ECE 3793
Test 1

Friday, March 11, 2005
6:00 PM - 9:00 PM

Spring 2005 | Name: SOLUTION

Dr. Havlicek | Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________________ Date: ____________________

1. **25 pts.** A continuous-time system H has input $x(t)$ and output $y(t)$ related by

$$y(t) = x(-2t + 1) + 1.$$

(a) **2 pts.** Is the system H memoryless? Justify your answer.

When $t=-2$, the output is $y(-2) = x(5) + 1$.

Thus, the output when $t=-2$ depends on the future value $x(5)$ of the input at $t=5$. Since the output at $t=-2$ depends on the input from a different time, H is NOT memoryless.

(b) **3 pts.** Is the system H causal? Justify your answer.

As shown in part (a), when $t=-2$ the output depends on the future input $x(5)$. Therefore, the system is NOT causal.

Problem 1, cont...

(c) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = x_1(-2t+1) + 1$

Let $y_2(t) = H\{x_2(t)\} = x_2(-2t+1) + 1$

Let $x_3(t) = a x_1(t) + b x_2(t)$, where $a, b \in \mathbb{C}$ are constants. Then

$$y_3(t) = H\{x_3(t)\} = x_3(-2t+1) + 1$$

$$= a x_1(-2t+1) + b x_2(-2t+1) + \underline{\underline{1}}.$$

But $a y_1(t) + b y_2(t) = a x_1(-2t+1) + b x_2(-2t+1) + \underline{\underline{(a+b)}}$

Since $y_3(t) \neq a y_1(t) + b y_2(t)$, the system is <u>NOT</u> linear.

(d) **5 pts.** Is the system H time invariant? Justify your answer.

Let $y_1(t) = H\{x_1(t)\} = x_1(-2t+1) + 1$. To find $y_1(t-t_0)$, use the "rule of thumb" on p. 1.86 of the course notes:

1st xform: $t \mapsto -2t+1$; 2nd xform: $t \mapsto t - t_0$.

$$\underline{\underline{y_1(t-t_0)}} = x_1(-2\theta+1)+1 \Big|_{\theta = t - t_0} = x_1(-2t+2t_0+1)+1.$$

Now let $x_2(t) = x_1(t-t_0)$. Then $y_2(t) = H\{x_2(t)\} = x_2(-2t+1)+1$.

Use the "rule of thumb" again with:

1st xform: $t \mapsto t - t_0$; 2nd xform: $t \mapsto -2t+1$.

$$y_2(t) = x_1(\theta - t_0)+1 \Big|_{\theta = -2t+1} = x_1(-2t+1-t_0)+1.$$

The system is <u>NOT</u> time invariant because

$y_2(t) \neq y_1(t-t_0)$.

Problem 1, cont...

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

Let $x(t)$ be a bounded input. Then $\exists\, B \in \mathbb{R}, B>0$, s.t. $|x(t)| < B \ \forall\, t \in \mathbb{R}$. Let $y(t) = H\{x(t)\}$.

Then $|y(t)| = |x(-2t+1)+1| \leq |x(-2t+1)| + 1$

$< B+1$.

So $y(t)$ is a banded signal with bound $B+1$.

Since every bounded input produces a bounded output, the system <u>is</u> <u>BIBO</u> <u>stable</u>.

(f) **5 pts.** Is the system H invertible? Justify your answer.

By interchanging names, we obtain a relation between the input and output of the inverse system (if it exists). If we can solve this for the output of the inverse system, then it will show that H is invertible and it will give us the inverse system I/O equation. For H, $y(t) = x(-2t+1)+1$, so for the inverse system,

$$x(t) = y(-2t+1)+1$$

$$y(-2t+1) = x(t) - 1 \quad (*)$$

Let $\theta = -2t+1$

Then $\theta - 1 = -2t$

$$t = -\tfrac{1}{2}\theta + \tfrac{1}{2} \quad (**)$$

plug $(**)$ into $(*)$:

$$y(\theta) = x(-\tfrac{1}{2}\theta + \tfrac{1}{2}) - 1$$

change θ to t:

$$\boxed{y(t) = x(-\tfrac{1}{2}t + \tfrac{1}{2}) - 1}$$

The system is invertible and this is the I/O relation of the inverse system.

