac{\pi}{4}n\right) - \cos\left(\tfrac{\pi}{4}(n+8)\right)$$

$$= \cos\left(\tfrac{\pi}{4}n\right) - \cos\left(\tfrac{\pi}{4}n + 2\pi\right)$$

$$= \cos\left(\tfrac{\pi}{4}n\right) - \cos\left(\tfrac{\pi}{4}n\right) = \underline{\underline{0}}$$

So $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} 0 \cdot x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 0 = \underline{\underline{0}}.$$

$$\boxed{y[n] = 0}$$