ECE 3793
Test 1

Wednesday, October 29, 2003
7:00 PM - 10:00 PM

Fall 2003

Dr. Havlicek

Name: SOLUTION

Student Num: __________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

1. **25 pts.** The output of a good audio amplifier is equal to a constant times the input (this constant is called the "gain" of the amp). For all practical amps, however, there is a limit to the output voltage that can be generated. When the absolute value of the input becomes too large, the amp cannot generate a sufficient output voltage and the "peaks" get "clipped" off of the output waveform. This is called *clipping*. Although it is very bad for the speakers, it is sometimes done intentionally to subwoofers in a biamped system to add a "punchy" sound to the kick drum.

   A high-quality practical audio amplifier with a gain of two can be modeled as a continuous-time system H with input $x(t)$ and output $y(t)$ related by

$$y(t) = \begin{cases} 2x(t), & |x(t)| \le 60, \\ -120, & x(t) < -60, \\ 120, & x(t) > 60. \end{cases}$$

   (a) **2 pts.** Is the system H memoryless? Justify your answer.

For $|x(t)| \leq 60$, $y(t)$ depends on $x(t)$ but <u>not</u> on $x(t-t_0)$ for values $t_0 \neq 0$ ✓

For $|x(t)| > 60$, $y(t)$ doesn't depend on $x(t)$ at all, so it certainly doesn't depend on $x(t-t_0)$ for values $t_0 \neq 0$.

$\Rightarrow$ So $y(t)$ sometimes depends on the current input, but <u>never</u> on past or future values of the input.

<u>H IS MEMORYLESS</u>

   (b) **3 pts.** Is the system H causal? Justify your answer.

As shown in part (a), the system output will never depend on a future value of the input.

Therefore, <u>H IS CAUSAL</u>

Problem 1, cont...

(c) **5 pts.** Is the system H linear? Justify your answer.

Let $x_1(t) = 200$. Then $y_1(t) = H\{x_1(t)\} = 120$.

Let $x_2(t) = 300$. Then $y_2(t) = H\{x_2(t)\} = 120$.

$\rightarrow$ So $y_1(t) + y_2(t) = 240$.

Let $x_3(t) = x_1(t) + x_2(t) = 500$.

Then $y_3(t) = H\{x_3(t)\} = 120$.

Since $y_3(t) \neq y_1(t) + y_2(t)$, this system is <u>NOT</u> <u>LINEAR</u>.

(d) **5 pts.** Is the system H time invariant? Justify your answer.

Let the input be $x_1(t)$. Then the output is

$$y_1(t) = H\{x_1(t)\} = \begin{cases} -120, & x_1(t) < -60 \\ 2x_1(t), & |x_1(t)| \leq 60 \\ 120, & x_1(t) > 60 \end{cases}$$

Let $x_2(t) = x_1(t - t_0)$.

$$\text{Then } y_2(t) = H\{x_2(t)\} = \begin{cases} -120, & x_2(t) < -60 \\ 2x_2(t), & |x_2(t)| \leq 60 \\ 120, & x_2(t) > 60 \end{cases}$$

$$= \begin{cases} -120, & x_1(t-t_0) < -60 \\ 2x_1(t-t_0), & |x_1(t-t_0)| \leq 60 \\ 120, & x_1(t-t_0) > 60 \end{cases} = y_1(t-t_0) \checkmark$$

H <u>is</u> <u>shift</u> <u>invariant</u>.

Intuition: the output is equal to the input or is clipped. When you shift the input, the clipped regions in the output shift by the same amount.

Problem 1, cont...

(e) **5 pts**. Is the system H BIBO stable? Justify your answer.

Yes.

Proof: $|y(t)| \leq 120$. Therefore, the output is bounded by any $B \in \mathbb{R}$ such that $B > 120$.

→ Therefore, every input signal makes a bounded output.

→ Then, certainly, every bounded input must make a bounded output. QED.