2. **25 pts.** A discrete-time LTI system H has input $x[n]$ and output $y[n]$ related by

$$y[n] = 2x[n+1] - x[n-1].$$

(a) **5 pts.** Find the impulse response $h[n]$.

$$y[n] = 2x[n+1] - x[n-1]$$

$$= 2x[n] * \delta[n+1] - x[n] * \delta[n-1]$$

$$= x[n] * \left(2\delta[n+1] - \delta[n-1]\right)$$

$$\Rightarrow h[n] = 2\delta[n+1] - \delta[n-1]$$

Problem 2, cont...

(b) **5 pts.** Is the system H memoryless? Justify your answer.

A discrete-time LTI system is memoryless iff $\exists K \in \mathbb{C}$ s.t. $h[n] = K\delta[n]$. That's not the case here, since $h[n] = 2\delta[n+1] - \delta[n-1]$.

Therefore, this system is <u>not</u> memoryless.

(c) **5 pts.** Is the system H causal? Justify your answer.

When $n=-1$, we have $h[-1] = 2\delta[0] - \delta[-2] = 2 \neq 0$.

Since $h[n]$ is not zero $\forall\, n<0$, the system is <u>NOT</u> <u>causal</u>.

(d) **5 pts.** Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |2\delta[n+1] - \delta[n-1]|$$

$$\leq \sum_{n=-\infty}^{\infty} 2|\delta[n+1]| + |\delta[n-1]|$$

$$= 2\sum_{n=-\infty}^{\infty} |\delta[n+1]| + \sum_{n=-\infty}^{\infty} |\delta[n-1]|$$

$$= 2\cdot 1 + 1 = 3 < \infty.$$

Therefore the system <u>IS</u> BIBO stable.

(e) **5 pts.** Is the system invertible? Justify your answer.

Use the same strategy as in problem 1(f).

For the inverse system, $x[n] = 2y[n+1] - y[n-1]$

Let $m = n+1$; $n = m-1$: $x[m-1] = 2y[m] - y[m-2]$

write "n" again: $x[n-1] = 2y[n] - y[n-2]$

$$2y[n] = x[n-1] + y[n-2]$$

(let $n = m-2$) $\quad y[n] = \frac{1}{2}x[n-1] + \frac{1}{2}y[n-2]. \quad (*)$

Eq. $(*)$ implies that $y[n-2] = \frac{1}{2}x[n-3] + \frac{1}{2}y[n-4]. \quad (**)$

Plug $(**)$ into $(*)$ and continue in a similar manner:

$$\begin{aligned} y[n] &= \tfrac{1}{2}x[n-1] + \tfrac{1}{2}\left(\tfrac{1}{2}x[n-3] + \tfrac{1}{2}y[n-4]\right) \\ &= \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-3] + \tfrac{1}{4}y[n-4] \\ &= \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-3] + \tfrac{1}{4}\left(\tfrac{1}{2}x[n-5] + \tfrac{1}{2}y[n-6]\right) \\ &= \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-3] + \tfrac{1}{8}x[n-5] + \tfrac{1}{8}y[n-6] \ldots \end{aligned}$$

Now the pattern is apparent and we have

$$\begin{aligned} y[n] &= \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-3] + \tfrac{1}{8}x[n-5] + \tfrac{1}{16}x[n-7] + \cdots \\ &= \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k+1} x[n-2k-1] \end{aligned}$$

The system is invertible and this is the I/O relation of the inverse system.

