ECE 3793
Test 1

Thursday, March 14, 2002
7:00 PM - 10:00 PM

Spring 2002 Name: SOLUTION

Dr. Havlicek Student Num:__________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **five** problems. Work any **four** of them. Only **four** problems will be graded. Below, you must circle the numbers of the **four** problems you wish to have graded. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts.** A continuous-time LTI system H has impulse response

$$h(t) = \frac{4}{3}\left[u(t) - u(t-1)\right] - \frac{1}{3}\delta(t-2).$$

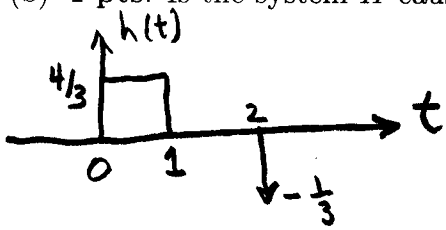
(a) **4 pts.** Is the system H memoryless? Justify your answer.

Note: This is text problem 2.22d), given on HW 3.

A continuous-time LTI system H is memoryless iff the impulse response is $K\delta(t)$, a constant times the Dirac Delta. That's not the case here.

Therefore, H is NOT MEMORYLESS

(b) **4 pts.** Is the system H causal? Justify your answer.



The system is causal, since $h(t) = 0 \ \forall \ t < 0$.

(c) **4 pts.** Is the system H BIBO stable? Justify your answer.

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \frac{4}{3}\left[u(t) - u(t-1)\right] + \frac{1}{3}\delta(t-2)\,dt$$

$$= \frac{4}{3}\int_{0}^{1} dt + \frac{1}{3}\int_{-\infty}^{\infty}\delta(t-2)\,dt$$

$$= \frac{4}{3}t\Big|_0^1 + \frac{1}{3} = \frac{4}{3}\left[1-0\right] + \frac{1}{3}$$

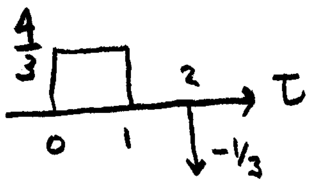
$$= \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < \infty.$$

Therefore H is BIBO stable.

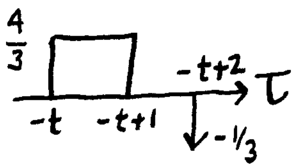
Problem 1, cont. . .

(d) **13 pts**. The system input is given by $x(t) = at + b$, where $a, b \in \mathbb{R}$ are constants. Find the system output $y(t)$.

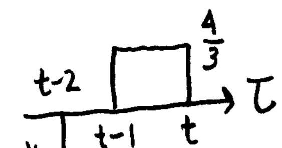
$h(\tau)$



$h(t+\tau)$



$h(t-\tau)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} (a\tau + b)\left\{\tfrac{4}{3}\left[u(t-\tau) - u(t-\tau-1)\right] - \tfrac{1}{3}\delta(t-\tau-2)\right\} d\tau$$

$$= \frac{4}{3}\int_{t-1}^{t} a\tau + b\,d\tau - \frac{1}{3}\int_{-\infty}^{\infty}(a\tau+b)\,\delta(t-\tau-2)\,d\tau$$

$$= \frac{4a}{3}\int_{t-1}^{t}\tau\,d\tau + \frac{4b}{3}\int_{t-1}^{t}d\tau - \frac{1}{3}\int_{-\infty}^{\infty}(a\tau+b)\,\delta([t-2]-\tau)\,d\tau$$

$$= \frac{4a}{3}\left[\tfrac{1}{2}\tau^2\right]_{t-1}^{t} + \frac{4b}{3}\tau\Big|_{t-1}^{t} - \frac{1}{3}\left[a\tau+b\right]_{\tau=t-2}$$

$$= \frac{4a}{6}\left[t^2 - (t-1)^2\right] + \frac{4b}{3}\left[t-(t-1)\right] - \frac{1}{3}\left[a(t-2)+b\right]$$

$$= \frac{4a}{6}\left[t^2 - (t^2-2t+1)\right] + \frac{4b}{3}[1] - \frac{1}{3}\left[at - 2a + b\right]$$

$$= \frac{4a}{6}\left[2t-1\right] + \frac{4b}{3} - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

$$= \frac{4}{3}at - \frac{2}{3}a + \frac{4}{3}b - \frac{1}{3}at + \frac{2}{3}a - \frac{1}{3}b$$

