ECE 3793
Test 1

Wednesday, October 23, 2002
7:00 PM - 10:00 PM

Fall 2002

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

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TOTAL (100):

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1. **25 pts.** A continuous-time system H has input $x(t)$ and output $y(t)$ related by

$$y(t) = t^2 x(t-1).$$

(a) **5 pts.** Is the system H memoryless? Justify your answer.

When $t=1$, $y(1) = 1^2 x(0)$, so the output $y(1)$ depends on the past input $x(0)$. Therefore the system is not memoryless

(b) **5 pts.** Is the system H causal? Justify your answer.

$y(t)$ depends on the past input $x(t-1)$, but not on future values of the input. Therefore the system is causal.

(c) **5 pts.** Is the system H BIBO stable? Justify your answer.

Let the input be $x(t) = 1$.
Then $|x(t)| < 2 \ \forall t \in \mathbb{R}$. Therefore, the input is bounded by $B=2$.

In this case, the output is given by
$y(t) = t^2$. Since $\nexists B \in \mathbb{R}$ s.t. $|y(t)| < B \ \forall t \in \mathbb{R}$, this output is unbounded.
Therefore a bounded input produced an unbounded output and the system is not BIBO stable.

Problem 1, cont...

(d) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = t^2 x_1(t-1)$.

Let $y_2(t) = H\{x_2(t)\} = t^2 x_2(t-1)$.

Let $a, b \in \mathbb{C}$ be constants.

Let $x_3(t) = a x_1(t) + b x_2(t)$.

Then $y_3(t) = H\{x_3(t)\} = t^2 x_3(t-1)$

$$= t^2 \{a x_1(t-1) + b x_2(t-1)\}$$

$$= a t^2 x_1(t-1) + b t^2 x_2(t-1) = \underline{a y_1(t) + b y_2(t)}.$$

$\Rightarrow$ The system <u>is</u> <u>linear</u>.

(e) **5 pts.** Is the system H time invariant? Justify your answer.

Let $x_1(t) = u(t)$ and let $y_1(t) = H\{x_1(t)\} = t^2 u(t-1)$.

Then $y_1(t-2) = (t-2)^2 u(t-3)$

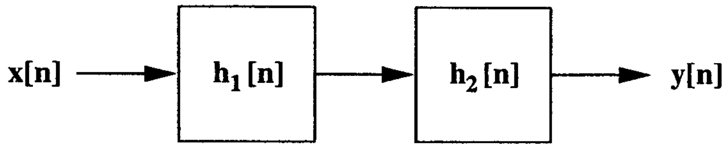
Now let $x_2(t) = x_1(t-2) = u(t-2)$.

Then $y_2(t) = H\{x_2(t)\} = t^2 x_2(t-1) = t^2 u(t-3)$.

Since $y_2(t) \neq y_1(t-2)$, the system H is

<u>NOT</u> <u>TIME</u> <u>INVARIANT</u>.

2. **25 pts.** A discrete-time LTI system H is formed by cascading two discrete-time LTI systems H_1 and H_2 as shown in the figure below:



The impulse response of the overall system H is given by

$$h[n] = \left(\frac{1}{4}\right)^{n+2} u[n+2].$$

The impulse response $h_1[n]$ of system H_1 is given by

$$h_1[n] = \left(\frac{1}{4}\right)^{n} u[n].$$

(a) **10 pts.** Find the impulse response $h_2[n]$ of system H_2.

$$h[n] = h_1[n] * h_2[n]$$

$$\left(\frac{1}{4}\right)^{n+2} u[n+2] = \left(\frac{1}{4}\right)^{n} u[n] * h_2[n] = h_1[n+2]$$

Now, for any $n_0 \in \mathbb{Z}$ and any $x[n] : \mathbb{Z} \to \mathbb{C}$,

$$x[n] * \delta[n-n_0] = x[n-n_0].$$

In particular, with $n_0 = -2$, we obtain

$$x[n] * \delta[n+2] = x[n+2]$$

So H_2 merely advances the signal by two time steps and

$$\underline{\underline{h_2[n] = \delta[n+2]}}$$

(b) **5 pts.** Is the system H memoryless? Justify your answer.

A discrete-time LTI system is memoryless iff the impulse response is a constant times the Kronecker delta: $h[n] = K\delta[n]$. That's <u>not</u> the case here where $h[n] = (\frac{1}{4})^{n+2} u[n+2]$. So H is <u>not</u> <u>memoryless</u>.

(c) **5 pts.** Is the system H causal? Justify your answer.

