ECE 3793
Test 1

Wednesday, October 29, 2003
7:00 PM - 10:00 PM

Fall 2003
Dr. Havlicek

Name: SOLUTION
Student Num: ____________________________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are four problems. Work all four. Formulas appear at the end of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:
1. (25) _______
2. (25) _______
3. (25) _______
4. (25) _______

__________________________________________

TOTAL (100):

__________________________________________
1. **25 pts.** The output of a good audio amplifier is equal to a constant times the input (this constant is called the "gain" of the amp). For all practical amps, however, there is a limit to the output voltage that can be generated. When the absolute value of the input becomes too large, the amp cannot generate a sufficient output voltage and the "peaks" get "clipped" off of the output waveform. This is called clipping. Although it is very bad for the speakers, it is sometimes done intentionally to subwoofers in a biamped system to add a "punchy" sound to the kick drum.

A high-quality practical audio amplifier with a gain of two can be modeled as a continuous-time system \( H \) with input \( x(t) \) and output \( y(t) \) related by

\[
y(t) = \begin{cases} 
2x(t), & |x(t)| \leq 60, \\
-120, & x(t) < -60, \\
120, & x(t) > 60.
\end{cases}
\]

(a) **2 pts.** Is the system \( H \) memoryless? Justify your answer.

For \( |x(t)| \leq 60 \), \( y(t) \) depends on \( x(t) \) but **not on** \( x(t-t_0) \) for values \( t_0 \neq 0 \) ✔

For \( |x(t)| > 60 \), \( y(t) \) **doesn't depend on** \( x(t) \) at all, so it certainy **doesn't depend on** \( x(t-t_0) \) for values \( t_0 \neq 0 \). ☠

\[\Rightarrow\] So \( y(t) \) **sometimes depends on the current input, but never on past or future values of the input.**

(b) **3 pts.** Is the system \( H \) causal? Justify your answer. **\( H \) IS MEMORYLESS**

As shown in part (a), the system output will **never depend on a future value of the input.**

Therefore, **\( H \) IS CAUSAL**

2
Problem 1, cont...

(c) 5 pts. Is the system $H$ linear? Justify your answer.

Let $x_1(t) = 200$. Then $y_1(t) = H\{x_1(t)^3\} = 120$.
Let $x_2(t) = 300$. Then $y_2(t) = H\{x_2(t)^3\} = 120$.

$\rightarrow$ So $y_1(t) + y_2(t) = 240$.
Let $x_3(t) = x_1(t) + x_2(t) = 500$.
Then $y_3(t) = H\{x_3(t)^3\} = 120$.

Since $y_3(t) \neq y_1(t) + y_2(t)$, this system is **not linear**.

(d) 5 pts. Is the system $H$ time invariant? Justify your answer.

Let the input be $x_1(t)$. Then the output is
\[ y_1(t) = H\{x_1(t)^3\} = \begin{cases} -120, & x_1(t) < -60 \\ 2x_1(t), & |x_1(t)| \leq 60 \\ 120, & x_1(t) > 60 \end{cases} \]

Let $x_2(t) = x_1(t-t_0)$.
Then $y_2(t) = H\{x_2(t)^3\} = \begin{cases} -120, & x_2(t) < -60 \\ 2x_2(t), & |x_2(t)| \leq 60 \\ 120, & x_2(t) > 60 \end{cases} $
\[ = \begin{cases} -120, & x_1(t-t_0) < -60 \\ 2x_1(t-t_0), & |x_1(t-t_0)| \leq 60 \\ 120, & x_1(t-t_0) > 60 \end{cases} = y_1(t-t_0) \checkmark \]

$H$ is **shift invariant**.

Intuition: the output is equal to the input or is clipped. When you shift the input, the clipped regions in the output shift by the same amount.
Problem 1, cont...

(e) 5 pts. Is the system $H$ BIBO stable? Justify your answer.

Yes.

Proof: $|y(t)| \leq 120$. Therefore, the output is bounded by any $B \in \mathbb{R}$ such that $B > 120$.

