ECE 3793
Test 1

Tuesday, February 22, 2000
6:00 PM - 9:00 PM

Spring 2000

Dr. Havlicek

Name: SOLUTION

Student Num:__________

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 180 minutes to complete the test. All work must be your own. You may use one 8.5 $\times$ 11 inch two-sided note sheet as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

1. (25) ________

2. (25) ________

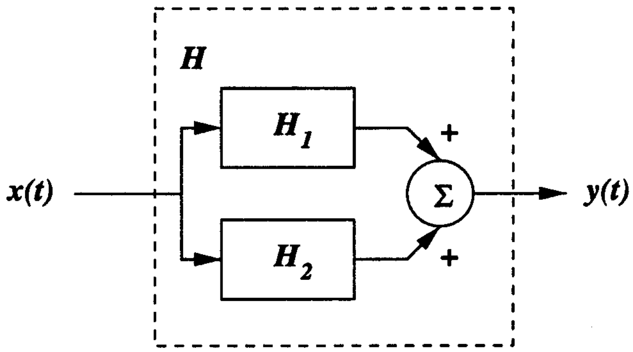
3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts.** Consider a continuous-time system H formed by connecting two systems H_1 and H_2 in parallel as shown in the figure below.



The input/output relationship for system H_1 is given by

$$y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau.$$

The input/output relationship for system H_2 is given by

$$y(t) = \frac{d}{dt}x(t).$$

(a) **5 pts.** Is the system H memoryless? Justify your answer.

when $t=2$, $y(2) = \int_{-\infty}^{4} x(\tau)\,d\tau + \frac{d}{dt}x(t)\Big|_{t=2}$.

Since $y(2)$ depends on the future input $x(3)$, the system is not memoryless.

(b) **5 pts.** Is the system H causal? Justify your answer.

As shown in (a) above, $y(2)$ depends on the future input $x(3)$, so the system is not causal.

Problem 1, cont...

(c) **5 pts.** Is the system H stable? Justify your answer.

Let $x(t) = u(-t)$. Then $|x(t)| \leq 1 \ \forall t$, so $x(t)$ is bounded by $B=1$.

In this case, $y(t) = \int_{-\infty}^{2t} u(-\tau)\,d\tau + \frac{d}{dt}u(-t)$

$$= \int_{-\infty}^{2t} u(-\tau)\,d\tau - \delta(t)$$

So $y(-1) = \int_{-\infty}^{-2} 1\,d\tau - 0 = \int_{-\infty}^{-2} d\tau = \tau\Big|_{-\infty}^{-2}$

$$= \lim_{A\to\infty} -2 + A \to \infty$$

So $|y(-1)| \to \infty$ and the system is <u>not</u> stable.

(d) **5 pts.** Is the system H time invariant? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = \int_{-\infty}^{2t} x_1(\tau)\,d\tau + \frac{d}{dt}x_1(t)$

$$= \int_{-\infty}^{2t} x_1(\tau)\,d\tau + \lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h}$$

So $y_1(t-t_0) = \int_{-\infty}^{2(t-t_0)} x_1(\tau)\,d\tau + \lim_{h\to 0} \frac{x_1(t-t_0+h) - x_1(t-t_0)}{h}$

Let $x_2(t) = x_1(t-t_0)$.

Then $y_2(t) = \int_{-\infty}^{2t} x_2(\tau)\,d\tau + \lim_{h\to 0} \frac{x_2(t+h) - x_2(t)}{h}$

$$= \int_{-\infty}^{2t} x_1(\tau - t_0)\,d\tau + \lim_{h\to 0} \frac{x_2(t-t_0+h) - x_1(t-t_0)}{h}$$

$\left\{\begin{matrix} \theta = \tau - t_0 \\ d\theta = d\tau \end{matrix}\right\} = \int_{-\infty}^{2t-t_0} x_1(\theta)\,d\theta \ {}^{3}+ \lim_{h\to 0} \frac{x_2(t-t_0+h) - x_1(t-t_0)}{h} \neq y_1(t-t_0).$

$\Rightarrow$ <u>NOT</u> <u>TIME</u> <u>INVARIANT</u>.

Problem 1, cont...