A discrete-time LTI system is causal iff $h[n]=0$ $\forall\, n<0$.

In this case, $h[-1] = (\frac{1}{4})^1 u[1] = \frac{1}{4} \neq 0$.

Therefore the system is <u>not</u> <u>causal</u>.

(d) **5 pts.** Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-2}^{\infty} \left(\frac{1}{4}\right)^{n+2}$$

$m = n+2$, $n = m-2$; $n=-2 \rightarrow m=0$, $n=\infty \rightarrow m=\infty$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m = \frac{1}{1-1/4} = \frac{1}{3/4} = \frac{4}{3} < \infty.$$

$\Rightarrow$ The system <u>is</u> <u>BIBO</u> <u>STABLE</u>.

3. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = u[n] - u[n-11].$$

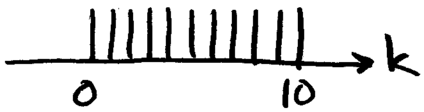
The system input is given by

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

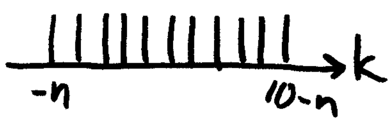
$h[k]$



I) $n<0$: $y[n]=0$

$h[n+k]$

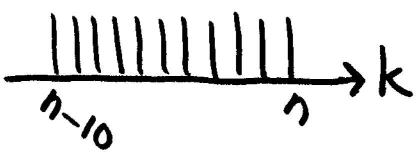
II) $n \geq 0$ and $n-10<0$: $0 \leq n < 10$:



$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k = \frac{\left(\frac{1}{4}\right)^0 - \left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}}$$

$h[n-k]$

$$= \frac{1-\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^n}{3\left(\frac{1}{4}\right)} = \frac{4-\left(\frac{1}{4}\right)^n}{3} = \frac{1}{3}\left[4-\left(\tfrac{1}{4}\right)^n\right]$$



III) $n \geq 10$:

$$y[n] = \sum_{k=n-10}^{n} \left(\tfrac{1}{4}\right)^k = \frac{\left(\frac{1}{4}\right)^{n-10} - \left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}}$$

$x[k]$

$$= \left(\tfrac{4}{3}\right)\left(\tfrac{1}{4}\right)^n \left[\left(\tfrac{1}{4}\right)^{-10} - \tfrac{1}{4}\right] = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{4}\right)^n (4)\left[4^{10} - \tfrac{1}{4}\right]$$

$$= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{4}\right)^n \left[4^{11} - 1\right]$$

All together:

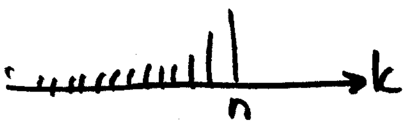
$$y[n] = \begin{cases} 0, & n<0 \\ \frac{1}{3}\left[4-\left(\frac{1}{4}\right)^n\right], & 0 \leq n < 10 \\ \frac{1}{3}\left(\frac{1}{4}\right)^n \left[4^{11}-1\right], & n \geq 10 \end{cases}$$

③ OTHER WAY...

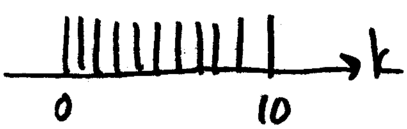
$x[k] = \left(\frac{1}{4}\right)^k u[k]$

$x[n+k] = \left(\frac{1}{4}\right)^{n+k} u[n+k]$

$x[n-k] = \left(\frac{1}{4}\right)^{n-k} u[n-k]$



$h[k]$



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

I) $n<0$: $y[n]=0$

II) $0 \leq n < 10$:

$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 4^k$$

$$= \left(\frac{1}{4}\right)^n \frac{4^0 - 4^{n+1}}{1-4} = \left(\frac{1}{4}\right)^n \frac{4^{n+1}-1}{3}$$

$$= \left(\frac{1}{3}\right)\left[4^{-n}4^{n+1} - \left(\frac{1}{4}\right)^n\right] = \frac{1}{3}\left[4 - \left(\frac{1}{4}\right)^n\right]$$

III) $n \geq 10$:

$$y[n] = \sum_{k=0}^{10} \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{10} 4^k$$

$$= \left(\frac{1}{4}\right)^n \frac{4^0 - 4^{11}}{1-4} = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right)\left[4^{11} - 1\right]$$

All Together;

$$y[n] = \begin{cases} 0, & n<0 \\ \frac{1}{3}\left[4 - \left(\frac{1}{4}\right)^n\right], & 0 \leq n < 10 \\ \frac{1}{3}\left(\frac{1}{4}\right)^n\left[4^{11}-1\right], & n \geq 10 \end{cases}$$

4. **25 pts**. A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-3t}u(t).$$

The system input is given by

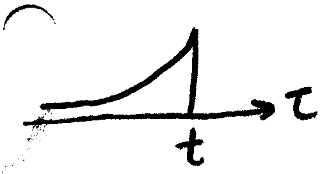
$$x(t) = e^{-4t}\cos(\omega_0 t)u(t),$$

where $\omega_0 \in \mathbb{R}$ is a constant. Find the system output $y(t)$.