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-3t}u(t).$$

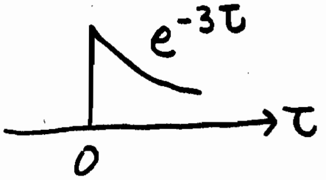
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The system input is given by

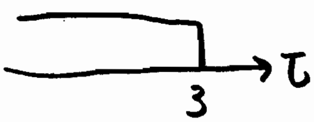
$$x(t) = u(-t+3).$$

Find the system output $y(t)$.

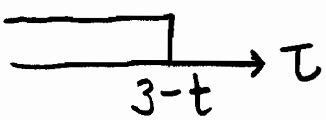
$h(\tau)$



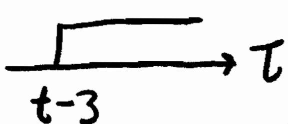
$x(\tau)$



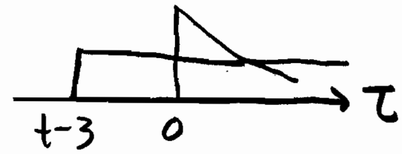
$x(\tau - -t) = x(t+\tau)$



$x(t-\tau)$



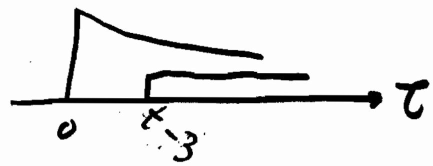
I) $t-3<0$; $t<3$:



$$y(t) = \int_0^{\infty} 1 \cdot e^{-3\tau}\, d\tau = -\frac{1}{3}\left[e^{-3\tau}\right]_{\tau=0}^{\infty}$$

$$= -\frac{1}{3}\lim_{A\to\infty}\left[e^{-3A} - 1\right] = \frac{1}{3}$$

II) $t-3 \geq 0$: $t \geq 3$:



$$y(t) = \int_{t-3}^{\infty} e^{-3\tau}\, d\tau$$

$$= -\frac{1}{3}\left[e^{-3\tau}\right]_{\tau=t-3}^{\infty}$$

$$= -\frac{1}{3}\lim_{A\to\infty}\left[e^{-3A} - e^{-3(t-3)}\right]$$

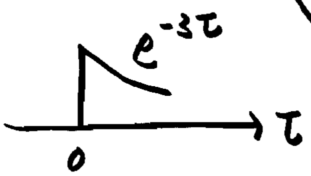
$$= -\frac{1}{3}\left[0 - e^{-3(t-3)}\right] = \frac{1}{3}e^{-3(t-3)}$$

All Together:

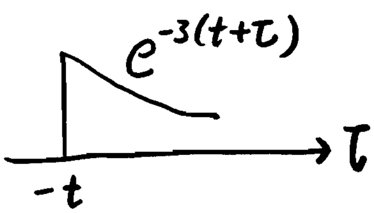
$$y(t) = \begin{cases} \frac{1}{3}, & t<3 \\ \frac{1}{3}e^{-3(t-3)}, & t \geq 3 \end{cases}$$

OTHER WAY: $y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$

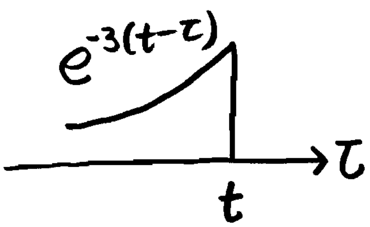
$h(\tau)$



$h(\tau - -t) = h(t+\tau)$

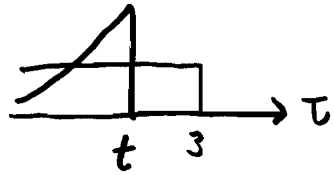


$h(t-\tau)$



$x(\tau)$

I) $t < 3$:

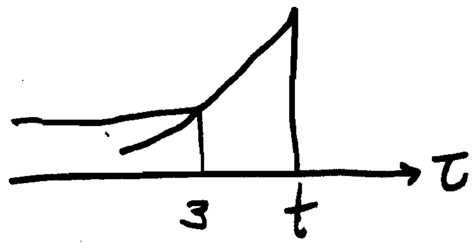


$$y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)} \cdot 1\, d\tau = e^{-3t} \int_{-\infty}^{t} e^{3\tau} d\tau$$

$$= \tfrac{1}{3} e^{-3t} \left[e^{3\tau} \right]_{\tau=-\infty}^{t} = \tfrac{1}{3} e^{-3t} \left[e^{3t} - 0 \right]$$

$$= \tfrac{1}{3}$$

II) $t \geq 3$:



$$y(t) = \int_{-\infty}^{3} e^{-3(t-\tau)} \cdot 1\, d\tau = e^{-3t} \int_{-\infty}^{3} e^{3\tau} d\tau$$

$$= \tfrac{1}{3} e^{-3t} \left[e^{3\tau} \right]_{\tau=-\infty}^{3}$$

$$= \tfrac{1}{3} e^{-3t} \left[e^{9} - 0 \right] = \tfrac{1}{3} e^{-3t} e^{9} = \tfrac{1}{3} e^{-3t+9}$$

$$= \tfrac{1}{3} e^{-3(t-3)}$$

All Together: $y(t) = \begin{cases} \frac{1}{3}, & t < 3 \\ \frac{1}{3} e^{-3(t-3)}, & t \geq 3 \end{cases}$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{3}\right)^n u[n+1].$$

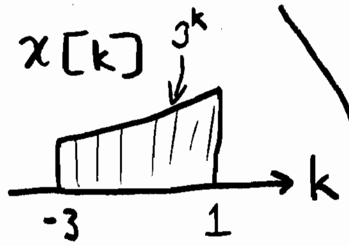
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

The system input is given by

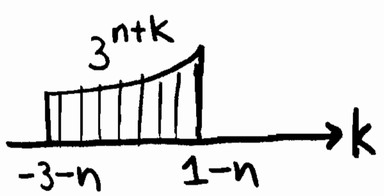
$$x[n] = 3^n\left\{u[n+3] - u[n-2]\right\} = \begin{cases} 3^n, & -3 \le n \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the system output $y[n]$.

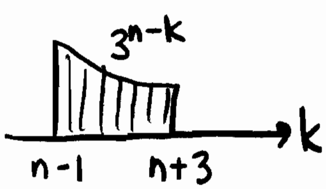
$x[k]$



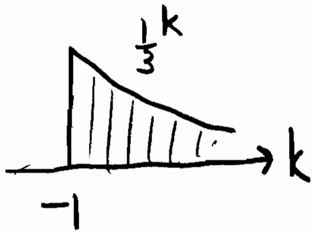
$x[k - -n] = x[n+k]$



$x[n-k]$



$h[k]$

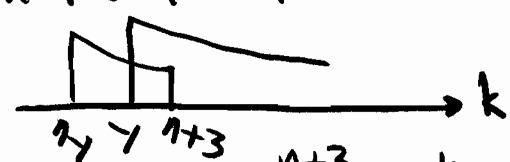


I) $n+3 < -1 \;:\; n < -4$

$h[k]x[n-k] = 0 \Rightarrow y[n] = 0$

II) $n+3 \geq -1$ and $n-1 < -1 \;:\; n \geq -4$ and $n < 0$

$-4 \le n < 0$



$$y[n] = \sum_{k=-1}^{n+3} h[k]\,x[n-k] = \sum_{k=-1}^{n+3} \left(\tfrac{1}{3}\right)^k (3)^{n-k}$$

$$= 3^n \sum_{k=-1}^{n+3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{3}\right)^k = 3^n \sum_{k=-1}^{n+3} \left(\tfrac{1}{9}\right)^k$$