(e) **5 pts**. Is the system H linear? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = \int_{-\infty}^{2t} x_1(\tau)d\tau + \frac{d}{dt}x_1(t)$

Let $y_2(t) = H\{x_2(t)\} = \int_{-\infty}^{2t} x_2(\tau)d\tau + \frac{d}{dt}x_2(t)$

Let a,b be constants and let $x_3(t) = ax_1(t) + bx_2(t)$.

Then $y_3(t) = H\{x_3(t)\} = \int_{-\infty}^{2t} x_3(\tau)d\tau + \frac{d}{dt}x_3(t)$

$$= \int_{-\infty}^{2t} ax_1(\tau) + bx_2(\tau)d\tau + \frac{d}{dt}\{ax_1(t) + bx_2(t)\}$$

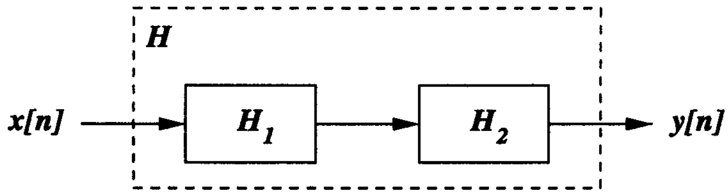
$$= a\int_{-\infty}^{2t} x_1(\tau)d\tau + b\int_{-\infty}^{2t} x_2(\tau)d\tau + a\frac{d}{dt}x_1(t) + b\frac{d}{dt}x_2(t)$$

$$= a\left\{\int_{-\infty}^{2t} x_1(\tau)\,d\tau + \frac{d}{dt}x_1(t)\right\} + b\left\{\int_{-\infty}^{2t} x_2(\tau)d\tau + \frac{d}{dt}x_2(t)\right\}$$

$$= ay_1(t) + by_2(t) \checkmark$$

The system <u>is</u> linear.

2. **25 pts**. Consider a discrete-time system H formed by connecting two LTI systems H_1 and H_2 in series as shown in the figure below.



The impulse response of LTI system H_1 is given by $h_1[n] = \left(\frac{1}{5}\right)^n u[n+2]$. The impulse response of LTI system H_2 is given by $h_2[n] = \delta[n-3]$.

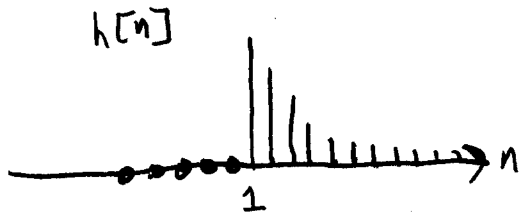
(a) **10 pts**. Is H an LTI system? Justify your answer by **either** stating a property proved in class **or** by showing directly that H is or is not LTI.

If H **is** LTI, then find the impulse response $h[n]$. Alternatively, if H is **not** LTI, then find the system output when the input is $\delta[n]$.

It was proven in class that the cascade connection of two LTI systems is an LTI system, Therefore, H is LTI.

$$h[n] = h_1[n] * h_2[n]$$
$$= h_1[n] * \delta[n-3]$$
$$= h_1[n-3]$$
$$= \left(\tfrac{1}{5}\right)^{n-3} u[n-3+2] = 125\left(\tfrac{1}{5}\right)^n u[n-1]$$

$$\boxed{h[n] = 125\left(\tfrac{1}{5}\right)^n u[n-1]}$$



Problem 2, cont...

(b) **5 pts.** Is H memoryless? Justify your answer.

An LTI discrete time system is memoryless iff $h[n] = K\delta[n]$, where K is a constant.

That is <u>not</u> the case here, so H is <u>not</u> memoryless.

(c) **5 pts.** Is H causal? Justify your answer.

The system <u>is</u> causal, because $h[n] = 0$ $\forall\, n < 0$.

