ECE 3793
Test 1

Wednesday, October 19, 2005
7:00 PM - 10:00 PM

Spring 2005

Dr. Havlicek

Name: SOLUTION

Student Num:__________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________ Date:__________

1. **25 pts.** A continuous-time system H has input $x(t)$ and output $y(t)$ related by

$$y(t) = |x(t)|.$$

(a) **5 pts.** Is the system H memoryless? Justify your answer.

For any time $t_0 \in \mathbb{R}$, the value $y(t_0)$ is determined from $x(t_0)$ alone and does not involve the input $x(t)$ for any times $t \neq t_0$. Therefore this system

IS MEMORYLESS

(b) **5 pts.** Is the system H linear? Justify your answer.

Let $x_1(t)$ be the constant signal $x_1(t) = 3$. Then $y_1(t) = |x_1(t)| = 3$ $(= x(t))$.

Let $x_2(t)$ also be a constant signal given by $x_2(t) = -2$. Then $y_2(t) = |x_2(t)| = 2$.

So the signal $y_1(t) + y_2(t)$ is a constant signal given by $y_1(t) + y_2(t) = 5$.

Now let $x_3(t) = x_1(t) + x_2(t) = 1$ (a constant signal).

Then $y_3(t) = |x_3(t)| = 1 \neq y_1(t) + y_2(t)$.

Therefore, this system IS NOT LINEAR

Problem 1, cont...

(c) **5 pts.** Is the system H time invariant? Justify your answer.

For an arbitrary input $x_1(t)$, the output is given by $y_1(t) = H\{x_1(t)\} = |x_1(t)|$. Shifting this output, we obtain $y_1(t-t_0) = |x_1(t-t_0)|$.

Now let $x_2(t) = x_1(t-t_0)$. Then

$$y_2(t) = |x_2(t)| = |x_1(t-t_0)| = y_1(t-t_0).$$

Therefore, this system IS TIME INVARIANT

(d) **5 pts.** Is the system H causal? Justify your answer.

It was shown in part (a) that $y(t)$ does not depend on inputs from times other than "t". Thus, in particular, $\forall t_0 \in \mathbb{R}$, $y(t_0)$ does not depend on future inputs $x(t)$ for which $t > t_0$.

Therefore, this system IS CAUSAL.

Problem 1, cont...

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

Suppose $x(t)$ is a bounded input. Then $\exists B \in \mathbb{R}$, $B > 0$, s.t. $|x(t)| \leq B \quad \forall t \in \mathbb{R}$.

Now,

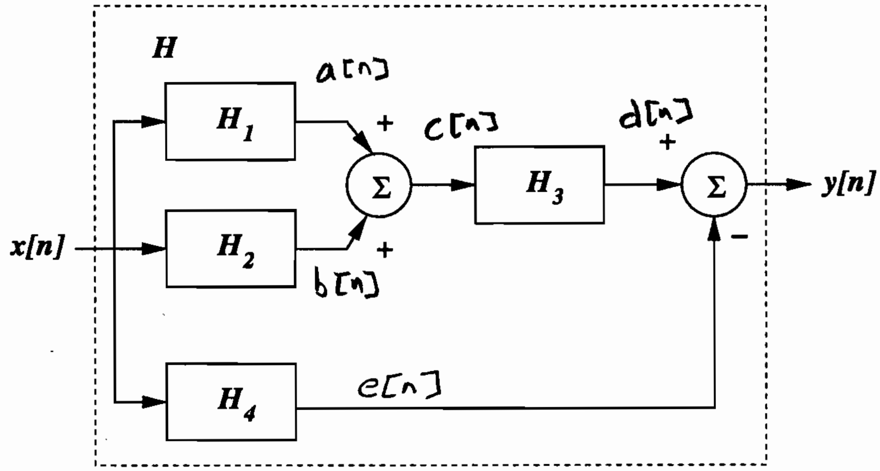
$$|y(t)| = \Big|\, |x(t)| \,\Big| = |x(t)| \leq B.$$

So the output $y(t)$ is also bounded by B.

Then every bounded input signal produces a bounded output signal.

This system IS BIBO STABLE

2. **25 pts.** The discrete-time system H is formed by connecting four LTI systems H_1, H_2, H_3, and H_4 as shown in the figure below.



It follows immediately from results proven in class that the overall system H is LTI. The impulse responses of the four LTI systems H_1 through H_4 are given by

$$\begin{aligned} h_1[n] &= u[n], \\ h_2[n] &= u[n+2] - u[n], \\ h_3[n] &= \delta[n-2], \\ h_4[n] &= \left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

(a) **10 pts.** Find the impulse response $h[n]$.