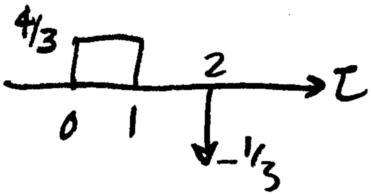
$$= \frac{4}{3}at - \frac{1}{3}at - \frac{2}{3}a + \frac{2}{3}a + \frac{4}{3}b - \frac{1}{3}b$$

$$= at + b = x(t).$$

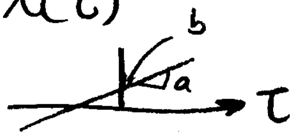
$$\boxed{y(t) = at + b}$$

1d) OTHER WAY:

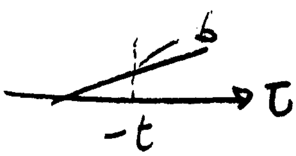
$h(\tau)$



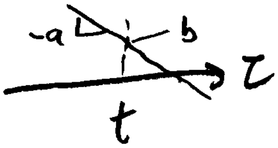
$x(\tau)$



$x(\tau+t)$



$x(t-\tau)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left\{ \tfrac{4}{3}[u(\tau) - u(\tau-1)] - \tfrac{1}{3}\delta(\tau-2) \right\} \left\{ a(t-\tau) + b \right\} d\tau$$

$$= \int_{-\infty}^{\infty} \tfrac{4}{3}[u(\tau) - u(\tau-1)]\,[a(t-\tau)+b]\, d\tau$$

$$- \tfrac{1}{3} \int_{-\infty}^{\infty} \delta(\tau-2)\,[a(t-\tau)+b]\, d\tau$$

$$= \tfrac{4}{3} \int_0^1 at - a\tau + b \, d\tau - \tfrac{1}{3}[a(t-2)+b]$$

$$= \tfrac{4}{3}\left[at\tau - \tfrac{1}{2}a\tau^2 + b\tau\right]_{\tau=0}^{1} - \tfrac{1}{3}[at - 2a + b]$$

$$= \tfrac{4}{3}\left[at - \tfrac{1}{2}a + b - 0\right] - \tfrac{1}{3}at + \tfrac{2}{3}a - \tfrac{1}{3}b$$

$$= \tfrac{4}{3}at - \tfrac{2}{3}a + \tfrac{4}{3}b - \tfrac{1}{3}at + \tfrac{2}{3}a - \tfrac{1}{3}b$$

$$= \underline{\underline{at + b}}$$

2. **25 pts**. The input $x[n]$ and output $y[n]$ of a discrete-time system H are related by

$$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k].$$

This is text problem 1.30 i), given on HW 3.

(a) **4 pts**. Is the system H memoryless? Justify your answer.

$y[0] = \sum_{k=-\infty}^{0} \left(\frac{1}{2}\right)^{-k} x[k]$, which depends on the past input $x[-1]$.

Therefore, the system is not memoryless.

(b) **4 pts**. Is the system H causal? Justify your answer.

$y[n]$ depends on inputs $x[k]$ for $k \in (-\infty, n]$.

→ Therefore $y[n]$ depends on the present input and on all past inputs, but not on future inputs.

→ Therefore the system is causal.

(c) **5 pts**. Is the system H linear? Justify your answer.

Let $y_1[n] = H\{x_1[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_1[k]$.

Let $y_2[n] = H\{x_2[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_2[k]$.