Therefore, every input signal makes a bounded output.

Then, certainly, every bounded input must make a bounded output. QED.

(f) 5 pts. Is the system $H$ invertible? Justify your answer.

No.

The input signals $x_1(t)$ and $x_2(t)$ in part (c) are different, but they both make the same output

$$y(t) = H\{x_1(t)\} = H\{x_2(t)\} = 120.$$
2. **25 pts.** A discrete-time system $H$ is formed by connecting two discrete-time systems $H_1$ and $H_2$ as shown in the figure below:

![Diagram of discrete-time system](image)

System $H_1$ is LTI and has impulse response $h_1[n] = (-1)^n u[n]$. System $H_2$ is linear and has input $x_2[n]$ and output $y_2[n]$ related by $y_2[n] = x_2[n + 1]$.

(a) **5 pts.** Is the system $H_1$ BIBO stable? Justify your answer.

$$
\sum_{n=-\infty}^{\infty} |h_1[n]| = \sum_{n=-\infty}^{\infty} |(-1)^n| |u[n]| = \sum_{n=0}^{\infty} 1
$$

$$
= \sum_{n=0}^{\infty} 1 = \sum_{n=0}^{\infty} 1
$$

$$
= \lim_{N \to \infty} \sum_{n=0}^{N-1} 1 = \lim_{N \to \infty} N \to \infty
$$

The system $H_1$ is **not** BIBO stable because the impulse response $h_1[n]$ is **not** absolutely summable.
(b) 5 pts. Is the overall system $H$ an LTI system? Justify your answer.

**Hint:** The overall system $H$ is equivalent to the system $\tilde{H}$ shown below:

Since $H$ and $\tilde{H}$ have the same input-output relation, you can analyze the block diagram for $\tilde{H}$ instead of the one for $H$ if you like.

$H_1$ is given as LTI. $H_2$ is given as linear.

Consider $H_2 : x[n] \rightarrow H_2 \rightarrow y[n]$.

The input $x[n]$ is not the same as in the picture above.

The output $y[n]$ is not the same as in the picture above.

When the input is $x_1[n]$, the output is $y_1[n] = H_2[x_1[n]] = x_1[n+1]$.

Let $x_2[n] = x_1[n-n_0]$.

Then $y_2[n] = H_2[x_2[n]] = x_2[n+1] = x_1[n-n_0+1] = x_1[n+1-n_0] = y_1[n-n_0]$.

Therefore $H_2$ is time invariant.

Therefore $H_1$ and $H_2$ are both LTI. As proved in class, the serial connection $\rightarrow H_1 \rightarrow H_2 \rightarrow$ is also LTI. So $\tilde{H}$ is the parallel connection of two LTI systems. Thus $\tilde{H}_2$ is LTI, and so is $H_1$. 
Problem 2, cont...

(c) 5 pts. If you answered yes in part (b), then find the impulse response $h[n]$ of the overall system. If instead you answered no in part (b), then give the input-output relation for $H$.

Consider the intermediate signals $a[n]$ and $b[n]$ shown in the block diagram on page 5. Let $d[n]$ be the input to $H$. Then the output $y[n]$ is the impulse response of $H$.

We have that $a[n] = H_1 \delta[n-2] = h_1[n] = (-1)^{n} u[n]$ and $b[n] = H_2 \{a[n-1]\} = h_2 \{(-1)^{n+1} u[n+1]\}$.

$$h[n] = a[n] + b[n] = (-1)^{n} u[n] + (-1)^{n+1} u[n+1] = (-1)^{n} u[n] + (-1)(-1)^{n+1}$$

(d) 5 pts. Is the system $H$ causal? Justify your answer.

$$h[n] = d[n+1]$$

$h[-1] = 1 \neq 0$.

The system is **NOT CAUSAL** because there is an $n<0$ such that $h[n] \neq 0$.

