ECE 3793
Test 1

Friday, February 27, 2004
5:30 PM - 8:30 PM

Spring 2004 | Name: SOLUTION

Dr. Havlicek | Student Num:

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

1. **25 pts**. A discrete time system H has input $x[n]$ and output $y[n]$ related by

$$y[n] = nx[n+1] + 1.$$

(a) **2 pts**. Is the system H memoryless? Justify your answer.

When $n=2$, $y[2] = 2x[3]+1$, which depends on the input at time $n=3$ ... i.e, $y[2]$ depends on the input from another time ... so <u>NOT MEMORYLESS</u>.

(b) **3 pts**. Is the system H causal? Justify your answer.

As in (a), $y[2]$ depends on $x[3]$, which is a <u>future</u> <u>input</u>.

So H is <u>NOT</u> <u>CAUSAL</u>.

Problem 1, cont...

(c) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1[n] = H\{x_1[n]\} = n x_1[n+1] + 1$
Let $y_2[n] = H\{x_2[n]\} = n x_2[n+1] + 1$

Let $c_1, c_2 \in \mathbb{C}$ be constants.

Then $c_1 y_1[n] + c_2 y_2[n] = n c_1 x_1[n+1] + n c_2 x_2[n] + (c_1 + c_2)$.

Let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then $y_3[n] = H\{x_3[n]\} = n x_3[n+1] + 1 = n\left(c_1 x_1[n+1] + c_2 x_2[n+1]\right) + 1$
$= n c_1 x_1[n+1] + n c_2 x_2[n+1] + 1$.

Since $(c_1 + c_2) \neq 1$ in general, we have

$y_3[n] \neq c_1 y_1[n] + c_2 y_2[n] \Rightarrow$ <u>NOT LINEAR</u>

(d) **5 pts.** Is the system H time invariant? Justify your answer.

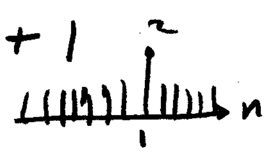
Let $x_1[n] = \delta[n-1]$.

Then $y_1[n] = H\{x_1[n]\} = n x_1[n+1] + 1 = n\delta[n] + 1 = 1$.

Then $y_1[n-1] = 1$.

Now let $x_2[n] = x_1[n-1] = \delta[n-2]$.

Then $y_2[n] = H\{x_2[n]\} = n x_2[n+1] + 1 = n\delta[n-1] + 1$
$= 1 \cdot \delta[n-1] + 1 = \delta[n-1] + 1$



Since $y_2[n] \neq y_1[n-1]$, H is <u>NOT</u> <u>TIME INVARIANT</u>

Problem 1, cont...

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

Let $x[n] = u[n-1]$, Then $x[n]$ is a bounded input, since $|x[n]| \leq 1 \;\forall n \in \mathbb{Z}$.

But $|y[n]| = |n\,x[n+1] + 1| = |n\,u[n] + 1|$

$$= \begin{cases} 1, & n \leq 0 \\ n+1, & n > 0 \end{cases}.$$

$\Rightarrow$ So $|y[n]|$ grows without bound as $n \to \infty$.

$\Rightarrow$ So $y[n]$ is not bounded.

$\rightarrow$ A bounded input produced an unbounded output.

$\Longrightarrow$ NOT BIBO STABLE.

(f) **5 pts.** Is the system H invertible? Justify your answer.

Let $x_1[n] = 0$. Then $y_1[n] = H\{x_1[n]\} = n\,x_1[n+1] + 1 = 1$.

Let $x_2[n] = \delta[n-1]$.

Then $y_2[n] = H\{x_2[n]\} = n\,x_2[n+1] + 1 = n\,\delta[n] + 1 = 1$.

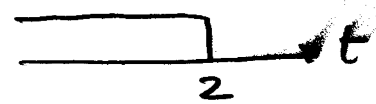
$\Rightarrow$ The two distinct inputs $x_1[n]$ and $x_2[n]$ both produced the same output.

$\Rightarrow$ The system is NOT INVERTIBLE

2. **25 pts.** A continuous time LTI system H has impulse response

$u(-t+2)$

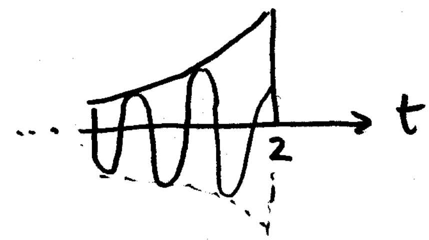
$$h(t) = e^{2t}\cos(2t)u(2-t).$$



(a) **4 pts.** Is the system H memoryless? Justify your answer.