Let $x_3[n] = ax_1[n] + bx_2[n]$, where a and b are constants.

Then $y_3[n] = H\{x_3[n]\} = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_3[k]$

$$= \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} \{ax_1[k] + bx_2[k]\}$$

$$= a\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_1[k] + b\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x_2[k]$$

$= ay_1[n] + by_2[n]$ ✓ The system is linear.

Problem 2, cont...

(d) **6 pts**. Is the system H BIBO stable? Justify your answer.

Let $x[n]$ be a bounded input. Then $\exists B \in \mathbb{R}$, $B \geq 0$, such that $|x[n]| \leq B \ \forall n \in \mathbb{Z}$.

Then $|y[n]| = \left|\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k]\right| = \left(\frac{1}{2}\right)^{n} \left|\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k} x[k]\right|$

$$\leq \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} |x[k]| \leq \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} B = B\left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k}$$

$$= B\left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} 2^{k} = B\left(\tfrac{1}{2}\right)^{n} \lim_{A \to \infty} \frac{2^{-A} - 2^{n+1}}{1-2} = B(2)2^{n}\left(\tfrac{1}{2}\right)^{n} = 2B < \infty.$$

Then $y[n]$ is bounded $\rightarrow$ Every bounded input produces a bounded output. Therefore the system <u>IS</u> <u>STABLE</u>

(e) **6 pts**. Is the system H invertible? Justify your answer. If it is, then construct the inverse system. If it is not, then find two distinct input signals that produce the same output signal.

$$y[n] = \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{n-k} x[k] = \left(\tfrac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\tfrac{1}{2}\right)^{-k} x[k]$$

$$= \left(\tfrac{1}{2}\right)^{n} \Bigg[\underbrace{\left(\tfrac{1}{2}\right)^{-n} x[n]}_{k=n \text{ term}} + \underbrace{\sum_{k=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{-k} x[k]}_{\text{rest}} \Bigg]$$

$$= \underbrace{\left(\tfrac{1}{2}\right)^{n}\left(\tfrac{1}{2}\right)^{-n}}_{1} x[n] + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{n-1} \sum_{k=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{-k} x[k]$$

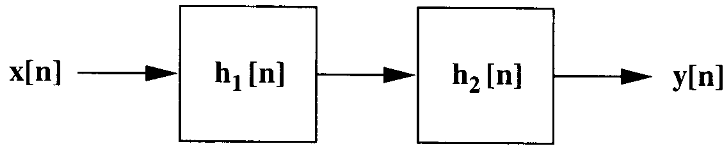
$$= x[n] + \tfrac{1}{2} \underbrace{\sum_{k=-\infty}^{n-1} \left(\tfrac{1}{2}\right)^{(n-1)-k} x[k]}_{y[n-1]} = x[n] + \tfrac{1}{2} y[n-1]$$

$\Rightarrow$ So $x[n] = y[n] - \frac{1}{2} y[n-1]$

Then the I/O relation for the inverse system is given by

$$\boxed{y[n] = x[n] - \tfrac{1}{2} x[n-1]}$$

3. **25 pts.** A discrete-time system H is formed by cascading two discrete-time LTI systems H_1 and H_2, as shown in the figure below:



The unit pulse response $h_1[n]$ is given by

$$h_1[n] = \sin 8n,$$

while the unit pulse response $h_2[n]$ is given by

$$h_2[n] = \alpha^n u[n],$$

where $|\alpha| < 1$. Find the system output $y[n]$ when the input is

$$x[n] = \delta[n] - \alpha\delta[n-1].$$

The overall system H has impulse response $h[n] = h_1[n] * h_2[n]$. But $h_1[n]$ is ugly and will probably make this a difficult calculation. For the same reason, we would like to avoid having to calculate $y_1[n] = h_1[n] * x[n]$.