METHOD I; by inspection,

$$h[n] = \left(h_1[n] + h_2[n]\right) * h_3[n] - h_4[n]$$

$$= \left(u[n] + u[n+2] - u[n]\right) * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n+2] * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= \left[1 - \left(\tfrac{1}{2}\right)^n\right] u[n]$$

METHOD II: let the input be $x[n]=\delta[n]$. Then the output $y[n]$ is by definition the impulse response $h[n]$.

Problem 2, cont...

Note the labeled signals $a[n]$, $b[n]$, $c[n]$, $d[n]$, and $e[n]$ on page 5. We have $a[n] = \delta[n] * h_1[n] = \delta[n] * u[n] = u[n]$.
$b[n] = \delta[n] * h_2[n] = \delta[n] * (u[n+2] - u[n]) = u[n+2] - u[n]$.
$c[n] = a[n] + b[n] = u[n] + u[n+2] - u[n] = u[n+2]$.
$d[n] = c[n] * h_3[n] = u[n+2] * \delta[n-2] = u[n]$.
$e[n] = \delta[n] * h_4[n] = h_4[n] = (\frac{1}{2})^n u[n]$. $h[n] = y[n] = d[n] - e[n]$
$= u[n] - (\frac{1}{2})^n u[n]$
$= [1 - (\frac{1}{2})^n] u[n]$

(b) **5 pts**. Is the system H memoryless? Justify your answer.

A discrete time LTI system is memoryless iff the impulse response is a constant times $\delta[n]$.

That's not the case here (as shown in part (a)), so this system is NOT MEMORYLESS

(c) **5 pts**. Is the system H causal? Justify your answer.

We have $h[n] = [1 - (\frac{1}{2})^n] u[n]$,

so $h[n] = 0 \quad \forall n < 0$.

Therefore, this system IS CAUSAL

Problem 2, cont...

(d) **5 pts.** Is the system H BIBO stable? Justify your answer.

- I guess that it is not BIBO stable. Why? Because $h[n] = [1-(\frac{1}{2})^n]\,u[n]$ is asymptotically approaching 1 as $n \to \infty$. Graph of $h[n]$:  1

So $\sum_{n=-\infty}^{\infty} |h[n]|$ will look like $\sum_{n=\text{Big}}^{\infty} 1 \to \infty$ for large n.

- Based on this guess, I will try to show that $\sum_{n=-\infty}^{\infty} |h[n]|$ diverges. If I can do this, it will prove that the system is not BIBO stable.

- NOTE: $h[n] = [1-(\frac{1}{2})^n]u[n] = \left[\frac{2^n}{2^n} - \frac{1}{2^n}\right]u[n]$

$$= \left[\frac{2^n - 1}{2^n}\right]u[n].$$

$\geq 0 \quad \forall \; n \geq 0$

We have:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left|\frac{2^n-1}{2^n}u[n]\right| = \sum_{n=0}^{\infty}\left|\frac{2^n-1}{2^n}\right| = \sum_{n=0}^{\infty}\frac{2^n-1}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{2^n} - \frac{1}{2^n} = \sum_{n=0}^{\infty}\frac{2^n}{2^n} - \sum_{n=0}^{\infty}\frac{1}{2^n} = \lim_{A\to\infty}\left[\sum_{n=0}^{A} 1 - \sum_{n=0}(\tfrac{1}{2})^n\right]$$

$$= \lim_{A\to\infty}\left[(A+1) - \frac{1}{1-\frac{1}{2}}\right] = \lim_{A\to\infty}\left[A+1-2\right]$$

$$= \lim_{A\to\infty}\left[A-1\right] \longrightarrow \underline{\infty}.$$

<u>NOT BIBO STABLE</u>

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-2t}u(t).$$

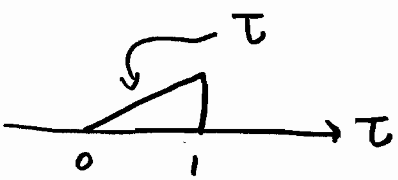
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

The system input is given by

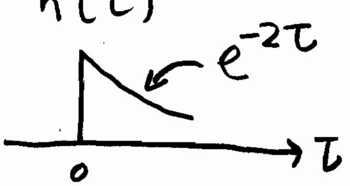
$$x(t) = \begin{cases} t, & 0 \le t \le 1, \\ 0, & \text{otherwise} \end{cases} = t\,[u(t) - u(t-1)].$$

Find the system output $y(t)$.

