ECE 3793
Test 1

Tuesday, October 6, 1998
12:00 PM - 1:15 PM

Fall 1998 Name: SOLUTION

Dr. Havlicek Student Num: ____________

Directions: You have 75 minutes to complete the test. All work must be your own. You may use one 8.5 $\times$ 11 inch two-sided note sheet as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (15) ________
3. (20) ________
4. (20) ________
5. (20) ________

TOTAL (100):

____

1. **25 pts.** True or False. Mark *True* only if the statement is **always** true.

TRUE	FALSE	
	X	(a) **2 pts.** Any discrete time signal $x[n]$ can be written uniquely as the product of its even and odd parts: $x[n] = \mathcal{E}v\{x[n]\} \times \mathcal{O}d\{x[n]\}$.
	X	(b) **2 pts.** If a system is shift invariant and BIBO stable, then it is also causal.
X		(c) **2 pts.** If a function $f(t)$ is differentiable at a point $t_0 \in \mathbb{R}$, then it is also continuous at t_0.
	X	(d) **2 pts.** If the Lebesgue integral of a function exists, then it is equal to the Riemann integral.
X		(e) **2 pts.** If $\omega_0 \neq 0$, then the fundamental period of the signal $x(t) = \cos(\omega_0 t)$ is $T_0 = \lvert 2\pi/\omega_0 \rvert$.
X		(f) **2 pts.** A discrete time LSI system is BIBO stable if and only if its unit pulse response is absolutely summable.
	X	(g) **2 pts.** If H is a discrete time linear system, and the output is $h[n]$ when the input is $\delta[n]$, then the output for an arbitrary signal $x[n]$ is given by $x[n] * h[n]$. Need LSI
X		(h) **2 pts.** If H is a system formed by cascading two LSI systems with unit pulse responses $h_1[n]$ and $h_2[n]$, then the unit pulse response of H is given by $h[n] = h_1[n] * h_2[n]$.
	X	(i) **2 pts.** The parallel connection of two discrete time LSI systems is a system that is linear, shift invariant, and causal.
X		(j) **2 pts.** If $\varphi(t)$ is in the space $\mathcal{D}$ of testing functions, then $\varphi(t)$ is differentiable.
X		(k) **2 pts.** A distribution is a continuous linear functional on $\mathcal{D}$.
X		(l) **3 pts.** ECE 3793 is more fun than any class you have ever taken before or ever will take in the future.

2. **15 pts.** Consider a discrete time system with output $y[n]$ that is related to the input $x[n]$ by

$$y[n] = -2x[n+2] + 4x[n] - 2x[n-2].$$

(a) **3 pts.** Is the system linear? Justify your answer.

Let $x_1[n]$ and $x_2[n]$ be inputs with corresponding outputs $y_1[n] = -2x_1[n+2] + 4x_1[n] - 2x_1[n-2]$
and $y_2[n] = -2x_2[n+2] + 4x_2[n] - 2x_2[n-2]$.

Let $x_3[n] = ax_1[n] + bx_2[n]$, where a and b are constants.

Then $y_3[n] = -2x_3[n+2] + 4x_3[n] - 2x_3[n-2]$

$= -2(ax_1[n+2] + bx_2[n+2]) + 4(ax_1[n] + bx_2[n]) - 2(ax_1[n-2] + bx_2[n-2])$

$= -2ax_1[n+2] + 4ax_1[n] - 2ax_1[n-2] - 2bx_2[n+2] + 4bx_2[n] - 2bx_2[n-2]$

$= ay_1[n] + by_2[n]$.

Therefore, the system <u>is</u> <u>linear</u>.

(b) **3 pts.** Is the system shift invariant? Justify your answer.

Let $x_1[n]$ and $y_1[n]$ be as above in part (a).

Let $x_2[n] = x_1[n-n_0]$, where n_0 is a constant.

Then $y_2[n] = -2x_2[n+2] + 4x_2[n] - 2x_2[n-2]$

$= -2x_1[n+2-n_0] + 4x_1[n-n_0] - 2x_1[n-2-n_0]$.

$= y_1[n-n_0]$.

Therefore, the system <u>is</u> <u>shift invariant</u>.

Problem 2, cont...

(c) **3 pts.** If the system is LSI, find the unit pulse response $h[n]$. Otherwise, find the output when the input is the unit step sequence $u[n]$.