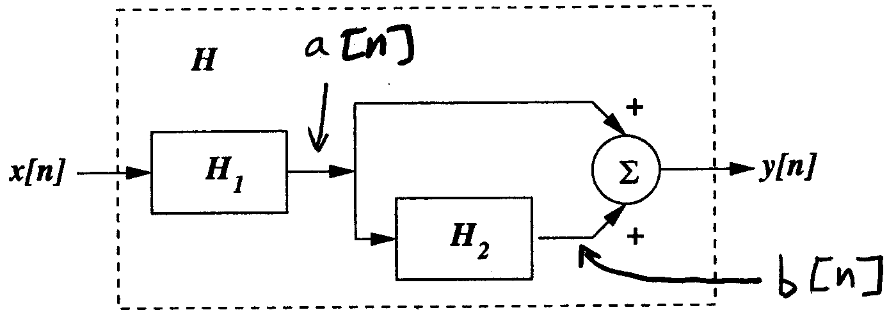
(f) **5 pts**. Is the system H invertible? Justify your answer.

NO.

The input signals $x_1(t)$ and $x_2(t)$ in part (c) are different, but they both make the same output

$$y(t) = H\{x_1(t)\} = H\{x_2(t)\} = 120.$$

2. **25 pts.** A discrete-time system H is formed by connecting two discrete-time systems H_1 and H_2 as shown in the figure below:



System H_1 is LTI and has impulse response $h_1[n] = (-1)^n u[n]$. System H_2 is linear and has input $x_2[n]$ and output $y_2[n]$ related by $y_2[n] = x_2[n+1]$.

(a) **5 pts.** Is the system H_1 BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h_1[n]| = \sum_{n=-\infty}^{\infty} |(-1)^n| \, |u[n]|$$

$$= \sum_{n=0}^{\infty} |-1| \cdot 1 = \sum_{n=0}^{\infty} 1$$

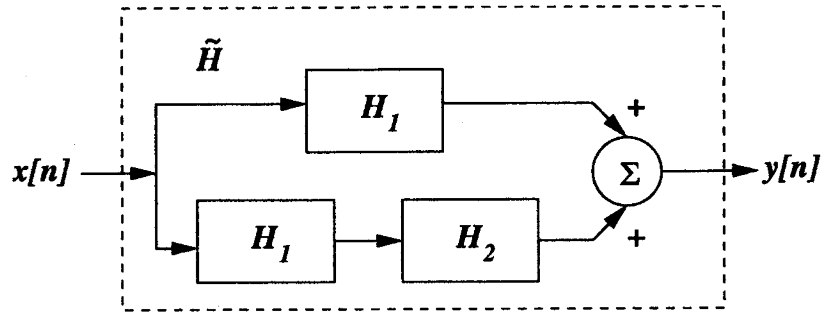
$$= \lim_{N\to\infty} \sum_{n=0}^{N-1} 1 = \lim_{N\to\infty} N \rightarrow \underline{\underline{\infty}}$$

The system H_1 is <u>not</u> BIBO stable because the impulse response $h_1[n]$ is <u>not</u> absolutely summable.

Problem 2, cont...

(b) **5 pts.** Is the overall system H an LTI system? Justify your answer.

Hint: The overall system H is equivalent to the system $\widetilde{H}$ shown below:



Since H and $\widetilde{H}$ have the same input-output relation, you can analyze the block diagram for $\widetilde{H}$ instead of the one for H if you like.

H_1 is given as LTI. H_2 is given as linear.

This part is about H_2 only

Consider H_2 : $x[n] \to \boxed{H_2} \to y[n]$

not the same $x[n]$ as in the picture above

not the same $y[n]$ as in the picture above.

When the input is $x_1[n]$, the output is $y_1[n] = H_2\{x_1[n]\}$
$= x_1[n+1]$,

Let $x_2[n] = x_1[n-n_0]$.

Then $y_2[n] = H\{x_2[n]\} = x_2[n+1] = x_1[n-n_0+1]$
$= x_1[n+1-n_0]$
$= y_1[n-n_0]$.