A continuous-time LTI system is memoryless iff $h(t)$ is a constant times $\delta(t)$. That's <u>not</u> the case here

$h(t)$



So the system is <u>NOT MEMORYLESS</u>.

(b) **5 pts.** Is the system H causal? Justify your answer.

A continuous-time LTI system is causal iff $h(t)=0 \ \forall \ t<0$. For this system, we have

$$h(-\pi) = e^{-2\pi}\cos(-2\pi)\cdot 1 = e^{-2\pi} \neq 0.$$

$\Rightarrow$ <u>NOT</u> <u>CAUSAL</u>

(c) **5 pts.** Is the system H BIBO stable? Justify your answer.

A continuous-time LTI system is BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|dt < \infty$.

In this case,

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{2} |e^{2t}\cos(2t)|\,dt = \int_{-\infty}^{2} e^{2t}|\cos(2t)|dt$$

$$\leq \int_{-\infty}^{2} e^{2t}\cdot 1\,dt = \int_{-\infty}^{2} e^{2t}dt$$

$$= \lim_{A\to\infty} \frac{1}{2}e^{2t}\Big|_{t=-A}^{2} = \lim_{A\to\infty} \frac{1}{2}\left[e^{4} - e^{-2A}\right]$$

$$= \frac{1}{2}e^{4} < \infty.$$ Therefore $\int_{-\infty}^{\infty} |h(t)|dt < \infty$.

The system <u>IS</u> <u>BIBO</u> <u>STABLE</u>.

Problem 2, cont...

(d) **6 pts.** Find the system input-output relation that tells how the output $y(t)$ is related to the input $x(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{2} x(t-\tau)\, e^{2\tau} \cos(2\tau)\, d\tau$$

(e) **5 pts.** Find the system output $y(t)$ when the input is given by $x(t) = 1$.

In this case,

$$y(t) = \int_{-\infty}^{2} 1 \cdot e^{2\tau} \cos(2\tau)\, d\tau = \int_{-\infty}^{2} e^{2\tau} \cos(2\tau)\, d\tau$$

$$= \lim_{A\to\infty} \frac{1}{2^2+2^2} \left\{ e^{2\tau} \left[2\cos(2\tau) + 2\sin(2\tau) \right] \right\}_{\tau=-A}^{2}$$

$$= \frac{1}{8} \Big\{ 0 \left[2 \lim_{A\to\infty} \cos(-2A) + 2 \lim_{A\to\infty} \sin(-2A) \right] + e^{4} \left[2\cos(4) + 2\sin(4) \right] \Big\}$$

$$= \frac{1}{8} \cdot e^4 \cdot 2 \left[\cos(4) + \sin(4) \right] = \frac{1}{4} e^4 \left[\cos(4) + \sin(4) \right]$$

$$\boxed{y(t) = \frac{1}{4} e^4 \left[\cos(4) + \sin(4) \right] \approx -19.2519}$$

3. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(-\frac{1}{6}\right)^n u[n].$$

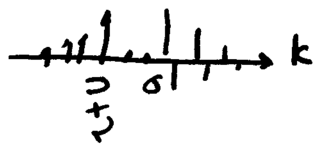
The system input is given by

$$x[n] = \left(\frac{1}{2}\right)^n u[n+2].$$

Find the system output $y[n]$.

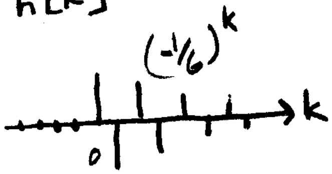
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

