ECE 3793
Test 1

Tuesday, March 13, 2001
7:00 PM - 10:00 PM

Spring 2001 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **five** problems. Work any **four** of them. Only **four** problems will be graded. Below, you must circle the numbers of the **four** problems you wish to have graded. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

1. (25) ________

2. (25) ________

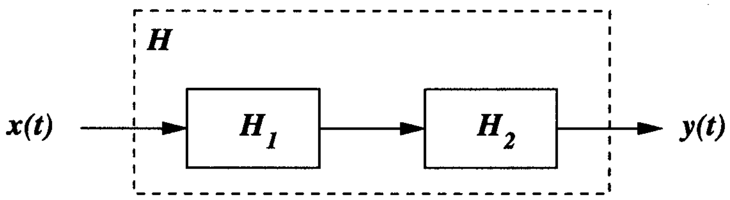
3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts**. Consider a continuous-time system H formed by connecting two systems H_1 and H_2 in series as shown in the figure below. System H has input $x(t)$ and output $y(t)$.



The input/output relation for system H_1 is $y(t) = tx(t+2) + 5$. The output of H_1 is fed directly into system H_2. System H_2 is an ideal differentiator with input/output relation $y(t) = \frac{d}{dt}x(t)$.

Hint: In calculating the input/output relation for the overall system H, be sure to apply the chain rule carefully when you differentiate the output of system H_1.

(a) **5 pts**. Is the system H memoryless? Justify your answer.

The I/O relation for H is

$$y(t) = \frac{d}{dt}\{t\,x(t+2) + 5\} = t\frac{d}{dt}x(t+2) + x(t+2)$$

$\Rightarrow$ since $y(0)$ depends on the future input $x(2)$, the system <u>is not memoryless</u>

(b) **5 pts**. Is the system H causal? Justify your answer.

H is <u>not causal</u> for the same reason as in part (a):

$y(0)$ depends on the <u>future</u> input $x(2)$.

Problem 1, cont...

(c) **5 pts.** Is the system H BIBO stable? Justify your answer.

Let the input be $x(t) = u(t)$. Then $|x(t)| \leq 1$, so $x(t)$ is a bounded input signal.

The output in this case is

$$y(t) = t\frac{d}{dt}u(t+2) + u(t+2)$$

$$= t\delta(t+2) + u(t+2) = u(t+2) - 2\delta(t+2)$$

Since $y(-2) \to -\infty$, the output is <u>not</u> <u>bounded</u>.

Since a bounded input produced an unbounded output, the system is <u>not</u> <u>stable</u>.

(d) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1(t)$ be the output when $x_1(t)$ is the input:

$$y_1(t) = H\{x_1(t)\} = t\frac{d}{dt}x_1(t+2) + x_1(t+2).$$

Let $y_2(t)$ be the output when $x_2(t)$ is the input:

$$y_2(t) = H\{x_2(t)\} = t\frac{d}{dt}x_2(t+2) + x_2(t+2).$$

Let $x_3(t) = ax_1(t) + bx_2(t)$ where a & b are constants.

Then $y_3(t) = t\frac{d}{dt}x_3(t+2) + x_3(t+2)$

$$= t\frac{d}{dt}\{ax_1(t+2) + bx_2(t+2)\} + ax_1(t+2) + bx_2(t+2)$$

$$= at\frac{d}{dt}x_1(t+2) + bt\frac{d}{dt}x_2(t+2) + ax_1(t+2) + bx_2(t+2)$$

$$= a\left\{t\frac{d}{dt}x_1(t+2) + x_1(t+2)\right\} + b\left\{t\frac{d}{dt}x_2(t+2) + x_2(t+2)\right\}$$

$$= ay_1(t) + by_2(t) \checkmark$$

The system <u>is</u> <u>linear</u>.

Problem 1, cont...

(e) **5 pts.** Is the system H time invariant? Justify your answer.

I think that H is <u>not</u> time invariant because of the "t" out front in the term $t\frac{d}{dt}x(t+2)$.