The system is LSI, as shown in (a) and (b).

$$h[n] = -2\delta[n+2] + 4\delta[n] - 2\delta[n-2].$$

→ If you got (a) or (b) wrong, the answer would be

$$y[n] = -2u[n+2] + 4u[n] - 2u[n-2].$$

(d) **3 pts.** Is the system causal? Justify your answer.

The system is not causal because

$$h[-2] = -2 \neq 0.$$

(e) **3 pts.** Is the system BIBO stable? Justify your answer.

The system is BIBO stable because

$$\|h[n]\|_{\ell^1} = \sum_{n=-\infty}^{\infty} |h[n]| = 2+4+2 = 8 < \infty.$$

3. **20 pts.** A discrete time LSI system has unit pulse response $h[n] = 3^{-n}u[n]$. Find the output $y[n]$ when the input is $x[n] = 2^{-2n}u[n]$.

Hint: Write a general expression for $y[n]$ in terms of $h[n]$ and $x[n]$, plug in the given expressions for $h[n]$ and $x[n]$, and carefully simplify to a form where you can apply your summation formulas.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-2k} u[k]\, 3^{-(n-k)} u[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-2k} 3^{-n} 3^{k} u[k]\, u[n-k]$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{1}{4}\right)^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n} \left(\tfrac{3}{4}\right)^k$$

This is Zero if $n<0$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{1-\left(\tfrac{3}{4}\right)^{n+1}}{1-\tfrac{3}{4}}\right] u[n]$$

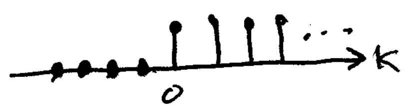
$$= 4\left(\tfrac{1}{3}\right)^n \left[1-\left(\tfrac{3}{4}\right)^{n+1}\right] u[n]$$

$$= \left[4\left(\tfrac{1}{3}\right)^n - 4\left(\tfrac{1}{3}\right)^n\left(\tfrac{3}{4}\right)^{n+1}\right] u[n]$$

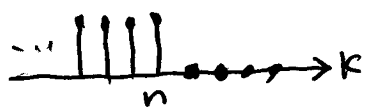
$$= \left[4\left(\tfrac{1}{3}\right)^n - 4\left(\tfrac{1}{3}\right)^n\left(\tfrac{3}{4}\right)^{n}\left(\tfrac{3}{4}\right)\right] u[n]$$

$$= \underline{\underline{\left[4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n\right] u[n]}}$$

$u[k]$



$u[n-k]$

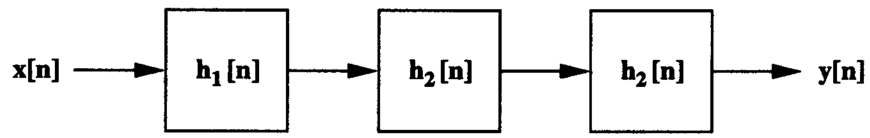


$$\boxed{\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a},\quad a\neq 1}$$

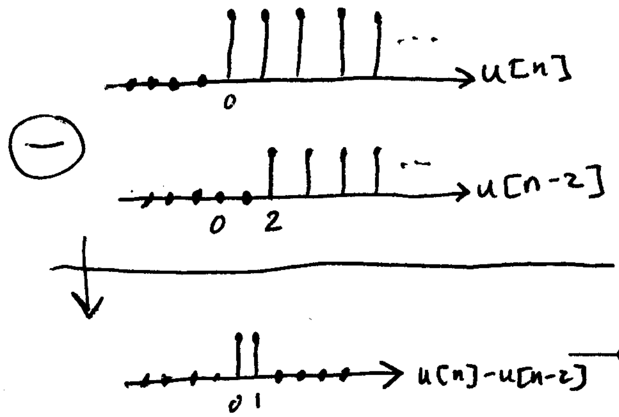
$$y[n] = \left[4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n\right] u[n]$$

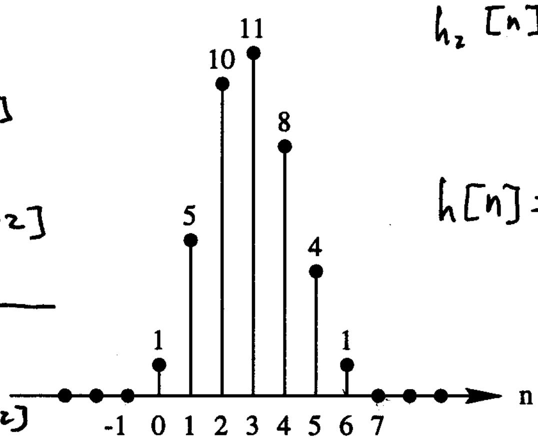
$$\boxed{y[n] = \left(\frac{4}{3^n} - \frac{3}{4^n}\right) u[n]}$$

4. **20 pts.** Consider the cascade interconnection of three causal LSI systems illustrated below.