Case I) $n+2<0 \;:\; n<-2$

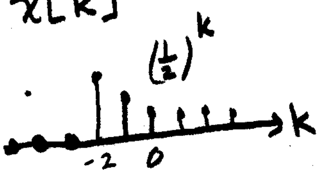


$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0.$$

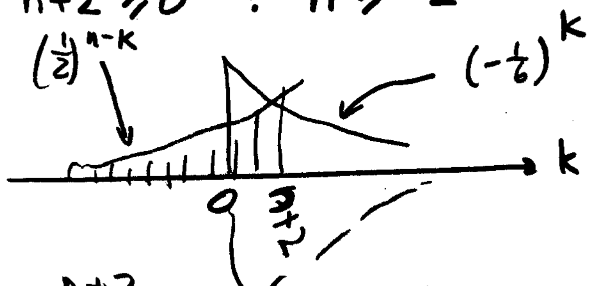
$h[k]$



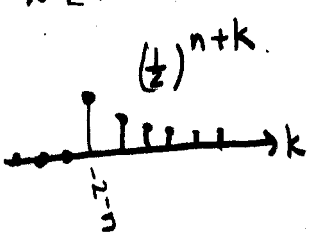
$x[k]$



Case II) $n+2 \geq 0 \;:\; n \geq -2$



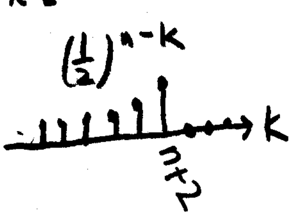
$x[n+k]$



$$y[n] = \sum_{k=0}^{n+2} \left(\frac{1}{2}\right)^{n-k} \left(-\frac{1}{6}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n+2} \left(\frac{1}{2}\right)^{-k} \left(-\frac{1}{6}\right)^k = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n+2} 2^k \left(-\frac{1}{6}\right)^k$$

$x[n-k]$



$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n+2} \left(-\frac{2}{6}\right)^k = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n+2} \left(-\frac{1}{3}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \frac{\left(-\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{n+3}}{1-\left(-\frac{1}{3}\right)} = \left(\frac{1}{2}\right)^n \frac{1-\left(-\frac{1}{3}\right)^n\left(-\frac{1}{3}\right)^3}{4/3}$$

$$= \left[\left(\frac{1}{2}\right)^n + \frac{1}{27}\left(\frac{1}{2}\right)^n\left(-\frac{1}{3}\right)^n\right]\frac{3}{4} = 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^n + \frac{3}{108}\left(\frac{1}{2}\cdot -\frac{1}{3}\right)^n$$

$$= 3\left(\frac{1}{2}\right)^{n+2} + \frac{3}{108}\left(-\frac{1}{6}\right)^n = 3\left(\frac{1}{2}\right)^{n+2} + \frac{3}{3\cdot 6\cdot 6}\left(-\frac{1}{6}\right)^n$$

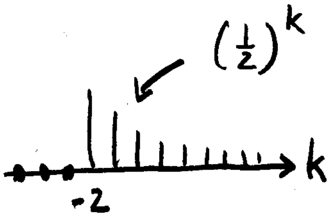
$$= 3\left(\frac{1}{2}\right)^{n+2} + \left(-\frac{1}{6}\right)\left(-\frac{1}{6}\right)\left(-\frac{1}{6}\right)^n = 3\left(\frac{1}{2}\right)^{n+2} + \left(-\frac{1}{6}\right)^n.$$

All Together: $\underline{y[n] = \left[3\left(\frac{1}{2}\right)^{n+2} + \left(-\frac{1}{6}\right)^n\right] u[n+2]}$

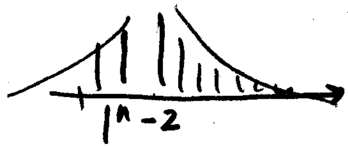
OTHER WAY

More Workspace for Problem 3... $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

$x[k]$

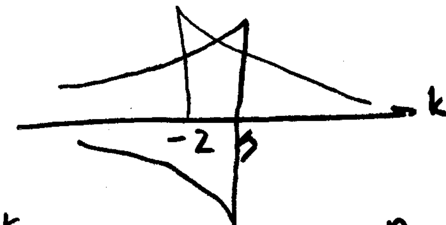


case I) $n<-2$:

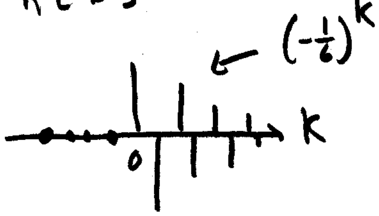


$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0.$

case II) $n \geq -2$:



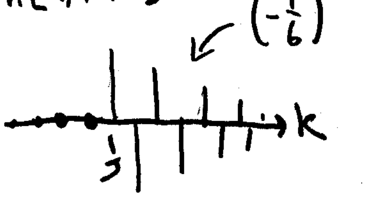
$h[k]$



$$y[n] = \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^k \left(-\tfrac{1}{6}\right)^{n-k} = \left(-\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^k \left(-\tfrac{1}{6}\right)^{-k}$$

$$= \left(-\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} \left(\tfrac{1}{2}\right)^k (-6)^k = \left(-\tfrac{1}{6}\right)^n \sum_{k=-2}^{n} (-3)^k$$