<u>Proof</u>: Let $x_1(t) = u(t)$.

$$\text{Then } y_1(t) = t\frac{d}{dt}u(t+2) + u(t+2)$$
$$= t\delta(t+2) + u(t+2)$$
$$= u(t+2) - 2\delta(t+2).$$

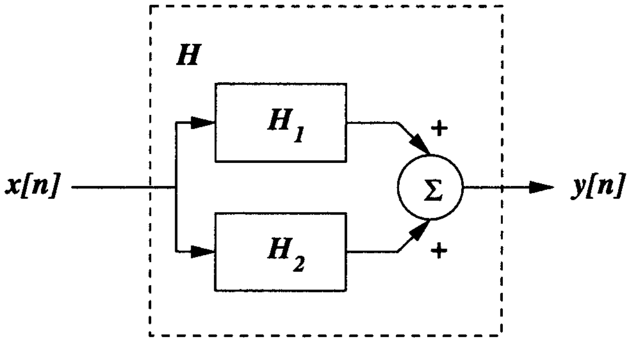
Let $t_0 = 3$. Then $y_1(t-t_0) = y_1(t-3) = u(t-3+2) - 2\delta(t-3+2)$
$$= u(t-1) - 2\delta(t-1).$$

Now let $x_2(t) = x_1(t-t_0) = u(t-3)$.

$$\text{Then } y_2(t) = t\frac{d}{dt}x_2(t+2) + x_2(t+2)$$
$$= t\frac{d}{dt}u(t-1) + u(t-1)$$
$$= t\delta(t-1) + u(t-1)$$
$$= u(t-1) + \delta(t-1) \neq y_1(t-t_0).$$

Since $y_2(t) \neq y_1(t-t_0)$, the system is <u>not</u> time invariant.

2. **25 pts**. The discrete-time system H shown in the figure below is formed by connecting two LTI systems H_1 and H_2 in parallel.



The impulse response of LTI system H_1 is given by

$$h_1[n] = \left(-\frac{1}{2}\right)^n u[n-2]$$

and the impulse response of LTI system H_2 is given by

$$h_2[n] = (1.01)^n u[1-n].$$

(a) **10 pts**. Is H an LTI system? If you say **yes**, then state the reason and find the impulse response $h[n]$. If instead you say **no**, then prove that it is not.

H is an LTI system, because, as we proved in class, the parallel connection of any two LTI systems is itself another LTI system.

The impulse response of H is given by

$$h[n] = h_1[n] + h_2[n]$$
$$= (1.01)^n u[1-n] + \left(-\tfrac{1}{2}\right)^n u[n-2]$$

Problem 2, cont...

(b) **5 pts.** Is the system H memoryless? Justify your answer.

A discrete-time LTI system H is memoryless iff the impulse response is a constant times $\delta[n]$.

In this case, $h[n]$ is nonzero $\forall n \in \mathbb{Z}$, so the system is <u>not</u> memoryless.

(c) **5 pts.** Is the system H causal? Justify your answer.

A discrete-time LTI system is causal iff the impulse response is zero $\forall n < 0$. In this case,

$$h[-1] = (1.01)^{-1} = \frac{1}{1.01} \neq 0.$$

So the system is <u>not</u> causal.

(d) **5 pts.** Is the system H BIBO stable? Justify your answer.

A discrete-time LTI system is stable iff the impulse response is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{1} (1.01)^n + \sum_{n=2}^{\infty} \left|\left(-\tfrac{1}{2}\right)^n\right|$$

$$= \sum_{n=-\infty}^{0} (1.01)^n + (1.01)^1 + \sum_{n=2}^{\infty} \left(\tfrac{1}{2}\right)^n$$

$$= 1.01 + \sum_{n=0}^{\infty} \frac{1}{1.01} + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n - 1 - \tfrac{1}{2}$$

$$= \frac{1}{1 - \frac{1}{1.01}} + \frac{1}{1 - \frac{1}{2}} - 0.49 = 101 + 2 - 0.49$$

$$= 102.51 < \infty$$

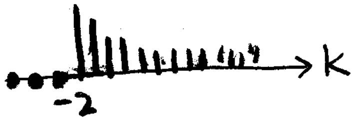
The system <u>is</u> stable.

3. **25 pts**. Consider a discrete-time LTI system H with impulse response

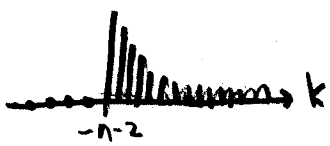
$$h[n] = \left(\frac{1}{4}\right)^{-n} u[-n+2].$$

The system input is given by $x[n] = \left(\frac{1}{2}\right)^n u[n+2]$. Find the system output $y[n]$.

