ECE 3793
Test 1

Wednesday, October 20, 2004
7:00 PM - 10:00 PM

Fall 2004 | Name: SOLUTION

Dr. Havlicek | Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________________ Date: ____________________

1. **25 pts.** A continuous-time system H has input $x(t)$ and output $y(t)$ related by

$$y(t) = \begin{cases} x(t+1), & x(t+1) \geq 0, \\ 1, & x(t+1) < 0. \end{cases}$$

(a) **2 pts.** Is the system H memoryless? Justify your answer.

Let $x(t) = t$. Then $x(2) = 2 \geq 0$.

So $y(1) = x(2)$.

Since $1 \neq 2$, this shows that the system output can depend on inputs from other times. NOT MEMORYLESS

(b) **3 pts.** Is the system H causal? Justify your answer.

As shown in part (a), when $x(t) = t$ the output $y(1)$ depends on the future input $x(2)$.

NOT CAUSAL

Problem 1, cont...

(c) **5 pts.** Is the system H linear? Justify your answer.

Let $x_1(t) = -1$. Then $y_1(t) = H\{x_1(t)\} = 1$

Let $x_2(t) = -2$. Then $y_2(t) = H\{x_2(t)\} = 1$

Let $x_3(t) = x_1(t) + x_2(t) = -3$.

Then $y_3(t) = H\{x_3(t)\} = 1 \neq y_1(t) + y_2(t)$.

NOT LINEAR

(d) **5 pts.** Is the system H time invariant? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = \begin{cases} x_1(t+1), & x(t+1) \geq 0 \\ 1, & x(t+1) < 0 \end{cases}$

Then $y_1(t-t_0) = \begin{cases} x_1(t-t_0+1), & x(t-t_0+1) \geq 0 \\ 1, & x(t-t_0+1) < 0 \end{cases}$

Now let $x_2(t) = x_1(t-t_0)$.

Then $y_2(t) = H\{x_2(t)\} = \begin{cases} x_2(t+1), & x_2(t+1) \geq 0 \\ 1, & x_2(t+1) < 0 \end{cases}$

$$= \begin{cases} x_1(t-t_0+1), & x_1(t-t_0+1) \geq 0 \\ 1, & x_1(t-t_0+1) < 0 \end{cases}$$

$= y_1(t-t_0)$ ✓

IT IS TIME INVARIANT

Problem 1, cont...

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

Suppose that $x(t)$ is a bounded input.
Then $\exists B \in \mathbb{R}$, $B>0$, s.t. $|x(t)| \leq B \;\; \forall t \in \mathbb{R}$.

Now, for each $t \in \mathbb{R}$, $y(t) = x(t+1)$ or $y(t) = 1$.

Then $|y(t)| = |x(t+1)| \leq B$ or $|y(t)| = 1$.

So, $\forall t \in \mathbb{R}$, $|y(t)| \leq \max(B, 1)$.

Then $y(t)$ is bounded by $\max(B, 1)$.

The system is BIBO stable, since all bounded inputs produce bounded outputs.

(f) **5 pts.** Is the system H invertible? Justify your answer.

As shown in part (c), the distinct input signals $x_1(t) = -1$ and $x_2(t) = -2$ both produce the same output signal $y(t) = 1$.

NOT INVERTIBLE

2. **25 pts.** A discrete-time LTI system H_1 has impulse response

$$h_1[n] = 2^n u[-n].$$

(a) **5 pts.** Is the system H_1 memoryless? Justify your answer.

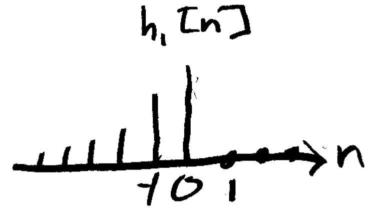
A discrete-time LTI system is memoryless iff $h_1[n]$ is a constant times $\delta[n]$. That's clearly not the case here, since $h_1[-1] = 2^{-1} = \frac{1}{2}$ while $\delta[-1] = 0$.

$\Rightarrow$ NOT MEMORYLESS

(b) **5 pts.** Is the system H_1 causal? Justify your answer.

A discrete-time LTI system is causal iff $h_1[n] = 0 \; \forall \; n < 0$. As shown in part (a), we have here that $h[-1] = \frac{1}{2} \neq 0$, so this system is NOT CAUSAL.