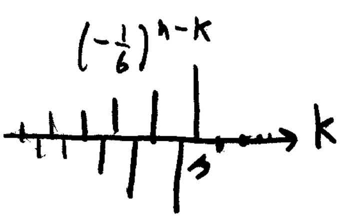
$h[n+k]$



$$= \left(-\tfrac{1}{6}\right)^n \frac{(-3)^{-2} - (-3)^{n+1}}{1-(-3)} = \tfrac{1}{4}\left(-\tfrac{1}{6}\right)^n \left[\left(-\tfrac{1}{3}\right)^2 - (-3)(-3)^n\right]$$

$$= \tfrac{1}{4}\left(-\tfrac{1}{6}\right)^n \left[\left(-\tfrac{1}{3}\right)^2 + 3(-3)^n\right]$$

$h[n-k]$



$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{6}\right)^n + \tfrac{3}{4}\left(-\tfrac{-3}{6}\right)^n$$

$$= \left(-\tfrac{1}{6}\right)^2 \left(-\tfrac{1}{6}\right)^n + \tfrac{3}{4}\left(\tfrac{1}{2}\right)^n$$

$$= 3\left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^n + \left(-\tfrac{1}{6}\right)^{n+2}$$

$$= 3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}$$

All Together $y[n] = \begin{cases} 0, & n<-2 \\ 3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}, & n \geq -2 \end{cases}$

$$= \left[3\left(\tfrac{1}{2}\right)^{n+2} + \left(-\tfrac{1}{6}\right)^{n+2}\right] u[n+2]$$

4. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = u(t) - u(t+2) = \begin{cases} -1, & -2 \le t \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

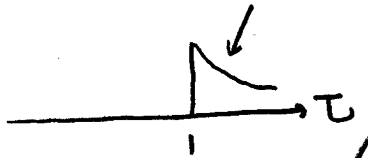
The system input is given by

$$x(t) = e^{-5(t-1)}u(t-1).$$

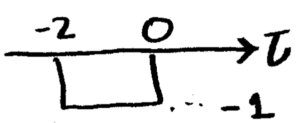
Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

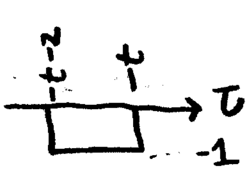
$x(\tau)$ $e^{-5(\tau-1)}$



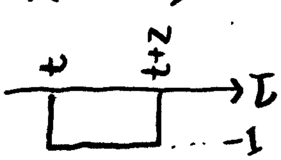
$h(\tau)$



$h(t+\tau)$



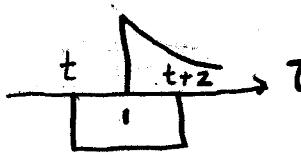
$h(t-\tau)$



case I) $t+2<1$: $t<-1$

$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

case II) $t<1$ and $t+2 \geq 1$: $t<1$ and $-1 \le t$: $-1 \le t < 1$



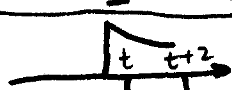
$$y(t) = \int_{1}^{t+2} (-1) e^{-5(\tau-1)}\, d\tau$$

$$= -\int_{1}^{t+2} e^{-5\tau} e^{5}\, d\tau = -e^{5}\int_{1}^{t+2} e^{-5\tau}\, d\tau$$

$$= -e^{5}\left(-\tfrac{1}{5}\right)\left[e^{-5\tau}\right]_{\tau=1}^{t+2} = \frac{e^{5}}{5}\left[e^{-5(t+2)} - e^{-5}\right]$$

$$= \tfrac{1}{5}\left[e^{5}e^{-5t}e^{-10} - 1\right] = \tfrac{1}{5}\left[e^{-5t}e^{-5} - 1\right] = \tfrac{1}{5}\left[e^{-5(t+1)} - 1\right]$$

case III) $t \geq 1$:



$$y(t) = -\int_{t}^{t+2} e^{-5\tau} e^{5}\, d\tau = -e^{5}\int_{t}^{t+2} e^{-5\tau}\, d\tau = \frac{e^{5}}{5}\left[e^{-5\tau}\right]_{\tau=t}^{t+2}$$

$$= \frac{e^{5}}{5}\left[e^{-5(t+2)} - e^{-5t}\right] = \frac{e^{5}}{5} e^{-5t}\left[e^{-10} - 1\right] = \tfrac{1}{5}e^{-5t}\left[e^{-5} - e^{5}\right]$$

$$= \tfrac{1}{5}\left[e^{-5t}e^{-5} - e^{-5t}e^{5}\right] = \tfrac{1}{5}\left[e^{-5(t+1)} - e^{-5(t-1)}\right]$$