Look for another approach:

$$y[n] = \left(x[n] * h_1[n]\right) * h_2[n]$$

$$= \underbrace{\left(x[n] * h_2[n]\right)}_{\text{maybe this will be easier.}} * h_1[n]$$

$$\left(x[n] * h_2[n]\right) = \left(\delta[n] - \alpha\delta[n-1]\right) * h_2[n]$$

$$= h_2[n] - \alpha h_2[n-1]$$

$$= \alpha^n u[n] - \alpha\left(\alpha^{n-1} u[n-1]\right) \longrightarrow$$

More Workspace for Problem 3...

$$(x[n] * h_2[n]) = \dots = \alpha^n u[n] - \alpha^n u[n-1]$$

$$= \alpha^n \left[u[n] - u[n-1] \right]$$

$$= \alpha^n \delta[n] = \alpha^0 \delta[n] = \delta[n]$$

So $y[n] = (x[n] * h_2[n]) * h_1[n]$

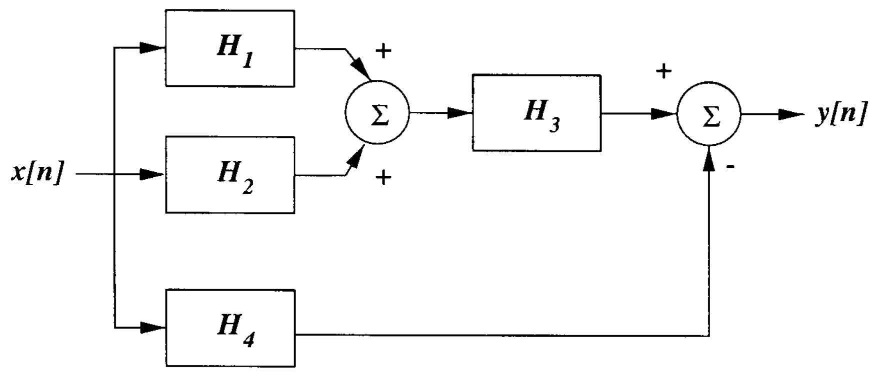
$$= \delta[n] * h_1[n]$$

$$= h_1[n]$$

$$= \sin 8n.$$

$$\boxed{y[n] = \sin 8n}$$

4. **25 pts.** The discrete-time system H is formed by connecting four LTI systems H_1, H_2, H_3, and H_4 as shown in the figure below.



The impulse responses of the four LTI systems H_1 through H_4 are given by

$$\begin{aligned} h_1[n] &= u[n], \\ h_2[n] &= u[n+2] - u[n], \\ h_3[n] &= \delta[n-2], \\ h_4[n] &= \left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

Find the **unit step response** $s[n]$ of the system H.

H is an LTI system with impulse response

$$h[n] = \left(h_1[n] + h_2[n]\right) * h_3[n] - h_4[n]$$

$$= \left(u[n] + u[n+2] - u[n]\right) * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n+2] * \delta[n-2] - \left(\tfrac{1}{2}\right)^n u[n]$$

$$= u[n] - \left(\tfrac{1}{2}\right)^n u[n] = \left[1 - \left(\tfrac{1}{2}\right)^n\right] u[n].$$

The step response is given by

$$s[n] = \sum_{k=-\infty}^{n} h[n] = \sum_{k=-\infty}^{n} \left[1 - \left(\tfrac{1}{2}\right)^k\right] u[k]$$

$$= \sum_{k=0}^{n} \left[1 - \left(\tfrac{1}{2}\right)^k\right]$$

→

if $n<0$, there are no terms in the sum and $s[n]=0$.