(e) 5 pts. Is the system $H$ BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |d[n+1]| = \sum_{n=-\infty}^{\infty} d[n+1] = 1$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$, the system is **BIBO stable**.
3. 25 pts. A continuous-time LTI system \( H \) has impulse response \( h(t) \) given by

\[
h(t) = u(t + 2) - u(t - 2) = \begin{cases} 
1, & |t| \leq 2, \\
0, & \text{otherwise}.
\end{cases}
\]

The system input is given by

\[
x(t) = e^{-3(t-1)}u(t-1).
\]

Find the system output \( y(t) \).

\[
y(t) = \sum_{\tau = -\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\]

\(\text{I}) \quad t + 2 < 1 \quad \Rightarrow \quad t < -1:\)

\[
y(t) = 0.
\]

\(\text{II}) \quad t + 2 \geq 1 \text{ and } t - 2 < 1 \quad \Rightarrow \quad t > -1 \text{ and } t < 3 : -1 \leq t < 3 :\)

\[
y(t) = \int_{-2}^{1} e^{-(t-\tau)} \tau d\tau = e^{3} \int_{1}^{t+2} e^{-3\tau} d\tau
\]

\[
e^{3} (-\frac{1}{3}) \left[ e^{-3\tau} \right]_{t=1}^{t+2} = -\frac{e^{3}}{3} \left[ e^{-3(t+2)} - e^{-3} \right]
\]

\[
= -\frac{1}{3} \left[ e^{3} e^{-3t} e^6 - 1 \right] = \frac{1}{3} \left[ 1 - e^{-3(t+1)} \right]
\]

\(\text{III}) \quad t - 2 \geq 1 \quad \Rightarrow \quad t > 3 :\)

\[
y(t) = \int_{t-2}^{t+2} e^{-(t-\tau)} \tau d\tau = e^{3} \int_{t-2}^{t+2} e^{-3\tau} d\tau
\]

\[
= -\frac{e^{3}}{3} \left[ e^{-3\tau} \right]_{\tau=t-2}^{t+2} = -\frac{e^{3}}{3} \left[ e^{-3(t+2)} - e^{-3(t-2)} \right]
\]

\[
= -\frac{e^{3}}{3} \left[ e^{3} e^{-6} - e^{-3t} e^{6} \right] = \frac{1}{3} \left[ e^{-3t} e^9 - e^{-3t} e^{-3} \right]
\]

\[
\text{ALL TOGETHER: } y(t) = \begin{cases} 
\frac{1}{3} e^{-3(t-1)}, & t < -1 \\
\frac{1}{3} e^{-3t} [e^9 - e^{-3}], & -1 \leq t < 3 \\
\frac{1}{3} e^{-3} [e^9 - e^{-3}], & t \geq 3
\end{cases}
\]
\[
\text{OTHER WAY:} \\
y(t) = h(t) * x(t) = \int_\infty^t h(t) x(t-t) \, dt
\]

I) \( t-1 < -2 : t < -1 \) : \\
\[ y(t) = 0 \]

II) \(-2 \leq t-1 < 2 : -1 \leq t < 3 \) : \\
\[ y(t) = \int_{-2}^{t-1} e^{-3(t-t-\tau)} \, d\tau = \int_{-2}^{t-1} e^{-3t} e^{3\tau} e^{-3} \, d\tau \]
\[ = e^{-3t} e^{3} \int_{-2}^{t-1} e^{3\tau} \, d\tau = e^{-3(t-1)} \left( \frac{1}{3} \right) [e^{3\tau}]_{\tau=-2}^{t-1} \]
\[ = \frac{1}{3} e^{-3(t-1)} \left[ e^{3(t-1)} - e^{-6} \right] \]
\[ = \frac{1}{3} e^{-3t} e^{3} \left[ e^{3t} e^{-3} - e^{-6} \right] = \frac{1}{3} \left[ 1 - e^{-3t} e^{-3} \right] \]
\[ = \frac{1}{3} \left[ 1 - e^{-3(t+1)} \right] \]