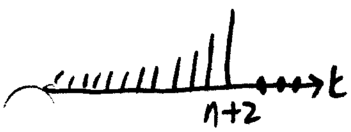
$x[k]$



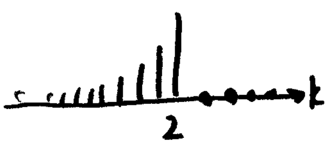
$x[n+k]$



$x[n-k]$



$h[k]$



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

I) $n+2 < 2 \Rightarrow n < 0$:

$$y[n] = \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\tfrac{1}{8}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} 8^k$$

$$= \left(\tfrac{1}{2}\right)^n \lim_{A \to \infty} \frac{8^{-A} - 8^{n+3}}{1-8} = \left(\tfrac{1}{2}\right)^n \frac{0 - 8^{n+3}}{-7}$$

$$= \left(\tfrac{1}{2}\right)^n 8^n \frac{8^3}{7} = \frac{8^3}{7}\left(\tfrac{1}{2} \cdot 8\right)^n = \frac{8^3}{7} 4^n.$$

II) $n+2 \geq 2 \Rightarrow n \geq 0$:

$$y[n] = \sum_{k=-\infty}^{2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{n-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{2} \left(\tfrac{1}{4}\right)^{-k} \left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{2} \left(\tfrac{1}{8}\right)^{-k} = \left(\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{2} 8^k$$

$$= \left(\tfrac{1}{2}\right)^n \lim_{A \to \infty} \frac{8^{-A} - 8^3}{1-8} = \left(\tfrac{1}{2}\right)^n \frac{0 - 8^3}{-7} = \frac{8^3}{7}\left(\tfrac{1}{2}\right)^n.$$

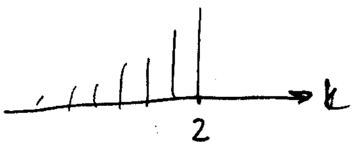
<u>All Together</u>

$$y[n] = \begin{cases} \frac{8^3}{7} 4^n, & n < 0 \\ \frac{8^3}{7}\left(\frac{1}{2}\right)^n, & n \geq 0 \end{cases}$$

③ The other way:

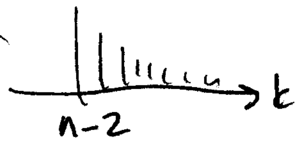
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

h[k]

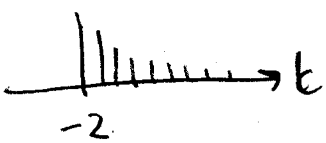


h[n+k]

h[n-k]



x[k]



I) $n-2 < -2 \Rightarrow n < 0$:

$$y[n] = \sum_{k=-2}^{\infty} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^{k-n} = 4^n \sum_{k=-2}^{\infty} \left(\tfrac{1}{8}\right)^k$$

$$= 4^n \frac{\left(\frac{1}{8}\right)^{-2} - \left(\frac{1}{8}\right)^{\infty}}{1-\frac{1}{8}}$$

$$= 4^n \left(\tfrac{8}{7}\right) 8^2 = \frac{8^3}{7} 4^n$$

II) $n-2 \geq -2 \Rightarrow n \geq 0$:

$$y[n] = \sum_{k=n-2}^{\infty} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^{k-n} = 4^n \sum_{k=n-2}^{\infty} \left(\tfrac{1}{8}\right)^k$$

$$= 4^n \frac{\left(\frac{1}{8}\right)^{n-2} - \left(\frac{1}{8}\right)^{\infty}}{1-\frac{1}{8}} = 4^n \left(\tfrac{8}{7}\right) 8^{2-n}$$

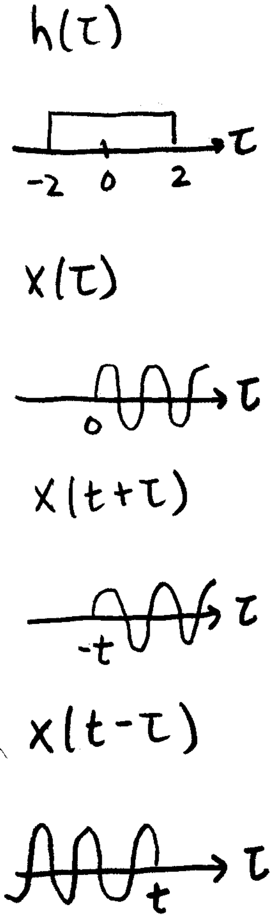
$$= \frac{8^3}{7} 4^n \left(\tfrac{1}{8}\right)^n = \frac{8^3}{7} \left(\tfrac{1}{2}\right)^n$$

All Together:

$$y[n] = \begin{cases} \frac{8^3}{7} 4^n, & n < 0 \\ \frac{8^3}{7} \left(\frac{1}{2}\right)^n, & n \geq 0. \end{cases}$$

4. **25 pts.** Consider a continuous-time LTI system H with impulse response $h(t) = u(t+2) - u(t-2)$. The system input is given by $x(t) = e^{j\pi t}u(t)$. Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



I) $t < -2$: $y(t) = 0$.

