ECE 3793
Test 1

Wednesday, March 12, 2003
7:00 PM - 10:00 PM

Spring 2003

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

____

TOTAL (100):

____

1. **25 pts**. A discrete-time system H has input $x[n]$ and output $y[n]$ related by

$$y[n] = nx[n+1] - 1.$$

(a) **2 pts**. Is the system H memoryless? Justify your answer.

when $n=1$, $y[1] = x[2] - 1$, which depends on the input $x[2]$ from another time.
The system is not memoryless because the current output depends on an input from another time.

(b) **3 pts**. Is the system H causal? Justify your answer.

As shown in (a) above, when $n=1$ the output $y[1]$ depends on the future input $x[2]$. Therefore the system is NOT CAUSAL

(c) **5 pts**. Is the system H linear? Justify your answer.

- I guess "no" because of the $nx[n+1] \underline{-1}$ part.
- Try to construct a counterexample to linearity:

Let $x_1[n] = 1$. Then $y_1[n] = H\{x_1[n]\} = n-1$.

Let $x_2[n] = 0$. Then $y_2[n] = H\{x_2[n]\} = -1$.

So $y_1[n] + y_2[n] = n-1-1 = n-2$.

Let $x_3[n] = x_1[n] + x_2[n] = 1+0 = 1$ $(= x_1[n])$.

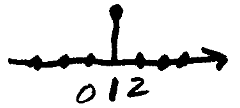
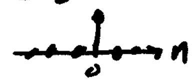
Then $y_3[n] = H\{x_3[n]\} = n-1 \neq y_1[n] + y_2[n]$.

Therefore the system is NOT LINEAR

Problem 1, cont...

(d) **5 pts.** Is the system H time invariant? Justify your answer.

I guess "no" because of $\underline{n}\, x[n+1] - 1$. Try to construct a counterexample:

Let $x_1[n] = \delta[n-1] =$ . Then $x_1[n+1] = \delta[n]$ $=$ 

So $y_1[n] = n\delta[n] - 1 = -1$.

Now shift $y_1[n]$ by 1 : $y_1[n-1] = -1$.

Let $x_2[n] = x_1[n-1] = \delta[n-2]$. Then $x_2[n+1] = \delta[n-1]$

Then $y_2[n] = n\delta[n-1] - 1 = \delta[n-1] - 1 \neq y_1[n-1]$. NOT TIME INVARIANT

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

I guess "no" because of $\underline{n}\, x[n+1] - 1$.

Let $x[n] = 1$. Then $x[n]$ is bounded, e.g., by $B = 2$ since $|x[n]| < 2 \ \forall n \in \mathbb{Z}$. Now $y[n] = H\{x[n]\} = n - 1$.
Then $|y[n]| = |n-1|$, and $\lim_{n\to\infty} |y[n]| \to \infty$ (grows without bound). Therefore, $\nexists B \in \mathbb{R}$ s.t. $|y[n]| < B \ \forall n \in \mathbb{Z}$.
The system is NOT BIBO STABLE since a bounded input produced an unbounded output.

(f) **5 pts.** Is the system H invertible? Justify your answer.

I guess "no" because of $\underline{n}\, x[n+1] - 1$.

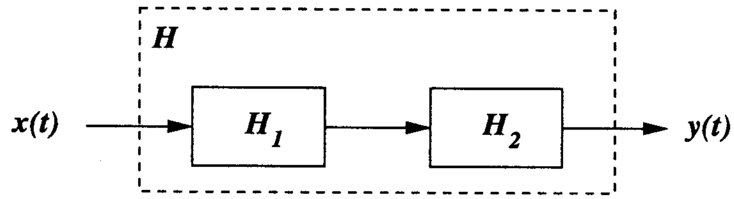
Let $x_1[n] = 0$. Then $y_1[n] = H\{x_1[n]\} = -1$.

Let $x_2[n] = \delta[n-1]$. Then $x_2[n+1] = \delta[n]$ and

$y_2[n] = H\{x_2[n]\} = n\delta[n] - 1 = -1 = y_1[n]$.

The system is NOT INVERTIBLE since the two distinct inputs $x_1[n]$ and $x_2[n]$ both produced the same output.

2. **25 pts.** A continuous-time system H is formed by cascading two continuous-time systems H_1 and H_2 as shown in the figure below:



System H_1 is LTI and has impulse response $h_1(t) = e^{-3t}u(t)$. System H_2 is linear and has input $x(t)$ and output $y(t)$ related by $y(t) = x(t) - x(t-2)$.