III) \( t-1 > 2 : t > 3 \) : \\
\[ y(t) = \int_{-2}^{2} e^{-3(t-t-\tau)} \, d\tau = e^{-3(t-1)} \int_{-2}^{2} e^{3\tau} \, d\tau \]
\[ = \frac{1}{3} e^{-3(t-1)} [e^{3\tau}]_{-2}^{2} = \frac{1}{3} e^{-3(t-1)} [e^{6} - e^{-6}] \]
\[ = \frac{1}{3} e^{-3t} e^{3} [e^{6} - e^{-6}] = \frac{1}{3} e^{-3t} [e^{9} - e^{-3}] \]

ALL TOGETHER:
\[ y(t) = \begin{cases} \\
0 & t < -1 \\
\frac{1}{3} [1 - e^{-3(t+1)}] & -1 \leq t < 3 \\
\frac{1}{3} e^{-3t} [e^{9} - e^{-3}] & t > 3 
\end{cases} \]
4. **25 pts.** A discrete-time LTI system $H$ has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n u[n].$$

The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n-1].$$

Find the system output $y[n]$.

\[ y[n] = \sum_{k=\infty}^{\infty} x[k] h[n-k] \]

I) $n < 1$:

\[ y[n] = 0. \]

II) $n \geq 1$:

\[ y[n] = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^k = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^k \]

\[ = \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \]

\[ = \left(\frac{1}{6}\right)^n \left[2 - 2^{n+1} \right] = \left(\frac{1}{6}\right)^n \left[2 - 2 \right] \]

\[ = 2 \left(\frac{1}{6}\right)^n \left[2^n - 1\right] = 2 \left[\left(\frac{1}{6}\right)^n 2^n - \left(\frac{1}{6}\right)^n\right] \]

\[ = 2 \left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{6}\right)^n\right] \]

**ALL TOGETHER:**

\[ y[n] = \begin{cases} 
0, & n < 1 \\
2\left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{6}\right)^n\right], & n \geq 1 
\end{cases} \]

\[ = 2\left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{6}\right)^n\right] u[n-1] \]
OTHER WAY!

More Workspace for Problem 4... \[ y[n] = h[n] \cdot x[n] = \sum_{k=0}^{\infty} h[k] \cdot x[n-k] \]

\begin{align*}
I) \quad n < 0 : \quad n < 1 : \quad & \quad 0 \quad \rightarrow \quad k \quad y[n] = 0 \\
II) \quad n \geq 0 : \quad n \geq 1 : \quad & \quad 0 \quad \rightarrow \quad k \\
\end{align*}

\[ y[n] = \sum_{k=0}^{n-1} \left( \frac{1}{6} \right)^k \left( \frac{1}{3} \right)^{n-k} = \left( \frac{1}{3} \right)^n \sum_{k=0}^{n-1} \left( \frac{1}{3} \right)^k = \left( \frac{1}{3} \right)^n \sum_{k=0}^{n-1} \left( \frac{1}{3} \right)^k = \left( \frac{1}{3} \right)^n \sum_{k=0}^{n-1} \left( \frac{1}{3} \right)^k = \left( \frac{1}{3} \right)^n \left[ \frac{1 - \left( \frac{1}{3} \right)^n}{1 - \frac{1}{3}} \right] = 2 \left( \frac{1}{3} \right)^n \left[ 1 - \left( \frac{1}{3} \right)^n \right]
\]

\[ = 2 \left[ \left( \frac{1}{3} \right)^n - \left( \frac{1}{3} \right)^n \left( \frac{1}{3} \right)^n \right] = 2 \left[ \left( \frac{1}{3} \right)^n - \left( \frac{1}{3} \right)^n \right]
\]

**ALL TOGETHER:**

\[ y[n] = \begin{cases} 0, & n < 1 \\ 2 \left[ \left( \frac{1}{3} \right)^n - \left( \frac{1}{3} \right)^n \right], & n \geq 1 \end{cases}
\]

\[ = 2 \left[ \left( \frac{1}{3} \right)^n - \left( \frac{1}{3} \right)^n \right] u[n-1]
\]