$$= 3^n \frac{\left(\frac{1}{9}\right)^{-1} - \left(\frac{1}{9}\right)^{n+4}}{1 - \frac{1}{9}} = 3^n \frac{9 - \left(\frac{1}{9}\right)^{n+4}}{8/9}$$

$$= \frac{9}{8}\left[9 \cdot 3^n - \left(\tfrac{1}{3}\right)^{2n+8} \cdot 3^n\right] = \frac{9}{8}\left[3^{n+2} - \left(\tfrac{1}{3}\right)^{2n+8}\left(\tfrac{1}{3}\right)^{-n}\right]$$

$$= \frac{9}{8}\left[3^{n+2} - \left(\tfrac{1}{3}\right)^{2n-n+8}\right] = \frac{9}{8}\left[3^{n+2} - \left(\tfrac{1}{3}\right)^{n+8}\right]$$

$$= \frac{1}{8}\left[3^2 3^{n+2} - 3^2\left(\tfrac{1}{3}\right)^{n+8}\right] = \frac{1}{8}\left[3^{n+4} - \left(\tfrac{1}{3}\right)^{-2}\left(\tfrac{1}{3}\right)^{n+8}\right]$$

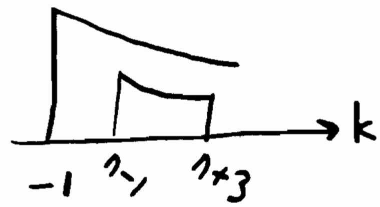
$$= \frac{1}{8} 3^{n+4} - \frac{1}{8}\left(\tfrac{1}{3}\right)^{n+6}$$

$$= \frac{81}{8} 3^n - \frac{1}{8 \cdot 729}\left(\tfrac{1}{3}\right)^n = \frac{81}{8} 3^n - \frac{1}{5832}\left(\tfrac{1}{3}\right)^n$$

$\longrightarrow$ cont.

More Workspace for Problem 4. . .

III) $n-1 \geq -1$: $n \geq 0$;



$$y[n] = \sum_{k=n-1}^{n+3} h[k]x[n-k] = \sum_{k=n-1}^{n+3} \left(\tfrac{1}{3}\right)^k (3)^{n-k} = 3^n \sum_{k=n-1}^{n+3} \left(\tfrac{1}{3}\right)^k (3)^{-k}$$

$$= 3^n \sum_{k=n-1}^{n+3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{3}\right)^k = 3^n \sum_{k=n-1}^{n+3} \left(\tfrac{1}{3}\right)^{2k} = 3^n \sum_{k=n-1}^{n+3} \left(\tfrac{1}{9}\right)^k$$

$$= 3^n \frac{\left(\frac{1}{9}\right)^{n-1} - \left(\frac{1}{9}\right)^{n+4}}{1 - 1/9} = 3^n \left(\tfrac{9}{8}\right) \left[\left(\tfrac{1}{9}\right)^n \cdot 9 - \left(\tfrac{1}{9}\right)^n \left(\tfrac{1}{9}\right)^4\right]$$

$$= \frac{1}{8}\left[\left(\tfrac{3}{9}\right)^n \cdot 9^2 - \left(\tfrac{3}{9}\right)^n \left(\tfrac{1}{9}\right)^3\right] = \frac{1}{8}\left[81\left(\tfrac{1}{3}\right)^n - \left(\tfrac{1}{3}\right)^n \tfrac{1}{729}\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{81}{8} - \frac{1}{8 \cdot 729}\right] = \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{8}\right)\left[81 - \tfrac{1}{729}\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{81}{8} - \frac{1}{5832}\right]$$

All Together:

$$y[n] = \begin{cases} 0 & , \quad n < -4 \\ \frac{81}{8} 3^n - \frac{1}{5832}\left(\frac{1}{3}\right)^n & , \quad -4 \leq n < 0 \\ \left[\frac{81}{8} - \frac{1}{5832}\right]\left(\frac{1}{3}\right)^n & , \quad n \geq 0 \end{cases}$$