(d) **5 pts.** Is H stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = 125 \sum_{n=-\infty}^{\infty} \left| \left(\tfrac{1}{5}\right)^n u[n-1] \right|$$

$$= 125 \sum_{n=1}^{\infty} \left(\tfrac{1}{5}\right)^n$$

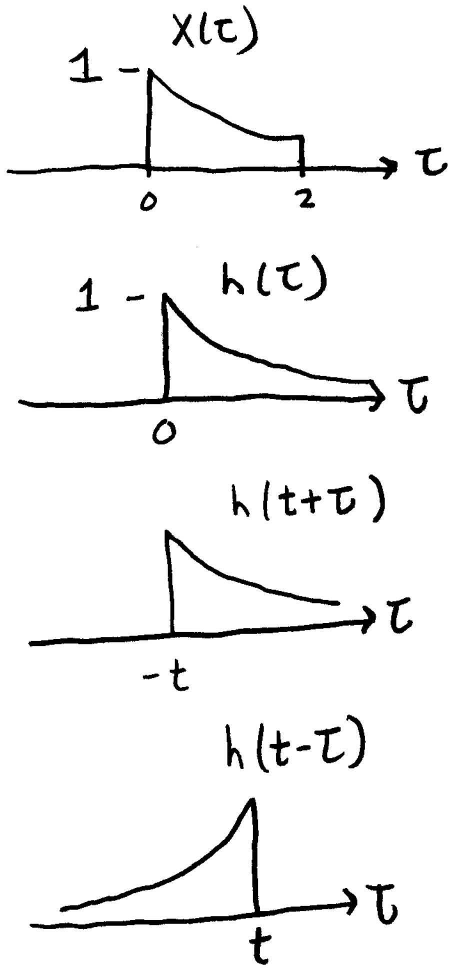
$$= 125 \lim_{A \to \infty} \frac{\frac{1}{5} - \left(\frac{1}{5}\right)^{A+1}}{1 - \frac{1}{5}}$$

$$= 125 \cdot \frac{\frac{1}{5}}{\frac{4}{5}} = 125 \cdot \frac{5}{4} \cdot \frac{1}{5} = \frac{125}{4} < \infty.$$

The system <u>is</u> stable.

3. **25 pts.** Consider a continuous-time LTI system with impulse response $h(t) = e^{-t}u(t)$. The system input is given by $x(t) = e^{-3t}[u(t) - u(t-2)]$. Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$



$\Rightarrow$ when $t<0$, $y(t) = 0$

$\Rightarrow$ when $0 \le t < 2$,

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^t e^{-3\tau}e^{\tau-t}\,d\tau$$

$$= e^{-t}\int_0^t e^{-2\tau}\,d\tau$$

$$= e^{-t}\left(-\tfrac{1}{2}\right)\left[e^{-2\tau}\right]_{\tau=0}^{t}$$

$$= -\tfrac{1}{2}e^{-t}\left[e^{-2t} - 1\right]$$

$$= \tfrac{1}{2}\left[1 - e^{-2t}\right]e^{-t}.$$

$\Rightarrow$ when $t \geq 2$, $y(t) = \int_0^2 x(\tau)h(t-\tau)\,d\tau = \int_0^2 e^{-3\tau}e^{\tau-t}\,d\tau$

$$= e^{-t}\int_0^2 e^{-2\tau}\,d\tau = -\frac{e^{-t}}{2}\left[e^{-2\tau}\right]_{\tau=0}^{2}$$

$$= -\frac{e^{-t}}{2}\left[e^{-4} - 1\right] = \tfrac{1}{2}\left[1 - e^{-4}\right]e^{-t}.$$

$$\boxed{\text{All together:} \quad y(t) = \begin{cases} 0 & , \; t<0 \\ \tfrac{1}{2}\left[1-e^{-2t}\right]e^{-t} & , \; 0 \le t < 2 \\ \tfrac{1}{2}\left[1-e^{-4}\right]e^{-t} & , \; 2 \le t \end{cases}}$$

More Workspace for Problem 3...

4. **25 pts.** Determine if the discrete-time signal

$$x[n] = \cos\left[\frac{\pi}{8}n^2\right]$$

is periodic. If the signal is periodic, then find the fundamental period.

This problem was given on homework 2.