(a) **10 pts.** Is the overall system H an LTI system? Justify your answer.

→ H_1 is given as LTI. Therefore, if we can show that H_2 is also LTI, it will be sufficient to prove that H is LTI.

→ Now H_2 is given as linear, so all that is needed is to show that H_2 is also time invariant.

→ Consider H_2: $x(t) \to \boxed{H_2} \to y(t) = x(t) - x(t-2)$.

Let the input to H_2 be $x_1(t)$. Then the output of H_2 is $y_1(t) = H_2\{x_1(t)\} = x_1(t) - x_1(t-2)$.

Now shift $y_1(t)$ by t_0: $y_1(t-t_0) = x_1(t-t_0) - x_1(t-t_0-2)$.

Let $x_2(t) = x_1(t-t_0)$.

Then
$$\begin{aligned} y_2(t) = H_2\{x_2(t)\} &= x_2(t) - x_2(t-2) \\ &= x_1(t-t_0) - x_1(t-t_0-2) \\ &= y_1(t-t_0) \checkmark \end{aligned}$$

Then H_2 is LTI, which implies that <u>H IS LTI</u>.

(b) **5 pts.** If you answered *yes* in part (a), then find the impulse response $h(t)$ of the overall system. If instead you answered *no* in part (a), give the input-output relation for H.

The impulse response $h(t)$ is what comes out of H when the input is $\delta(t)$. In this case, the output of H_1 is $\delta(t) * h_1(t) = h_1(t)$. This is input to H_2, which gives the overall output $y(t) = h_1(t) - h_1(t-2)$.

So $h(t) = h_1(t) - h_1(t-2) = \underline{\underline{e^{-3t}u(t) - e^{-3(t-2)}u(t-2)}}$

(c) **5 pts.** Is the system H causal? Justify your answer.

We have from (b) that $h(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2)$.
The first term "turns on" at $t=0$ and the second term "turns on" at $t=2$.

Therefore, $h(t) = 0 \ \forall\, t<0$ and the system <u>IS CAUSAL</u>.

(d) **5 pts.** Is the system H BIBO stable? Justify your answer.

The system is LTI, so it is BIBO Stable iff $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \left| e^{-3t}u(t) - e^{-3(t-2)}u(t-2) \right| dt$$

$$\leq \int_{-\infty}^{\infty} e^{-3t}u(t)\,dt + \int_{-\infty}^{\infty} e^{-3t+6}u(t-2)\,dt$$

$$= \int_{0}^{\infty} e^{-3t}\,dt + e^{6}\int_{2}^{\infty} e^{-3t}\,dt = -\frac{1}{3}e^{-3t}\Big|_{t=0}^{\infty} + e^{6}\left[-\frac{1}{3}e^{-3t}\right]_{t=2}^{\infty}$$

$$= -\frac{1}{3}[0-1] - \frac{e^{6}}{3}\left[0 - e^{-6}\right]$$

$$= \frac{1}{3} + \frac{e^{6}e^{-6}}{3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < \infty \checkmark$$

Therefore the system <u>IS</u> <u>BIBO</u> <u>STABLE</u>.

3. **25 pts**. A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = t\{u(t) - u(t-2)\} = \begin{cases} t, & 0 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

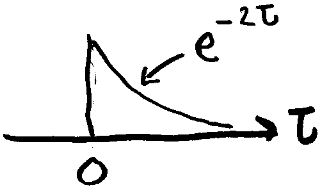
The system input is given by

$$x(t) = e^{-2t}u(t).$$

Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

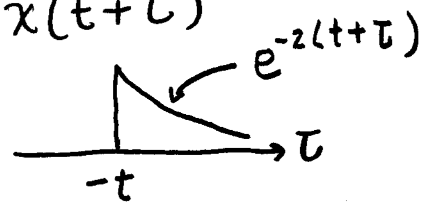
$x(\tau)$



Case I: $t<0$: $\underline{y(t) = 0}$ (no overlap)

Case II: $t \geq 0$ and $t<2$: $0 \le t < 2$:

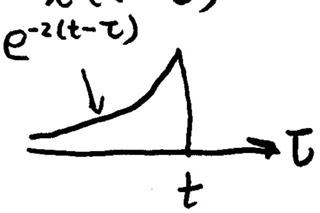
$x(t+\tau)$



$$y(t) = \int_0^t \tau e^{-2(t-\tau)} d\tau$$

$$= \int_0^t \tau e^{-2t} e^{2\tau} d\tau$$

$x(t-\tau)$



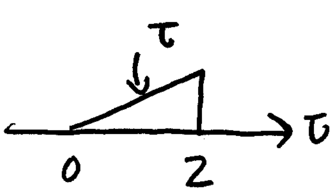
$$= e^{-2t} \int_0^t \tau e^{2\tau} d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \frac{1}{2}\right)\right]_{\tau=0}^{t}$$

$$= e^{-2t}\left[\frac{1}{2}e^{2t}\left(t-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)\right]$$