OTHER WAY

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{3}\right)^n u[n+1].$$

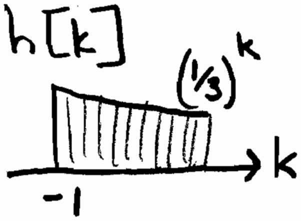
$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]\, x[k]$$

The system input is given by

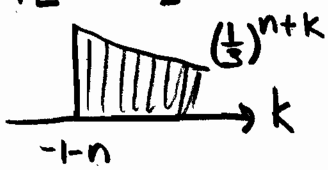
$$x[n] = 3^n \{u[n+3] - u[n-2]\} = \begin{cases} 3^n, & -3 \le n \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the system output $y[n]$.

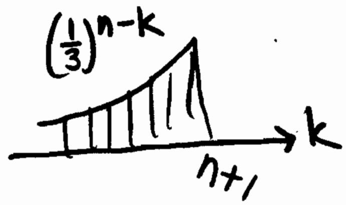
$h[k]$



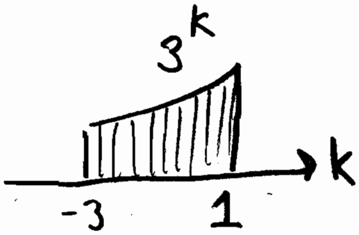
$h[k - -n] = h[n+k]$



$h[n-k]$



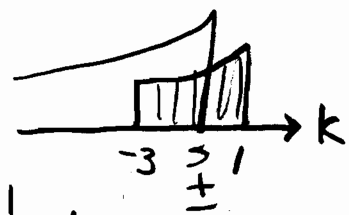
$x[k]$



I) $n+1 < -3 \;:\; n < -4 :$

$y[n] = 0.$

II) $-3 \le n+1 < 1 \;:\; -4 \le n < 0$



$$y[n] = \sum_{k=-3}^{n+1} \left(\tfrac{1}{3}\right)^{n-k} 3^k$$

$$= \left(\tfrac{1}{3}\right)^n \sum_{k=-3}^{n+1} 3^k 3^k = \left(\tfrac{1}{3}\right)^n \sum_{k=-3}^{n+1} 9^k$$

$$= \left(\tfrac{1}{3}\right)^n \frac{9^{-3} - 9^{n+2}}{1-9} = \frac{1}{8}\left(\tfrac{1}{3}\right)^n \left[81 \cdot 9^n - \frac{1}{9^3}\right]$$

$$= \frac{81}{8}\left(\frac{9}{3}\right)^n - \frac{1}{8 \cdot 729}\left(\tfrac{1}{3}\right)^n = \frac{81}{8} 3^n - \frac{1}{5832}\left(\tfrac{1}{3}\right)^n$$

III) $n+1 \ge 1 : n \ge 0 :$

$$y[n] = \sum_{k=-3}^{1} \left(\tfrac{1}{3}\right)^{n-k} 3^k = \left(\tfrac{1}{3}\right)^n \sum_{k=-3}^{1} 3^{2k} = \left(\tfrac{1}{3}\right)^n \sum_{k=-3}^{1} 9^k$$

$$= \left(\tfrac{1}{3}\right)^n \frac{9^{-3} - 9^2}{1-9} = \frac{1}{8}\left(\tfrac{1}{3}\right)^n \left[81 - \frac{1}{9^3}\right]$$

$$= \left(\tfrac{1}{3}\right)^n \left[\frac{81}{8} - \frac{1}{5832}\right]$$

All Together:

$$y[n] = \begin{cases} 0, & n < -4 \\ \frac{81}{8} 3^n - \frac{1}{5832}\left(\tfrac{1}{3}\right)^n, & -4 \le n < 0 \\ \left(\tfrac{1}{3}\right)^n \left[\frac{81}{8} - \frac{1}{5832}\right], & n \ge 0 \end{cases}$$