If $x[n]$ is periodic with period T_0, then

$$\cos\left[\frac{\pi}{8}n^2\right] = \cos\left[\frac{\pi}{8}(n+T_0)^2\right].$$

This is true only if $\frac{\pi}{8}n^2$ and $\frac{\pi}{8}(n+T_0)^2$ differ by an integer multiple of 2π, i.e., if

$$\frac{\pi}{8}(n+T_0)^2 - \frac{\pi}{8}n^2 = 2\pi k, \quad k \in \mathbb{Z}.$$

$$\frac{\pi}{8}n^2 + 2\frac{\pi}{8}nT_0 + \frac{\pi}{8}T_0^2 - \frac{\pi}{8}n^2 = 2\pi k$$

$$2\frac{\pi}{8}nT_0 + \frac{\pi}{8}T_0^2 = 2\pi k$$

$$\frac{nT_0}{8} + \frac{T_0^2}{16} = k = \text{any integer.}$$

This is true if T_0 is any integer multiple of 8, because if $m \in \mathbb{Z}$ and $T_0 = 8m$, we have

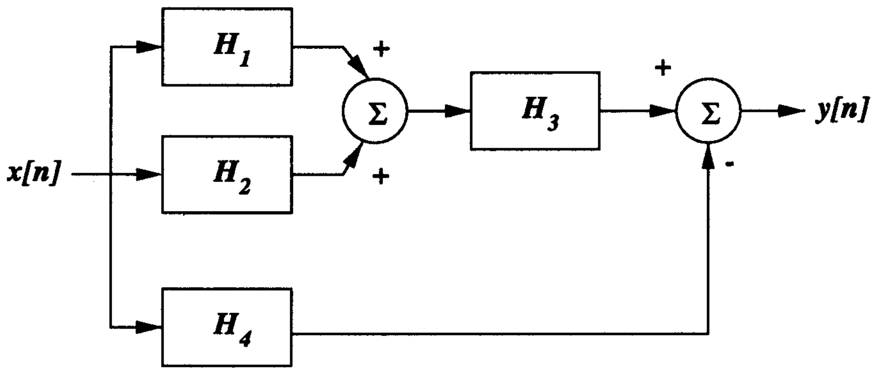
$$\frac{nT_0}{8} + \frac{T_0^2}{16} = \frac{8nm}{8} + \frac{64m^2}{16} = nm + 4m^2 = \text{integer.}$$

$\Rightarrow$ So the fundamental period is the smallest integer multiple of 8, or $\underline{\underline{8 \cdot 1}}$.

$$\boxed{T_0 = 8}$$

More Workspace for Problem 4...

5. **25 pts.** The discrete-time system H is formed by connecting four LTI systems H_1, H_2, H_3, and H_4 as shown in the figure below.



The impulse responses of the four LTI systems H_1 through H_4 are given by

$$\begin{aligned} h_1[n] &= u[n], \\ h_2[n] &= u[n+2] - u[n], \\ h_3[n] &= \delta[n-2], \\ h_4[n] &= \left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

Find the **unit step response** $s[n]$ of the system H.

The impulse response of H is

$$h[n] = \left(h_1[n] + h_2[n]\right) * h_3[n] - h_4[n]$$

$$= \left(u[n] + u[n+2] - u[n]\right) * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n+2] * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n] - \left(\tfrac{1}{2}\right)^n u[n] = \left[1 - \left(\tfrac{1}{2}\right)^n\right] u[n].$$

The step response of H is

$$s[n] = \sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{n} \left[1 - \left(\tfrac{1}{2}\right)^n\right] u[n]$$

$\longrightarrow$

If $n<0$, then $s[n] = \sum_{k=-\infty}^{n} 0 = 0$.

if $n \geq 0$, then

$$s[n] = \sum_{k=-\infty}^{n} \left[1-\left(\tfrac{1}{2}\right)^n\right] u[n] = \sum_{k=0}^{n} \left[1-\left(\tfrac{1}{2}\right)^n\right]$$

$$= \sum_{k=0}^{n} 1 - \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^n = n+1 - \frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$= n+1 - \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1/2} = n+1-2+2\left(\tfrac{1}{2}\right)^{n+1}$$

$$= n-1+\left(\tfrac{1}{2}\right)^{-1}\left(\tfrac{1}{2}\right)^{n+1}$$

$$= n-1+\left(\tfrac{1}{2}\right)^n$$

All together: $s[n] = \left\{n-1+2^{-n}\right\} u[n]$