if $n \geq 0$, then

$$s[n] = \sum_{k=0}^{n} 1 - \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k$$

$$= n+1 - \frac{(1/2)^0 - (1/2)^{n+1}}{1 - 1/2}$$

$$= n+1 - \frac{1 - \frac{1}{2}(\frac{1}{2})^n}{1/2} = n+1-2+\left(\tfrac{1}{2}\right)^n$$

$$= n-1+\left(\tfrac{1}{2}\right)^n$$

All together:

$$\boxed{s[n] = \left[n-1+\left(\tfrac{1}{2}\right)^n\right] u[n]}$$

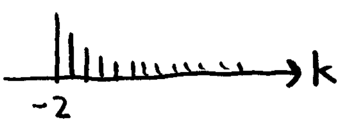
5. **25 pts.** Consider a discrete-time LTI system H with impulse response

$$h[n] = \left(\frac{1}{4}\right)^{-n} u[-n+2].$$

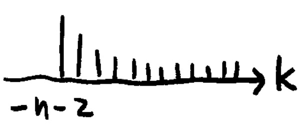
The system input is given by $x[n] = \left(\frac{1}{2}\right)^n u[n+2]$. Find the system output $y[n]$.

$h[k]$

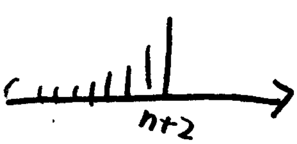
$x[k]$



$x[n+k]$



$x[n-k]$



I) $n+2 < 2 \longrightarrow n < 0$

$$y[n] = \sum_{k=-\infty}^{n+2} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{n+2} \left(\frac{1}{4}\right)^{-k} \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\frac{1}{4}\right)^{-k} \left(\frac{1}{2}\right)^{-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} \left(\frac{1}{8}\right)^{-k} = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{n+2} 8^k$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\right)^n \frac{8^{-A} - 8^{n+3}}{1-8} = 2^{-n}\left(\frac{1}{7}\right) 8^3 8^n$$

$$= \frac{8^3}{7} 2^{-n} 2^{3n} = \frac{512}{7} 2^{2n} = \frac{512}{7} 4^n$$

II) $n \geq 0$: $y[n] = \sum_{k=-\infty}^{2} \left(\frac{1}{4}\right)^{-k}\left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{2} 8^k$

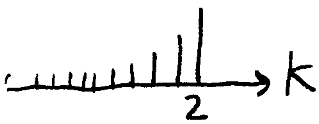
$$= \left(\frac{1}{2}\right)^n \lim_{A\to\infty} \frac{8^{-A} - 8^3}{1-8} = \left(\frac{1}{2}\right)^n \frac{8^3}{7} = \frac{512}{7}\left(\frac{1}{2}\right)^n$$

All Together:

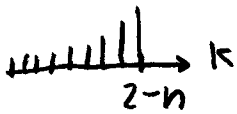
$$\boxed{y[n] = \begin{cases} \frac{512}{7} 4^n, & n < 0 \\ \frac{512}{7} \left(\frac{1}{2}\right)^n, & n \geq 0 \end{cases}}$$

(5) other way

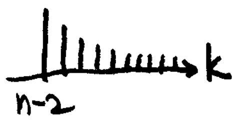
$h[k]$



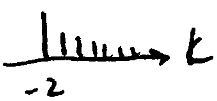
$h[n+k]$



$h[n-k]$



$x[k]$



I) $n-2 < -2$: $n < 0$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-2}^{\infty} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^{k-n}$$

$$= 4^n \sum_{k=-2}^{\infty} \left(\tfrac{1}{8}\right)^k = 4^n \frac{\left(\tfrac{1}{8}\right)^{-2} - 0}{7/8} = 4^n \cdot 8^2 \cdot 8 \cdot \tfrac{1}{7}$$

$$= \frac{8^3}{7} 4^n = \frac{512}{7} 4^n$$

II $n \geq 0$:

$$y[n] = \sum_{k=n-2}^{\infty} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{4}\right)^{k-n} = 4^n \sum_{k=n-2}^{\infty} \left(\tfrac{1}{8}\right)^k$$

$$= 4^n \left[8^{2-n} - 0\right] \frac{8}{7}$$

$$= 4^n 8^3 8^{-n} \frac{1}{7} = \frac{8^3}{7} 2^{2n} 2^{-3n}$$

$$= \frac{512}{7} 2^{-n} = \frac{512}{7} \left(\tfrac{1}{2}\right)^n$$

All Together:

$$y[n] = \begin{cases} \frac{512}{7} 4^n, & n < 0 \\ \frac{512}{7} \left(\tfrac{1}{2}\right)^n, & n \geq 0 \end{cases}$$