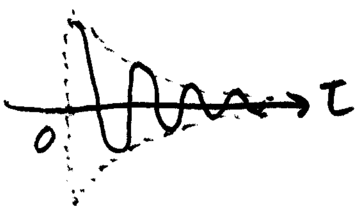
$h(\tau)$

$h(t+\tau)$

$h(t-\tau) = e^{-3(t-\tau)}u(t-\tau)$



$x(\tau) = e^{-4\tau}\cos\omega_0\tau\, u(\tau)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

I) $t<0$; $y(t)=0$.

II) $t \geq 0$; $y(t) = \int_0^t e^{-3(t-\tau)}e^{-4\tau}\cos\omega_0\tau\,d\tau$

$$= \int_0^t e^{-3t}e^{3\tau}e^{-4\tau}\cos\omega_0\tau\,d\tau$$

$$= e^{-3t}\int_0^t e^{-\tau}\cos\omega_0\tau\,d\tau$$

$$= e^{-3t}\left[\frac{e^{-\tau}(-\cos\omega_0\tau + \omega_0\sin\omega_0\tau)}{(-1)^2+\omega_0^2}\right]_{\tau=0}^{t}$$

$$= \frac{e^{-3t}}{1+\omega_0^2}\left[e^{-t}(-\cos\omega_0 t + \omega_0\sin\omega_0 t) - 1(-1+0)\right]$$

$$= \frac{e^{-4t}}{1+\omega_0^2}\omega_0\sin\omega_0 t - \frac{e^{-4t}}{1+\omega_0^2}\cos\omega_0 t + \frac{e^{-3t}}{1+\omega_0^2}.$$

All together:

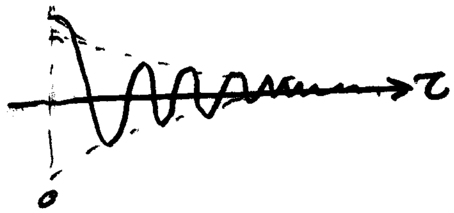
$$y(t) = \frac{\omega_0 e^{-4t}\sin\omega_0 t}{1+\omega_0^2}u(t) - \frac{e^{-4t}\cos\omega_0 t}{1+\omega_0^2}u(t) + \frac{e^{-3t}}{1+\omega_0^2}u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

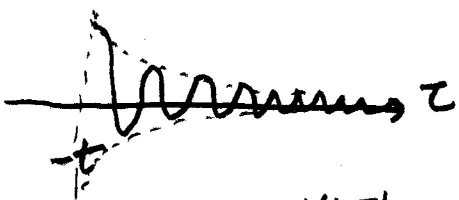
More Workspace for Problem 4...

OTHER WAY:

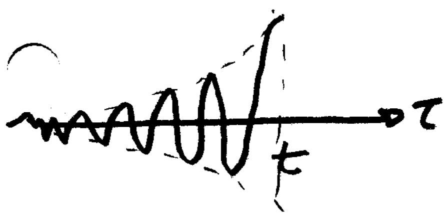
$x(\tau)$



$x(t+\tau)$



$$x(t-\tau) = e^{-4(t-\tau)} \cos\omega_0(t-\tau)\, u(t-\tau)$$



$$h(\tau) = e^{-3\tau} u(\tau)$$

I) $t<0$: $y(t) = 0$

II) $t \geq 0$:

$$y(t) = \int_0^t e^{-3\tau} e^{-4(t-\tau)} \cos[\omega_0(t-\tau)]\, d\tau$$

$\theta = t-\tau \qquad \tau = t-\theta$

$d\theta = -d\tau \qquad d\tau = -d\theta$

$$= \int_t^0 e^{-3(t-\theta)} e^{-4\theta} \cos\omega_0\theta\, (-d\theta)$$

$$= \int_0^t e^{-3(t-\theta)} e^{-4\theta} \cos\omega_0\theta\, d\theta$$

From here the solution is identical to what is shown on the previous page.