(c) **5 pts.** Is the system H_1 BIBO stable? Justify your answer.



$$\sum_{n=-\infty}^{\infty} |h_1[n]| = \sum_{n=-\infty}^{0} 2^n = \sum_{n=-\infty}^{0} \left(\tfrac{1}{2}\right)^{-n}$$

Let $m = -n$

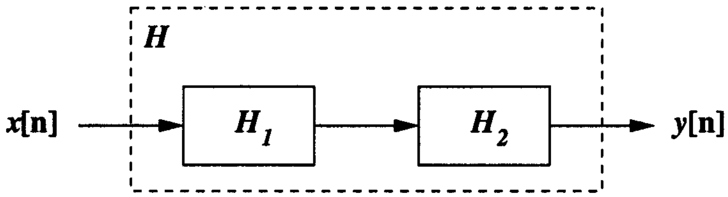
$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m = \frac{1}{1-\frac{1}{2}}$$

$$= 2 < \infty.$$

Since $\sum_{n=-\infty}^{\infty} |h_1[n]| < \infty$, the system IS BIBO STABLE.

Problem 2, cont...

(d) **10 pts**. A new system H is formed by connecting the system H_1 in series with a second discrete-time LTI system H_2 as shown in the figure below.

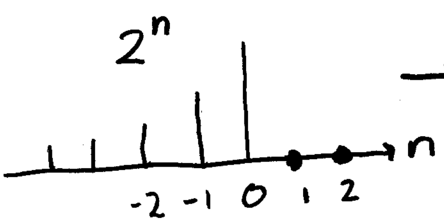


The impulse response of H_2 is given by $h_2[n] = \delta[n] - \frac{1}{2}\delta[n+1]$.

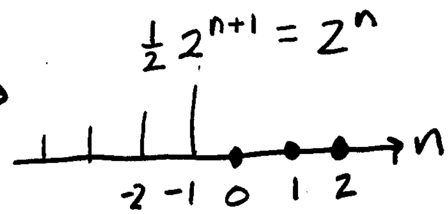
Find the impulse response $h[n]$ of the overall system H.

$$h[n] = h_1[n] * h_2[n]$$
$$= h_1[n] * \left(\delta[n] - \tfrac{1}{2}\delta[n+1]\right)$$
$$= h_1[n] * \delta[n] - \tfrac{1}{2} h_1[n] * \delta[n+1]$$
$$= h_1[n] - \tfrac{1}{2} h_1[n+1]$$

Graph of $h_1[n]$: Graph of $\frac{1}{2}h_1[n+1]$



$\rightarrow$ shift "right" by -1 $\rightarrow$



So $h[n] = 2^n u[-n] - \frac{1}{2} 2^{n+1} u[-(n+1)]$

$$= 2^n u[-n] - 2^n u[-n-1]$$

$$= \begin{cases} 2^n - 2^n, & n<0 \\ 2^0 - 0, & n=0 \\ 0 - 0, & n>0 \end{cases} = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases} = \delta[n]$$

$\boxed{h[n] = \delta[n]}$ (H_2 is the inverse of system H_1).

3. **25 pts**. A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{2t}u(-t).$$

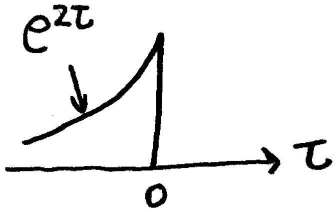
The system input is given by

$$x(t) = u(t-3).$$

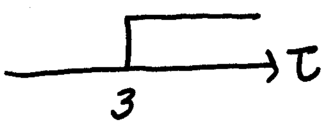
Find the system output $y(t)$.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