Therefore H_2 <u>is</u> <u>time</u> <u>invariant</u>

Therefore H_1 and H_2 are both LTI. As proved in class, the serial connection $\to \boxed{H_1} \to \boxed{H_2} \to$ is also LTI. So $\widetilde{H}$ is the parallel connection of two LTI systems. Thus $\widetilde{H}_2$ <u>is</u> LTI, and so is H.

Problem 2, cont...

(c) **5 pts.** If you answered *yes* in part (b), then find the impulse response $h[n]$ of the overall system. If instead you answered *no* in part (b), then give the input-output relation for H.

Consider the intermediate signals $a[n]$ and $b[n]$ shown in the block diagram on page 5.

Let $\delta[n]$ be the input to H. Then the output $y[n]$ is the impulse response of H.

We have that $a[n] = H_1\{\delta[n]\} = h_1[n] = (-1)^n u[n]$

$b[n] = H_2\{a[n]\} = H_2\{(-1)^n u[n]\} = (-1)^{n+1} u[n+1]$

$h[n] = a[n] + b[n] = (-1)^n u[n] + (-1)^{n+1} u[n+1] = (-1)^n u[n] + (-1)(-1)^n u[n+1]$

$= (-1)^n u[n] - (-1)^n \{\delta[n+1] + u[n]\}$

$= (-1)^n u[n] - (-1)^n u[n] - (-1)^n \delta[n+1]$

$= -(-1)^n \delta[n+1]$

$= \delta[n+1]$

$\boxed{h[n] = \delta[n+1]}$

(d) **5 pts.** Is the system H causal? Justify your answer.

$h[n] = \delta[n+1]$

$h[-1] = 1 \neq 0.$

The system is NOT CAUSAL because there is an $n<0$ such that $h[n] \neq 0$.

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n+1]| = \sum_{n=-\infty}^{\infty} \delta[n+1] = 1$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$, the system is BIBO stable.

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = u(t+2) - u(t-2) = \begin{cases} 1, & |t| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

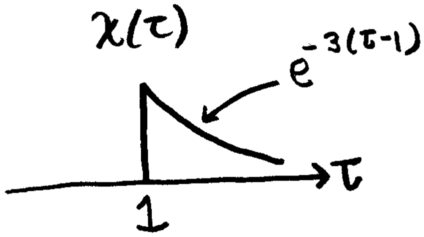
The system input is given by

$$x(t) = e^{-3(t-1)}u(t-1).$$

Find the system output $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

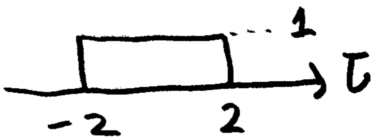
$x(\tau)$



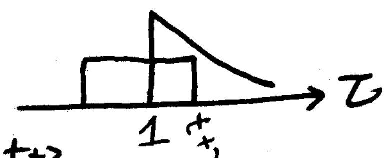
I) $t+2 < 1$: $t < -1$:

$y(t) = 0.$

$h(\tau)$



II) $t+2 \geq 1$ and $t-2 < 1$: $t \geq -1$ and $t < 3$: $-1 \leq t < 3$:

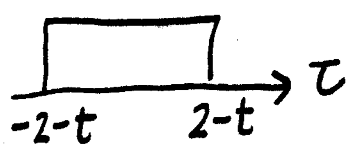


$$y(t) = \int_1^{t+2} e^{-3(\tau-1)}\, d\tau$$

$$= \int_1^{t+2} e^{-3\tau} e^{3}\, d\tau = e^3 \int_1^{t+2} e^{-3\tau}\, d\tau$$

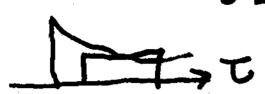
$h(\tau - -t) = h(t+\tau)$

$$= e^3\left(-\tfrac{1}{3}\right)\left[e^{-3\tau}\right]_{\tau=1}^{t+2} = -\frac{e^3}{3}\left[e^{-3(t+2)} - e^{-3}\right]$$



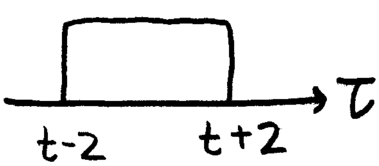
$$= -\tfrac{1}{3}\left[e^3 e^{-3t} e^{-6} - 1\right] = \tfrac{1}{3}\left[1 - e^{-3(t+1)}\right]$$