All Together:

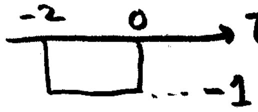
$$y(t) = \begin{cases} 0, & t<-1 \\ \tfrac{1}{5}\left[e^{-5(t+1)} - 1\right], & -1 \le t < 1 \\ \tfrac{1}{5}\left[e^{-5(t+1)} - e^{-5(t-1)}\right], & t \geq 1 \end{cases}$$

More Workspace for Problem 4...

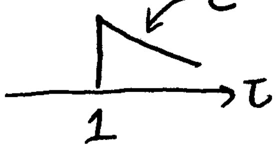
OTHER WAY

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

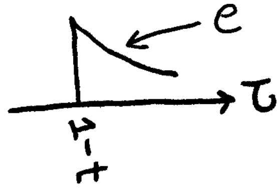
$h(\tau)$



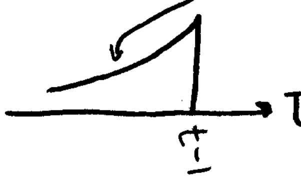
$x(\tau)$ $e^{-5(\tau-1)}$



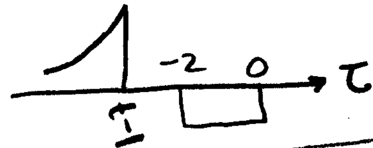
$x(t+\tau)$ $e^{-5(t+\tau-1)}$



$x(t-\tau)$ $e^{-5(t-\tau-1)}$

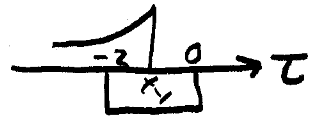


case I) $t-1 < -2$: $t < -1$:



$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

case II) $-2 \le t-1 < 0$: $-1 \le t < 1$:



$$y(t) = \int_{-2}^{t-1} (-1)\, e^{-5(t-\tau-1)}\, d\tau$$

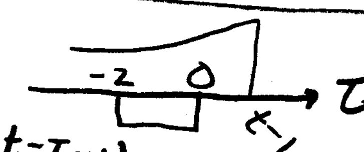
$$= -\int_{-2}^{t-1} e^{-5t} e^{5\tau} e^{5}\, d\tau = -e^{-5(t-1)} \int_{-2}^{t-1} e^{5\tau}\, d\tau$$

$$= -\frac{e^{-5(t-1)}}{5}\left[e^{5\tau}\right]_{\tau=-2}^{t-1} = -\frac{1}{5} e^{-5(t-1)}\left[e^{5(t-1)} - e^{-10}\right]$$

$$= \frac{1}{5}\left[e^{-5t} e^{5} e^{-10} - e^{-5(t-1)} e^{5(t-1)}\right]$$

$$= \frac{1}{5}\left[e^{-5t} e^{-5} - 1\right] = \frac{1}{5}\left[e^{-5(t+1)} - 1\right]$$

Case III) $t \ge 1$



$$y(t) = \int_{-2}^{0} (-1)\, e^{-5(t-\tau-1)}\, d\tau = -e^{-5(t-1)} \int_{-2}^{0} e^{5\tau}\, d\tau$$

$$= -\frac{1}{5} e^{-5(t-1)}\left[e^{5\tau}\right]_{\tau=-2}^{0} = -\frac{1}{5} e^{-5(t-1)}\left[1 - e^{-10}\right]$$

$$= \frac{1}{5} e^{-5(t-1)}\left[e^{-10} - 1\right] = \frac{1}{5}\left[e^{-5t} e^{5} e^{-10} - e^{-5(t-1)}\right]$$

$$= \frac{1}{5}\left[e^{-5t} e^{-5} - e^{-5(t-1)}\right] = \frac{1}{5}\left[e^{-5(t+1)} - e^{-5(t-1)}\right]$$

All together

$$y(t) = \begin{cases} 0 & , \ t < -1 \\ \frac{1}{5}\left[e^{-5(t+1)} - 1\right] & , \ -1 \le t < 1 \\ \frac{1}{5}\left[e^{-5(t+1)} - e^{-5(t-1)}\right] & , \ t \ge 1 \end{cases}$$