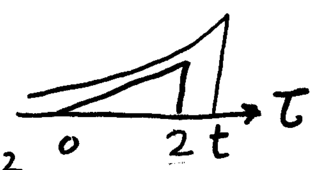
$$= e^{-2t}\left[\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}\right]$$

$$= \underline{\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}}.$$

$h(\tau)$



Case III: $t \geq 2$:



$$y(t) = \int_0^2 \tau e^{-2(t-\tau)} d\tau = \int_0^2 \tau e^{-2t} e^{2\tau} d\tau$$

$$= e^{-2t}\int_0^2 \tau e^{2\tau} d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \frac{1}{2}\right)\right]_{\tau=0}^{2}$$

$$= e^{-2t}\left[\frac{e^4}{2}\left(2-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)\right] = e^{-2t}\left[\frac{e^4}{2}\left(\frac{3}{2}\right) + \frac{1}{4}\right]$$

$$= \underline{\left[\frac{3e^4}{4} + \frac{1}{4}\right]e^{-2t}}$$

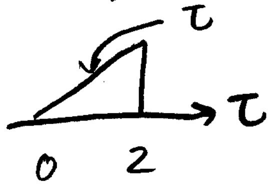
<u>All Together</u> :

$$y(t) = \begin{cases} 0, & t<0 \\ \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}, & 0 \le t < 2 \\ \left[\frac{3}{4}e^4 + \frac{1}{4}\right]e^{-2t}, & t \geq 2 \end{cases}$$

More Workspace for Problem 3... Other Way...

$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

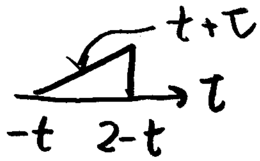
$h(\tau)$



case I: $t<0$: $y(t)=0$ (no overlap)

case II: $t \geq 0$ and $t-2<0$: $0 \leq t < 2$:

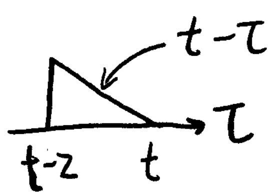
$h(t+\tau)$



$y(t) = \int_0^t e^{-2\tau}(t-\tau)\,d\tau$

$= t\int_0^t e^{-2\tau}d\tau - \int_0^t \tau e^{-2\tau}d\tau$

$h(t-\tau)$

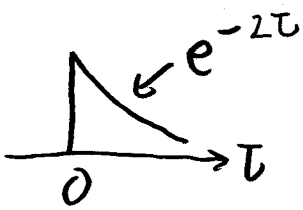


$= t(-\frac{1}{2})\left[e^{-2\tau}\right]_{\tau=0}^{t} - \left[\frac{e^{-2\tau}}{-2}(\tau+\frac{1}{2})\right]_{\tau=0}^{t}$

$= -\frac{1}{2}t\left[e^{-2t}-1\right] + \left[\frac{e^{-2t}}{2}(t+\frac{1}{2}) - \frac{1}{2}(\frac{1}{2})\right]$

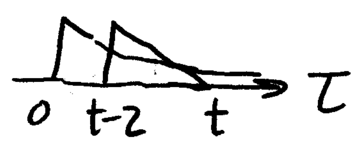
$x(\tau)$

$= \frac{1}{2}t - \frac{1}{2}te^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} - \frac{1}{4}$



$= \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$

case III: $t \geq 2$:



$y(t) = \int_{t-2}^{t} e^{-2\tau}(t-\tau)\,d\tau = t\int_{t-2}^{t} e^{-2\tau}d\tau - \int_{t-2}^{t} \tau e^{-2\tau}d\tau$

$= t\left[-\frac{1}{2}e^{-2\tau}\right]_{\tau=t-2}^{t} + \left[\frac{e^{-2\tau}}{2}(\tau+\frac{1}{2})\right]_{\tau=t-2}^{t}$

$= t\left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2(t-2)}\right] + \left[\frac{e^{-2t}}{2}(t+\frac{1}{2}) - \frac{e^{-2(t-2)}}{2}(t-2+\frac{1}{2})\right]$