III) $t-2 \geq 1$: $t \geq 3$:



$h(t-\tau)$

$$y(t) = \int_{t-2}^{t+2} e^{-3(\tau-1)}\, d\tau = e^3 \int_{t-2}^{t+2} e^{-3\tau}\, d\tau$$



$$= -\frac{e^3}{3}\left[e^{-3\tau}\right]_{\tau=t-2}^{t+2} = -\frac{e^3}{3}\left[e^{-3(t+2)} - e^{-3(t-2)}\right]$$

$$= -\frac{e^3}{3}\left[e^{-3t}e^{-6} - e^{-3t}e^{6}\right] = \tfrac{1}{3}\left[e^{-3t}e^{9} - e^{-3t}e^{-3}\right]$$

$$= \tfrac{1}{3}e^{-3t}\left[e^{9} - e^{-3}\right]$$

ALL TOGETHER:

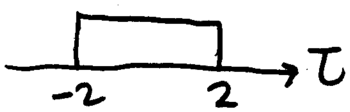
$$y(t) = \begin{cases} 0, & t < -1 \\ \tfrac{1}{3}\left[1 - e^{-3(t+1)}\right], & -1 \leq t < 3 \\ \tfrac{1}{3}e^{-3t}\left[e^{9} - e^{-3}\right] & t \geq 3 \end{cases}$$

More Workspace for Problem 3...

OTHER WAY:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

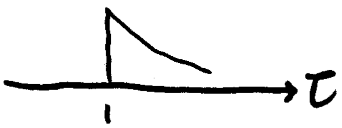
$h(\tau)$



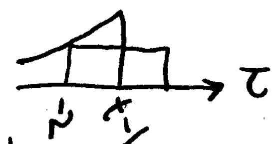
I) $t-1<-2$; $t<-1$:

$y(t)=0$

$x(\tau)$



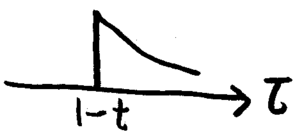
II) $-2 \le t-1 < 2$; $-1 \le t < 3$:



$$y(t) = \int_{-2}^{t-1} e^{-3(t-\tau-1)}\,d\tau = \int_{-2}^{t-1} e^{-3t}e^{3\tau}e^{3}\,d\tau$$

$$= e^{-3t}e^{3}\int_{-2}^{t-1} e^{3\tau}\,d\tau = e^{-3(t-1)}\left(\tfrac{1}{3}\right)\left[e^{3\tau}\right]_{\tau=-2}^{t-1}$$

$x(\tau+t)$

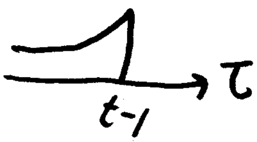


$$= \tfrac{1}{3}e^{-3(t-1)}\left[e^{3(t-1)} - e^{-6}\right]$$

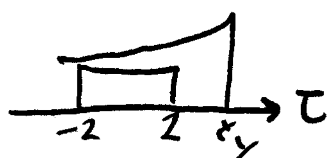
$$= \tfrac{1}{3}e^{-3t}e^{3}\left[e^{3t}e^{-3} - e^{-6}\right] = \tfrac{1}{3}\left[1 - e^{-3t}e^{-3}\right]$$

$x(t-\tau)$

$$= \tfrac{1}{3}\left[1 - e^{-3(t+1)}\right]$$



III) $t-1 \ge 2$: $t \ge 3$;



$$y(t) = \int_{-2}^{2} e^{-3(t-\tau-1)}\,d\tau = e^{-3(t-1)}\int_{-2}^{2} e^{3\tau}\,d\tau$$

$$= \tfrac{1}{3}e^{-3(t-1)}\left[e^{3\tau}\right]_{\tau=-2}^{2} = \tfrac{1}{3}e^{-3(t-1)}\left[e^{6}-e^{-6}\right]$$

$$= \tfrac{1}{3}e^{-3t}e^{3}\left[e^{6}-e^{-6}\right] = \tfrac{1}{3}e^{-3t}\left[e^{9}-e^{-3}\right]$$