II) $-2 \le t < 2$:

$$y(t) = \int_{-2}^{t} e^{j\pi(t-\tau)}\, d\tau = e^{j\pi t}\int_{-2}^{t} e^{-j\pi\tau}\, d\tau$$

$$= \frac{e^{j\pi t}}{-j\pi}\left[e^{-j\pi\tau}\right]_{\tau=-2}^{t}$$

$$= \frac{e^{j\pi t}}{-j\pi}\left[e^{-j\pi t} - e^{j2\pi}\right] = \frac{1 - \overbrace{e^{j2\pi}}^{1} e^{j\pi t}}{-j\pi}$$

$$= \frac{e^{j\pi t} - 1}{j\pi}$$

III) $t \ge 2$:

$$y(t) = \int_{-2}^{2} e^{j\pi(t-\tau)}\, d\tau = e^{j\pi t}\int_{-2}^{2} e^{-j\pi\tau}\, d\tau$$

$$= \frac{e^{j\pi t}}{-j\pi}\left[e^{-j\pi\tau}\right]_{\tau=-2}^{2} = \frac{e^{j\pi t}}{-j\pi}\left[e^{-j2\pi} - e^{j2\pi}\right]$$

$$= \frac{e^{j\pi t}}{-j\pi}\left[1 - 1\right] = 0.$$

$\therefore$ Together

$$y(t) = \begin{cases} 0, & t < -2 \\ \dfrac{e^{j\pi t} - 1}{j\pi}, & -2 \le t < 2 \\ 0, & t \ge 2 \end{cases} = \frac{e^{j\pi t}}{j\pi}\left[u(t+2) - u(t-2)\right]$$

5. **25 pts**. Determine if the discrete-time signal

$$x[n] = \cos\left[\frac{\pi}{8}n^2\right]$$

is periodic. If the signal is periodic, then find the fundamental period.

If $x[n]$ is periodic with period N, then

$$x[n] = x[n+N] \quad \forall n \in \mathbb{Z}$$

Then $\cos\left[\frac{\pi}{8}n^2\right] = \cos\left[\frac{\pi}{8}(n+N)^2\right] \quad \forall n \in \mathbb{Z}.$

Then $\frac{\pi}{8}n^2$ and $\frac{\pi}{8}(n+N)^2$ differ by an integer multiple of 2π $\forall n \in \mathbb{Z}$.

Then for some integer k, $\frac{\pi}{8}(n+N)^2 - \frac{\pi}{8}n^2 = k2\pi \quad \forall n \in \mathbb{Z}$

$$\frac{\pi}{8}(n^2 + 2nN + N^2) - \frac{\pi}{8}n^2 = k2\pi \quad \forall n \in \mathbb{Z}$$

$$\frac{\pi}{8}n^2 + \frac{2\pi}{8}nN + \frac{\pi}{8}N^2 - \frac{\pi}{8}n^2 = k2\pi \quad \forall n \in \mathbb{Z}$$

$$\frac{2\pi}{8}nN + \frac{\pi}{8}N^2 = k2\pi \quad \forall n \in \mathbb{Z}$$

$$\frac{2n}{8}N + \frac{1}{8}N^2 = 2k \text{ (integer)} \quad \forall n \in \mathbb{Z}$$

$$n\frac{N}{8} + \frac{N^2}{16} = \text{integer} \quad \forall n \in \mathbb{Z}.$$

$\Rightarrow$ N is the smallest positive integer that is divisible by 8 such that N^2 is divisible by 16.

Let $N = 8m$, $m \in \mathbb{Z}$.

Then $N^2 = 64m^2$ must be divisible by 8.

Since $64m^2$ is divisible by 8 for any integer m, choose $m=1$ to get the smallest satisfactory N.

$$\Rightarrow N = 8 \cdot m = 8 \cdot 1 = \underline{\underline{8}} \longrightarrow$$

$x[n]$ is periodic with fundamental period $N = 8$.