The unit pulse response $h_2[n]$ is $h_2[n] = u[n] - u[n-2]$, and the overall unit pulse response is shown below.





$$h_2[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$$

$$h[n] = h_1[n] * h_2[n] * h_2[n]$$

Find the unit pulse response $h_1[n]$.

$$h_2[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_2[k] h_2[n-k] = \sum_{k=-\infty}^{\infty} \left(\delta[k] + \delta[k-1]\right)\left(\delta[n-k] + \delta[n-k-1]\right)$$

$$= \sum_{k=-\infty}^{\infty} \delta[k]\delta[n-k] + \delta[k]\delta[n-k-1] + \delta[k-1]\delta[n-k-1] + \delta[k-1]\delta[n-k]$$

$$= \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-1] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$h[n] = h_1[n] * \left(h_2[n] * h_2[n]\right) = h_1[n] * \left(\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$$

$$= h_1[n] + 2h_1[n-1] + h_1[n-2]$$

$\Rightarrow$ It is given that $h_1[n]$ and $h_2[n]$ are causal.

$\Rightarrow h_1[n] = 0$ for all $n < 0$.

$$h[0] = h_1[0] + \underbrace{2h_1[-1]}_{0} + \underbrace{h_1[-2]}_{0} = h_1[0] = 1 \quad \left(\text{from graph}\right)$$

More Workspace for Problem 4...

$$h[1] = h_1[1] + 2h_1[0] + \underbrace{h_1[-1]}_{0} = h_1[1] + 2 = 5 \Rightarrow h_1[1] = 3$$

$$h[2] = h_1[2] + 2h_1[1] + h_1[0] = h_1[2] + 6 + 1 = 10 \Rightarrow h_1[2] = 3$$

$$h[3] = h_1[3] + 2h_1[2] + h_1[1] = h_1[3] + 6 + 3 = 11 \Rightarrow h_1[3] = 2$$

$$h[4] = h_1[4] + 2h_1[3] + h_1[2] = h_1[4] + 4 + 3 = 8 \Rightarrow h_1[4] = 1$$

$$h[5] = h_1[5] + 2h_1[4] + h_1[3] = h_1[5] + 2 + 2 = 4 \Rightarrow h_1[5] = 0$$

$$h[6] = h_1[6] + 2h_1[5] + h_1[4] = h_1[6] + 0 + 1 = 1 \Rightarrow h_1[6] = 0$$

$$\underbrace{h[n]}_{\text{all } n>6} = h_1[n] + 2h_1[n-1] + h_1[n-2] = h_1[n] + 0 + 0 = 0 \Rightarrow h_1[n] = 0, \quad n > 6$$

$$\boxed{h_1[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]}$$

5. **20 pts**. Suppose that $\mathcal{V}$ is a vector space and that the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ is an orthonormal basis for $\mathcal{V}$. Suppose further that an inner product is defined on $\mathcal{V}$ and that the inner product between two vectors $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ is written $\langle \mathbf{v}, \mathbf{w} \rangle$.

(a) **4 pts**. What is the dimension of the space $\mathcal{V}$?

Dimension = Number of vectors in Basis = 5.

(b) **4 pts**. What is the value of the inner product $\langle \mathbf{u}_2, \mathbf{u}_5 \rangle$?

Basis is orthonormal: $\langle \vec{u}_2, \vec{u}_5 \rangle = 0$.

(c) **4 pts**. What is the value of the inner product $\langle \mathbf{u}_3, \mathbf{u}_3 \rangle$?

Basis is orthonormal; $\langle \vec{u}_3, \vec{u}_3 \rangle = 1$

(d) **8 pts**. Suppose $\mathbf{v} \in \mathcal{V}$. Write $\mathbf{v}$ as a linear combination of the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$.

$$\vec{v} = \langle \vec{v}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{v}, \vec{u}_2 \rangle \vec{u}_2 + \langle \vec{v}, \vec{u}_3 \rangle \vec{u}_3 + \langle \vec{v}, \vec{u}_4 \rangle \vec{u}_4 + \langle \vec{v}, \vec{u}_5 \rangle \vec{u}_5$$