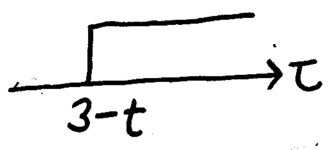
$h(\tau)$



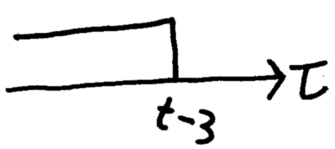
$x(\tau)$



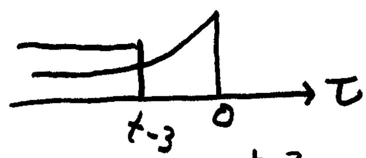
$x(\tau - -t) = x(t+\tau)$



$x(t-\tau)$



I) $t-3 < 0$: $t < 3$

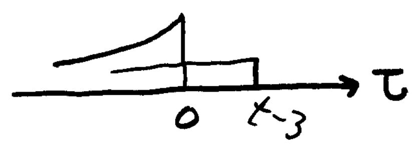


$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2}e^{2\tau}\Big|_{\tau=-\infty}^{t-3}$$

$$= \lim_{A\to\infty} \frac{1}{2}e^{2\tau}\Big|_{\tau=-A}^{t-3}$$

$$= \lim_{A\to\infty} \frac{1}{2}\left[e^{2(t-3)} - e^{-2A}\right] = \frac{1}{2}e^{2(t-3)}$$

II) $t-3 \geq 0$: $t \geq 3$



$$y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau$$

$$= \frac{1}{2}e^{2\tau}\Big|_{\tau=-\infty}^{0} = \lim_{A\to\infty} \frac{1}{2}e^{2\tau}\Big|_{\tau=-A}^{0}$$

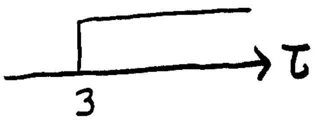
$$= \lim_{A\to\infty} \frac{1}{2}\left(e^{0} - e^{-2A}\right) = \frac{1}{2}$$

$$\boxed{\text{All Together: } y(t) = \begin{cases} \frac{1}{2}e^{2(t-3)}, & t<3 \\ 1/2, & t \geq 3 \end{cases}}$$

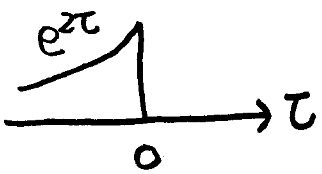
More Workspace for Problem 3...

OTHER WAY

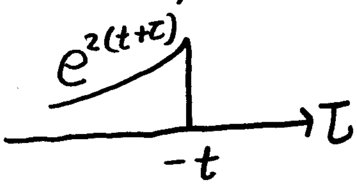
$x(\tau)$



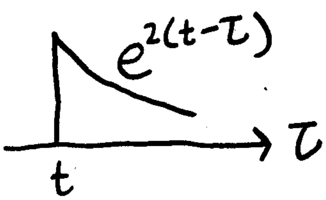
$h(\tau)$



$h(\tau - -t) = h(t+\tau)$

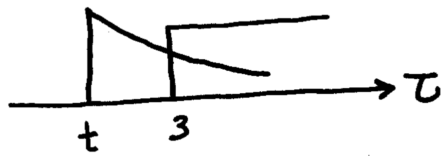


$h(t-\tau)$



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

I) $t < 3$



$$y(t) = \int_3^\infty e^{2(t-\tau)}\, d\tau = \int_3^\infty e^{2t} e^{-2\tau}\, d\tau$$

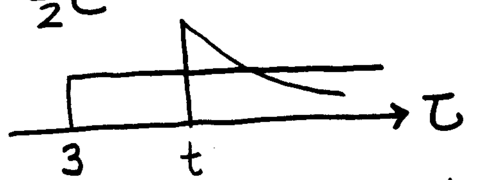
$$= \lim_{A\to\infty} e^{2t} \int_3^A e^{-2\tau}\, d\tau$$

$$= \lim_{A\to\infty} e^{2t} \left[-\tfrac{1}{2} e^{-2\tau}\right]_{\tau=3}^{A}$$

$$= -\tfrac{1}{2} e^{2t} \lim_{A\to\infty} \left[e^{-2A} - e^{-6}\right] = -\tfrac{1}{2} e^{2t}\left[0 - e^{-6}\right]$$

$$= \tfrac{1}{2} e^{2t} e^{-6} = \tfrac{1}{2} e^{2(t-3)}$$

II) $t \geq 3$



$$y(t) = \int_t^\infty e^{2(t-\tau)}\, d\tau = \lim_{A\to\infty} e^{2t} \int_t^A e^{-2\tau}\, d\tau$$

$$= -\tfrac{1}{2} e^{2t} \lim_{A\to\infty} \left[e^{-2\tau}\right]_{\tau=t}^{A} = -\tfrac{1}{2} e^{2t} \lim_{A\to\infty} \left[e^{-2A} - e^{-2t}\right]$$

$$= -\tfrac{1}{2} e^{2t}\left[0 - e^{-2t}\right] = \tfrac{1}{2} e^{2t} e^{-2t} = \tfrac{1}{2}$$

$$\boxed{\text{All Together: } \quad y(t) = \begin{cases} \tfrac{1}{2} e^{2(t-3)}, & t < 3 \\ \tfrac{1}{2}, & t \geq 3 \end{cases}}$$