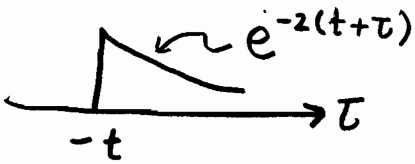
$x(\tau)$



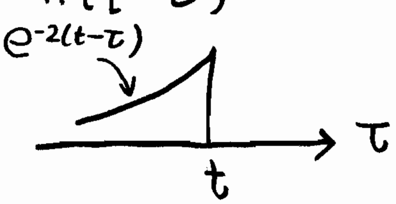
$h(\tau)$



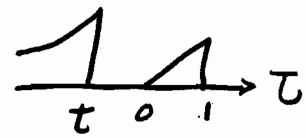
$h(\tau - -t) = h(t+\tau)$



$h(t-\tau)$



Case I) $t < 0$:



$$y(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$$

Case II) $t \geq 0$ and $t < 1$: $0 \leq t < 1$:

$$y(t) = \int_0^t e^{-2(t-\tau)}\,\tau\,d\tau = e^{-2t}\int_0^t \tau e^{2\tau}\,d\tau$$

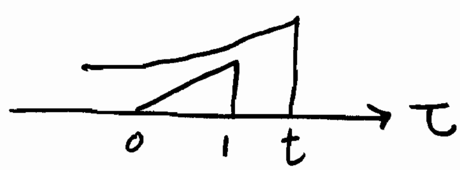
$$= e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \tfrac{1}{2}\right)\right]_{\tau=0}^{t}$$

$$= e^{-2t}\left[\frac{e^{2t}}{2}\left(t - \tfrac{1}{2}\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\right)\right]$$

$$= \tfrac{1}{2}\left(t - \tfrac{1}{2}\right) - \tfrac{1}{2}e^{-2t}\left(-\tfrac{1}{2}\right) = \tfrac{1}{2}t - \tfrac{1}{4} + \tfrac{1}{4}e^{-2t}$$

$$= \tfrac{1}{4}\left[e^{-2t} - 1\right] + \tfrac{1}{2}t.$$

Case III) $t \geq 1$:



$$y(t) = \int_0^1 e^{-2(t-\tau)}\,\tau\,d\tau = e^{-2t}\int_0^1 \tau e^{2\tau}\,d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \tfrac{1}{2}\right)\right]_{\tau=0}^{1}$$

$$= e^{-2t}\left[\frac{e^2}{2}\left(1 - \tfrac{1}{2}\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\right)\right] = e^{-2t}\left[\tfrac{1}{2}e^2\tfrac{1}{2} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\right]$$

$$= e^{-2t}\left[\tfrac{1}{4}e^2 + \tfrac{1}{4}\right] = \frac{1+e^2}{4}e^{-2t}$$

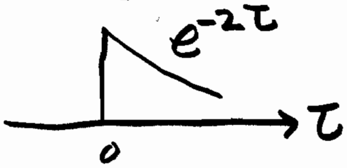
All Together:

$$y(t) = \begin{cases} 0, & t < 0 \\ \tfrac{1}{4}\left[e^{-2t} - 1\right] + \tfrac{1}{2}t, & 0 \leq t < 1 \\ \frac{1+e^2}{4}e^{-2t}, & t \geq 1 \end{cases}$$

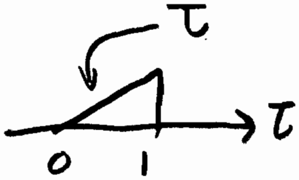
OTHER WAY: $y(t)=\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$

More Workspace for Problem 3...

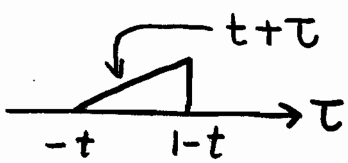
$h(\tau)$



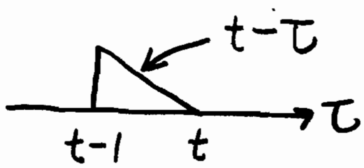
$x(\tau)$



$x(\tau--t)=x(t+\tau)$



$x(t-\tau)$



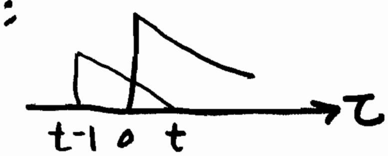
Case I) $t<0$:

$$y(t)=\int_{-\infty}^{\infty}0\,d\tau = 0.$$

Case II) $t\geq 0$ and $t-1<0$: $t\geq 0$ and $t<1$:

$0\leq t<1$:

$$y(t)=\int_0^t (t-\tau)e^{-2\tau}\,d\tau$$



$$=\int_0^t te^{-2\tau}-\tau e^{-2\tau}\,d\tau$$

$$=t\int_0^t e^{-2\tau}\,d\tau-\int_0^t \tau e^{-2\tau}\,d\tau$$

$$=t\left[-\tfrac{1}{2}e^{-2\tau}\right]_{\tau=0}^{t}-\left[\frac{e^{-2\tau}}{-2}\left(\tau-\frac{1}{-2}\right)\right]_{\tau=0}^{t}$$

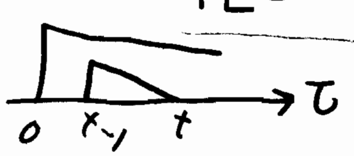
$$=t\left[-\tfrac{1}{2}e^{-2t}-\left(-\tfrac{1}{2}\right)\right]-\left[\frac{e^{-2t}}{-2}\left(t+\tfrac{1}{2}\right)-\left(-\tfrac{1}{2}\right)\left(0+\tfrac{1}{2}\right)\right]$$

$$=-\tfrac{1}{2}te^{-2t}+\tfrac{1}{2}t-\left[-\tfrac{1}{2}te^{-2t}-\tfrac{1}{4}e^{-2t}+\tfrac{1}{4}\right]$$

$$=-\tfrac{1}{2}te^{-2t}+\tfrac{1}{2}t+\tfrac{1}{2}te^{-2t}+\tfrac{1}{4}e^{-2t}-\tfrac{1}{4}$$

$$=\tfrac{1}{2}t-\tfrac{1}{4}+\tfrac{1}{4}e^{-2t}=\tfrac{1}{4}\left[e^{-2t}-1\right]+\tfrac{1}{2}t$$

Case III) $t\geq 1$



$$y(t)=\int_{t-1}^{t}(t-\tau)e^{-2\tau}\,d\tau$$

$$=t\int_{t-1}^{t}e^{-2\tau}\,d\tau-\int_{t-1}^{t}\tau e^{-2\tau}\,d\tau=t\left[-\tfrac{1}{2}e^{-2\tau}\right]_{\tau=t-1}^{t}-\left[\frac{e^{-2\tau}}{-2}\left(\tau-\frac{1}{-2}\right)\right]_{\tau=t-1}^{t}$$

$$=t\left[-\tfrac{1}{2}e^{-2t}-\left(-\tfrac{1}{2}\right)e^{-2(t-1)}\right]-\left[\frac{e^{-2t}}{-2}\left(t+\tfrac{1}{2}\right)-\frac{e^{-2(t-1)}}{-2}\left(t-1+\tfrac{1}{2}\right)\right]$$

$$=-\tfrac{1}{2}te^{-2t}+\tfrac{1}{2}te^{-2(t-1)}+\tfrac{1}{2}te^{-2t}+\tfrac{1}{4}e^{-2t}-\tfrac{1}{2}te^{-2(t-1)}+\tfrac{1}{4}e^{-2(t-1)}$$

$$=\tfrac{1}{4}e^{-2t}+\tfrac{1}{4}e^{-2(t-1)}=\tfrac{1}{4}\left[e^{-2t}+e^{-2t}e^{2}\right]=\tfrac{1}{4}e^{-2t}\left(1+e^{2}\right)=\frac{1+e^{2}}{4}e^{-2t}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

The system input is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n+1].$$

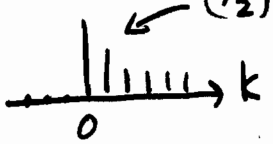
Find the system output $y[n]$.

$x[k]$

$\left(\frac{1}{4}\right)^k$

$h[k]$

$(1/2)^k$



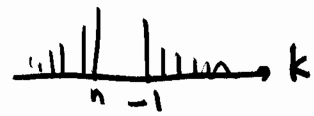
$h[k--n] = h[n+k]$

$\left(\frac{1}{2}\right)^{n+k}$

$h[n-k]$

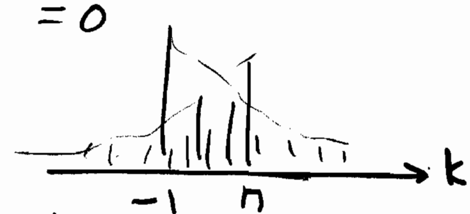
$\left(\frac{1}{2}\right)^{n-k}$

case I) $n < -1$:



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n \geq -1$:



$$y[n] = \sum_{k=-1}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^{n-k}$$

$$= \sum_{k=-1}^{n} \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-1}^{n} \left(\tfrac{1}{4}\right)^k 2^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=-1}^{n} \left(\tfrac{2}{4}\right)^k = \left(\tfrac{1}{2}\right)^n \sum_{k=-1}^{n} \left(\tfrac{1}{2}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \frac{\left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^{n+1}}{1 - 1/2} = \left(\tfrac{1}{2}\right)^n \frac{\left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^{n+1}}{1/2}$$

$$= \left(\tfrac{1}{2}\right)^{n-1} \left[\left(\tfrac{1}{2}\right)^{-1} - \left(\tfrac{1}{2}\right)^{n+1}\right]$$

$$= \left(\tfrac{1}{2}\right)^{n-1} \left(\tfrac{1}{2}\right)^{-1} - \left(\tfrac{1}{2}\right)^{n-1} \left(\tfrac{1}{2}\right)^{n+1}$$

$$= \left(\tfrac{1}{2}\right)^{n-1-1} - \left(\tfrac{1}{2}\right)^{n-1+n+1} = \left(\tfrac{1}{2}\right)^{n-2} - \left(\tfrac{1}{2}\right)^{2n}$$

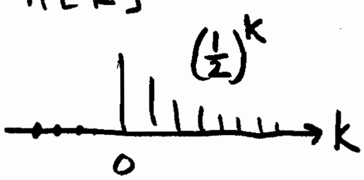
$$= \left(\tfrac{1}{2}\right)^n \left[4 - \left(\tfrac{1}{2}\right)^n\right]$$

All Together: $y[n] = \begin{cases} 0 & , n < -1 \\ \left(\frac{1}{2}\right)^n \left[4 - \left(\frac{1}{2}\right)^n\right] & , n \geq -1 \end{cases} = \left(\tfrac{1}{2}\right)^n \left[4 - \left(\tfrac{1}{2}\right)^n\right] u[n+1]$

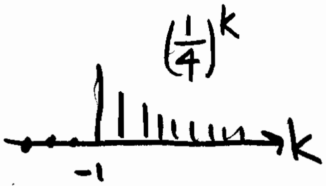
OTHER WAY: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

More Workspace for Problem 4...

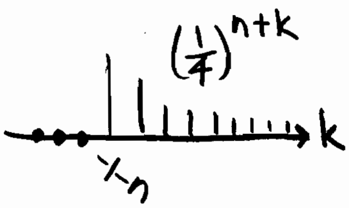
$h[k]$



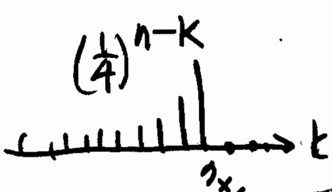
$x[k]$



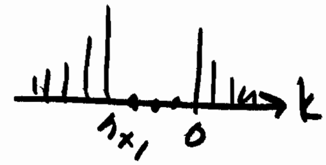
$x[k - -n] = x[n+k]$



$x[n-k]$

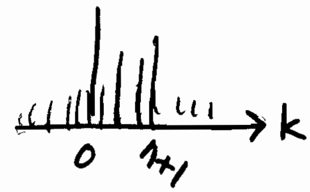


case I) $n+1 < 0$: $n < -1$:



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n+1 \geq 0$: $n \geq -1$:



$$y[n] = \sum_{k=0}^{n+1} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^{n-k}$$

$$= \sum_{k=0}^{n+1} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^n \left(\tfrac{1}{4}\right)^{-k} = \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n+1} \left(\tfrac{1}{2}\right)^k 4^k$$

$$= \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n+1} \left(\tfrac{4}{2}\right)^k = \left(\tfrac{1}{4}\right)^n \sum_{k=0}^{n+1} 2^k = \left(\tfrac{1}{4}\right)^n \frac{2^0 - 2^{n+2}}{1-2}$$

$$= \left(\tfrac{1}{4}\right)^n \frac{1 - 2^n 2^2}{-1} = \left(\tfrac{1}{4}\right)^n \left[4 \cdot 2^n - 1\right]$$

$$= \left(\tfrac{1}{2} \cdot \tfrac{1}{2}\right)^n \left[4 \cdot 2^n - 1\right] = \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{2}\right)^n \left[4 \cdot 2^n - 1\right]$$

$$= \left(\tfrac{1}{2}\right)^n \left[4 - \left(\tfrac{1}{2}\right)^n\right]$$

All Together: $$y[n] = \begin{cases} 0, & n < -1 \\ \left(\tfrac{1}{2}\right)^n \left[4 - \left(\tfrac{1}{2}\right)^n\right], & n \geq -1 \end{cases}$$

$$= \left(\tfrac{1}{2}\right)^n \left[4 - \left(\tfrac{1}{2}\right)^n\right] u[n+1]$$