$= -\frac{1}{2}te^{-2t} + \frac{1}{2}te^{-2t}e^{4} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}e^{4} + e^{-2t}e^{4} - \frac{1}{4}e^{-2t}e^{4}$

$= \left[\frac{3e^4}{4} + \frac{1}{4}\right]e^{-2t}$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{6}\right)^n u[n].$$

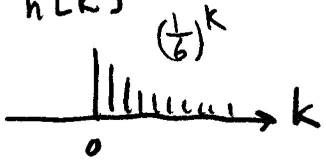
The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n-1].$$

Find the system output $y[n]$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

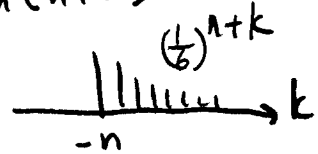
$h[k]$



case I) $n<1$: $y[n]=0$ (no overlap)

case II) $n \geq 1$:

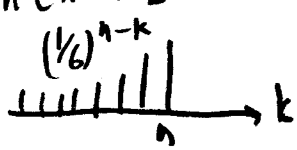
$h[n+k]$



$$y[n] = \sum_{k=1}^{n} x[k]\,h[n-k]$$

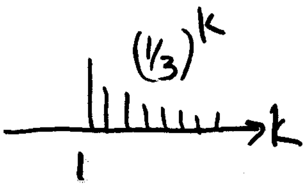
$$= \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{n-k}$$

$h[n-k]$



$$= \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{-k} = \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k 6^k$$

$x[k]$



$$= \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} \left(\frac{6}{3}\right)^k = \left(\frac{1}{6}\right)^n \sum_{k=1}^{n} 2^k$$

$$= \left(\frac{1}{6}\right)^n \frac{2^1 - 2^{n+1}}{1-2} = \left(\frac{1}{6}\right)^n \frac{2-2^{n+1}}{-1}$$

$$= \left(\frac{1}{2}\cdot\frac{1}{3}\right)^n \left[2^{n+1}-2\right] = \left(\frac{1}{3}\right)^n \left(\frac{1}{2}\right)^n 2\cdot 2^n - 2\left(\frac{1}{2}\cdot\frac{1}{3}\right)^n$$

$$= 2\left(\frac{1}{3}\right)^n - 2\left(\frac{1}{6}\right)^n = 2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right]$$

<u>All Together</u> ; $y[n] = \begin{cases} 0, & n<1 \\ 2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right], & n \geq 1 \end{cases}$

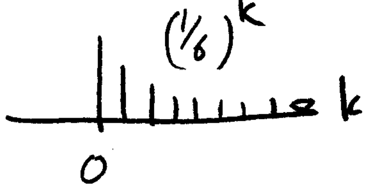
$$= 2\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right] u[n-1]$$

More Workspace for Problem 4...

Other way:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

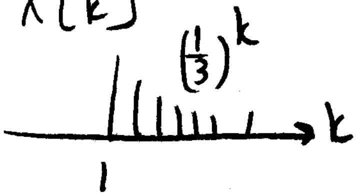
h[k]



case I) $n-1 < 0$; $n < 1$: $y[n] = 0$

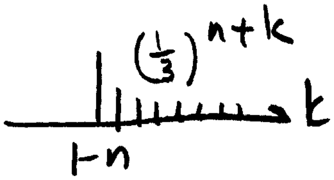
case II) $n \geq 1$:

x[k]



$$y[n] = \sum_{k=0}^{n-1} h[k]\, x[n-k] = \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{n-k}$$

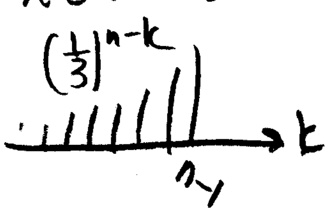
x[n+k]



$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k \left(\tfrac{1}{3}\right)^{-k} = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{6}\right)^k 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{3}{6}\right)^k = \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-1} \left(\tfrac{1}{2}\right)^k$$

x[n-k]



$$= \left(\tfrac{1}{3}\right)^n \frac{\left(\tfrac{1}{2}\right)^0 - \left(\tfrac{1}{2}\right)^n}{1-\tfrac{1}{2}} = \frac{\left(\tfrac{1}{3}\right)^n \left[1 - \left(\tfrac{1}{2}\right)^n\right]}{1/2}$$

$$= 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{2}\right)^n \left(\tfrac{1}{3}\right)^n\right] = 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right]$$

All Together:

$$y[n] = \begin{cases} 0, & n < 1 \\ 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right], & n \geq 1 \end{cases}$$

$$= 2\left[\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{6}\right)^n\right] u[n-1]$$