4. **25 pts**. A discrete-time LTI system H has impulse response $h[n]$ given by

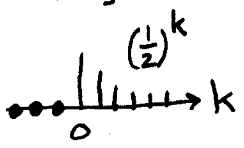
$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

The system input is given by

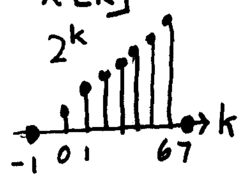
$$x[n] = 2^n\{u[n] - u[n-7]\} = \begin{cases} 2^n, & 0 \le n \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

Find the system output $y[n]$. $y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

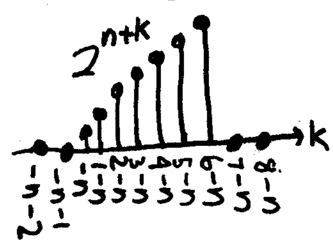
$h[k]$



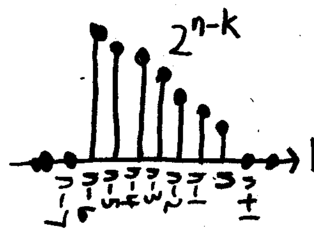
$x[k]$



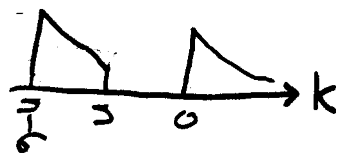
$x[k--n] = x[n+k]$



$x[n-k]$

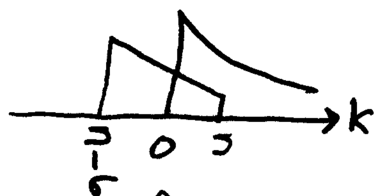


I) $n<0$:



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

II) $n \geq 0$ and $n-6<0$: $0 \le n < 6$:



$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k 2^{n-k}$$

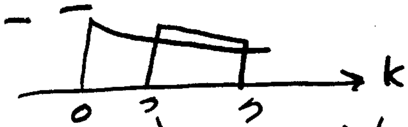
$$= \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k 2^n 2^{-k} = 2^n \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^k = 2^n \sum_{k=0}^{n} \left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right)^k$$

$$= 2^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k = 2^n \frac{\left(\frac{1}{4}\right)^0 - \left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}} = 2^n \frac{1-\frac{1}{4}\left(\frac{1}{4}\right)^n}{3/4}$$

$$= \frac{4}{3}\left[2^n - \tfrac{1}{4}2^n\left(\tfrac{1}{4}\right)^n\right] = \frac{4}{3}2^n - \frac{1}{3}\left(2\cdot\tfrac{1}{4}\right)^n = \frac{1}{3}\left[4(2)^n - \left(\tfrac{1}{2}\right)^n\right]$$

$$= \frac{1}{3}\left[2^{n+2} - \left(\tfrac{1}{2}\right)^n\right]$$

III) $n-6 \geq 0$: $n \geq 6$:



$$y[n] = \sum_{k=n-6}^{n} \left(\tfrac{1}{2}\right)^k 2^{n-k} = 2^n \sum_{k=n-6}^{n} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^k = 2^n \sum_{k=n-6}^{n} \left(\tfrac{1}{4}\right)^k = 2^n \frac{\left(\frac{1}{4}\right)^{n-6} - \left(\frac{1}{4}\right)^{n+1}}{1-1/4}$$

$$= \frac{4}{3}2^n\left[\left(\tfrac{1}{4}\right)^n\left(\tfrac{1}{4}\right)^{-6} - \left(\tfrac{1}{4}\right)^n\left(\tfrac{1}{4}\right)\right] = \frac{4}{3}2^n\left(\tfrac{1}{4}\right)^n\left[4^6 - 4^{-1}\right] = \frac{1}{3}\left(2\cdot\tfrac{1}{4}\right)^n\left[4^7-1\right]$$