Formulas from Trigonometry:

$\sin^2 A + \cos^2 A = 1$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$\cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan \frac{A}{2} = \frac{\sin A}{1+\cos A}$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\cos A - \cos B = 2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(B-A)$

$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

$\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$

Differentiation Formulas:

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$

$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \left(-\frac{\pi}{2} < \sin^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, \ (0 < \cos^{-1} u < \pi)$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}, \left(-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right)$

$\frac{d}{dx}e^u = e^u \frac{du}{dx}$

$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}, \ a \neq 0, 1$

Integration Formulas:

Integration by parts: $\int u\,dv = uv - \int v\,du$

$\int \frac{du}{u} = \ln|u|$

$\int e^u\,du = e^u$

$\int a^u\,du = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$

$\int \sin u\,du = -\cos u$

$\int \cos u\,du = \sin u$

$\int \tan u\,du = -\ln\cos u$

$\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

$\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

$\int \tan^2 u\,du = \tan u - u$

$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right)$

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$

$\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u + \sqrt{u^2+a^2})$

$\int \frac{du}{\sqrt{u^2-a^2}} = \ln(u + \sqrt{u^2-a^2})$

$\int x \sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$

$\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$

$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$\int x \cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$

$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$

$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$

$\int x e^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$

$\int \ln x\,dx = x\ln x - x$

$\int x \ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$

Summation Formulas:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad \alpha \neq 1 \qquad \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \quad |a| < 1 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1$$

$$\sum_{k=0}^{n} k a^k = \frac{a\{1-(n+1)a^n + na^{n+1}\}}{(1-a)^2}$$

Signals:

$$\delta(t) = \tfrac{d}{dt}u(t) \qquad \delta[n] = u[n] - u[n-1]$$

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt \qquad \langle x_1[n], x_2[n]\rangle = \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$$

$$\mathcal{E}v\{x(t)\} = \tfrac{1}{2}\{x(t) + x(-t)\} \qquad \mathcal{O}d\{x(t)\} = \tfrac{1}{2}\{x(t) - x(-t)\}$$

Complex Exponential Signals:

$e^{jw_0 t}$	$e^{jw_0 n}$
Distinct signals for distinct w_0	Identical signals for values of w_0 separated by multiples of 2π
Periodic for any choice of w_0	Periodic only if $w_0/(2\pi) = m/N \in \mathbb{Q}$
Fundamental frequency w_0	Fundamental frequency w_0/m
Fundamental period: $w_0 = 0$: undefined $w_0 \neq 0$: $2\pi/w_0$	Fundamental period: $w_0 = 0$: one $w_0 \neq 0$: $2\pi m/w_0$

Systems:

System H is linear if $H\{ax_1(t) + bx_2(t)\} = aH\{x_1(t)\} + bH\{x_2(t)\}$.

System H is time invariant if $H\{x(t-t_0)\} = y(t-t_0)$.

System H is memoryless if the current output depends only on the current input.

System H is invertible if distinct inputs produce distinct outputs.

System H is invertible if an inverse system G exists which "undoes" the action of H.

System H is causal if the current output depends only on the past and present inputs.

LTI system H is causal iff $h(t) = 0 \;\forall\; t < 0$.

Bounded: $x(t)$ is bounded if $\exists\; B \in \mathbb{R}$, $B > 0$, such that $|x(t)| \leq B \;\forall\; t \in \mathbb{R}$.

System H is BIBO stable if every bounded input produces a bounded output.

LTI system H is BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \qquad h(t) = \tfrac{d}{dt}s(t)$$

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad h[n] = s[n] - s[n-1]$$