ALL TOGETHER:

$$y(t) = \begin{cases} 0 & , \; t<-1 \\ \tfrac{1}{3}\left[1-e^{-3(t+1)}\right] & , \; -1 \le t < 3 \\ \tfrac{1}{3}e^{-3t}\left[e^{9}-e^{-3}\right] & , \; t \ge 3 \end{cases}$$

4. **25 pts**. A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n u[n].$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

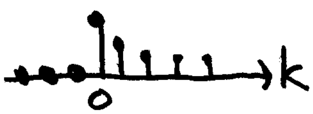
The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n-1].$$

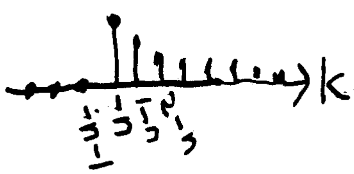
Find the system output $y[n]$.

$x[k]$

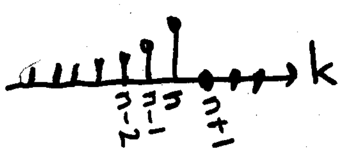
$h[k]$



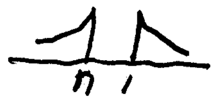
$h[k--n] = h[k+n]$



$h[n-k]$

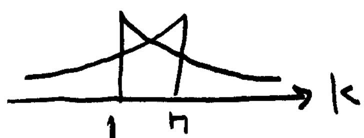


I) $n<1$:



; $y[n] = 0$.

II) $n \geq 1$:



$$y[n] = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k} = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^n \left(\frac{1}{6}\right)^{-k}$$

$$= \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k (6)^k = \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{6}{3}\right)^k = \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} 2^k$$

$$= \left(\frac{1}{6}\right)^n \left[\frac{2^1 - 2^{n+1}}{1-2}\right] = \left(\frac{1}{6}\right)^n \cdot 2\left[\frac{1-2^n}{-1}\right]$$

$$= 2\left(\frac{1}{6}\right)^n \left[2^n - 1\right] = 2\left[\left(\frac{1}{6}\right)^n 2^n - \left(\frac{1}{6}\right)^n\right]$$

$$= 2\left[\left(\frac{2}{6}\right)^n - \left(\frac{1}{6}\right)^n\right] = 2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right]$$

ALL TOGETHER :

$$y[n] = \begin{cases} 0, & n<1 \\ 2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right], & n \geq 1 \end{cases}$$

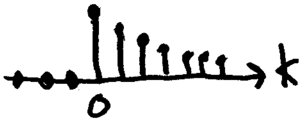
$$= \underline{2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right] u[n-1]}$$

OTHER WAY!

More Workspace for Problem 4...

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$h[k]$



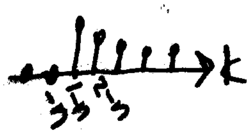
I) $n-1<0$: $n<1$:

$y[n]=0$

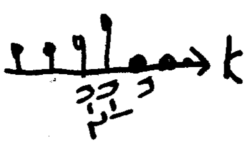
II) $n-1 \geq 0$: $n \geq 1$:

$x[k]$

$x[k--n]$



$x[n-k]$



$$y[n] = \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-k} = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{-k} = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{3}{6}\right)^k = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{3}\right)^n \left[\frac{\left(\tfrac{1}{2}\right)^0 - \left(\tfrac{1}{2}\right)^n}{1-\tfrac{1}{2}}\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{1-\left(\tfrac{1}{2}\right)^n}{1/2}\right] = 2\left(\tfrac{1}{3}\right)^n \left[1-\left(\tfrac{1}{2}\right)^n\right]$$

$$= 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{3}\right)^n\left(\tfrac{1}{2}\right)^n\right] = 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right]$$

ALL TOGETHER:

$$y[n] = \begin{cases} 0, & n<1 \\ 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right], & n \geq 1 \end{cases}$$

$$= 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right] u[n-1]$$