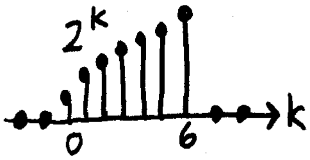
$$= \frac{1}{3}\left(\tfrac{1}{2}\right)^n\left[2^{14}-1\right] = \frac{16383}{3}\left(\tfrac{1}{2}\right)^n = 5461\left(\tfrac{1}{2}\right)^n,$$

ALL TOGETHER: $y[n] = \begin{cases} 0, & n<0 \\ \frac{1}{3}\left[2^{n+2} - \left(\frac{1}{2}\right)^n\right], & 0 \le n < 6 \\ 5461\left(\frac{1}{2}\right)^n, & n \geq 6 \end{cases}$

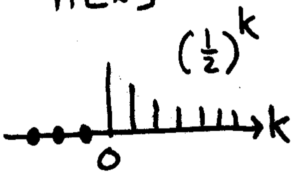
OTHER WAY $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

More Workspace for Problem 4...

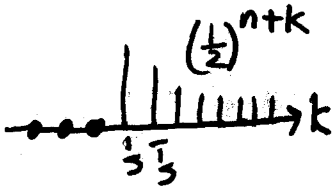
$x[k]$



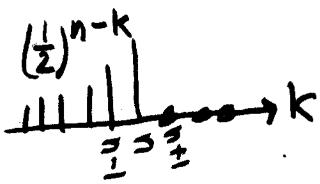
$h[k]$



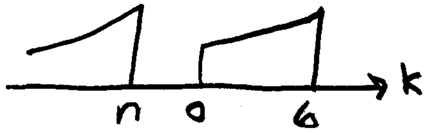
$h[k--n] = h[n+k]$



$h[n-k]$

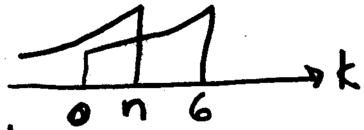


I) $n<0$:



$y[n] = 0$

II) $0 \le n < 6$:



$$y[n] = \sum_{k=0}^{n} 2^k \left(\tfrac{1}{2}\right)^{n-k}$$

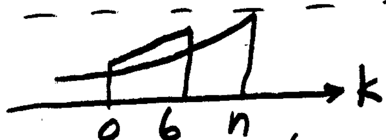
$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n} 2^k \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n} 2^k 2^k = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n} (2\cdot 2)^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{n} 4^k = \left(\tfrac{1}{2}\right)^n \frac{4^0 - 4^{n+1}}{1-4} = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n \left[1 - 4(4)^n\right]$$

$$= \tfrac{1}{3}\left(\tfrac{1}{2}\right)^n \left[4(2)^{2n} - 1\right] = \tfrac{1}{3}\left(\tfrac{1}{2}\right)^n \left[2^{2n+2} - 1\right]$$

$$= \tfrac{1}{3}\left[2^{-n}2^{2n+2} - \left(\tfrac{1}{2}\right)^n\right] = \tfrac{1}{3}\left[2^{n+2} - \left(\tfrac{1}{2}\right)^n\right]$$

III) $n \ge 6$:



$$y[n] = \sum_{k=0}^{6} 2^k \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{6} 2^k 2^k = \left(\tfrac{1}{2}\right)^n \sum_{k=0}^{6} 4^k$$

$$= \left(\tfrac{1}{2}\right)^n \frac{4^0 - 4^7}{1-4} = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n \left[1 - 4^7\right] = \tfrac{1}{3}\left(\tfrac{1}{2}\right)^n \left[4^7 - 1\right] = \frac{16383}{3}\left(\tfrac{1}{2}\right)^n$$

$$= 5461\left(\tfrac{1}{2}\right)^n$$

All Together:

$$y[n] = \begin{cases} 0, & n<0 \\ \tfrac{1}{3}\left[2^{n+2} - \left(\tfrac{1}{2}\right)^n\right], & 0 \le n < 6 \\ 5461\left(\tfrac{1}{2}\right)^n, & n \ge 6 \end{cases}$$