ECE 3793
Test 1

Friday, March 24, 2006
6:00 PM - 9:00 PM

Spring 2006

Dr. Havlicek

Name: SOLUTION

Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ______________________ Date: ______________________

1. **25 pts.** A continuous-time system H has input $x(t)$ and output $y(t)$ related by

$$y(t) = \begin{cases} tx(t) & x(t) \geq 0, \\ 0 & x(t) < 0. \end{cases}$$

(a) **3 pts.** Is the system H memoryless? Justify your answer.

- if $x(t) \geq 0$, then $x(t) = tx(t)$ which depends on the current input, but not on past or future values of the input.
- if $x(t)$ is not ≥ 0 (i.e. $x(t) < 0$), then $y(t) = 0$, which does not depend on past or future inputs.
- The decision depends on the current input, but not on past or future values of the input.
- Therefore, $y(t)$ depends on the current input $x(t)$, but not on $x(t+\tau)$ for any $\tau \neq 0$.
- Therefore the system **IS MEMORYLESS**

(b) **7 pts.** Is the system H linear? Justify your answer.

Let $x(t) = u(t)$.

Then $x(t) \geq t \;\; \forall t$, so

$y(t) = tx(t) = tu(t)$.

Now let $a = -1$ and $x_2(t) = ax_1(t)$
$= -u(t)$.

Then $y_2(t) = 0$.

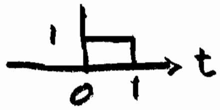
But $ay_1(t) = -y_1(t) = -tu(t)$.

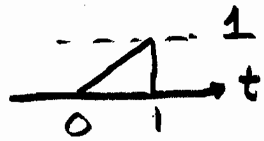
Since $y_2(t) \neq ay_1(t)$, the system is

NOT LINEAR

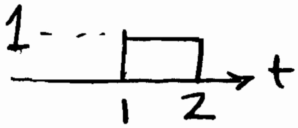
Problem 1, cont...

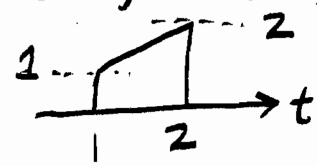
(c) **5 pts.** Is the system H time invariant? Justify your answer.

Let $x_1(t) =$  $= u(t) - u(t-1)$.

Then $y_1(t) = H\{x_1(t)\} = t\{u(t) - u(t-1)\} =$ 

Then $y_1(t-1) =$  $= (t-1)\{u(t-1) - u(t-2)\}$.

Now let $x_2(t) = x_1(t-1) =$  $= u(t-1) - u(t-2)$

Then $y_2(t) = t x_2(t) = t\{u(t-1) - u(t-2)\} =$ 

Since $y_1(t-1) \neq y_2(t)$, the system is

<u>NOT TIME INVARIANT</u>

(d) **3 pts.** Is the system H causal? Justify your answer.

That it <u>is</u> <u>causal</u> follows immediately from the fact that it is memoryless, as shown in part (a).

Problem 1, cont...

(e) **7 pts.** Is the system H BIBO stable? Justify your answer.

Let $x(t) = u(t)$. Then $|x(t)| \leq 1 \quad \forall\, t \in \mathbb{R}$,
so $x(t)$ is a bounded input.

The output is $y(t) = H\{x(t)\} = tu(t)$.

To show that this output is not bounded, we must prove $\nexists\, B \in \mathbb{R}$ s.t. $|y(t)| \leq B \;\; \forall\, t \in \mathbb{R}$. In other words, we must prove that $\forall\, B \in \mathbb{R}$ s.t. $B > 0$, $\exists\, t \in \mathbb{R}$ s.t. $|y(t)| > B$.

<u>Pf</u>: Let $B \in \mathbb{R}$ s.t. $B > 0$.

Let $t_0 = B + 1$.

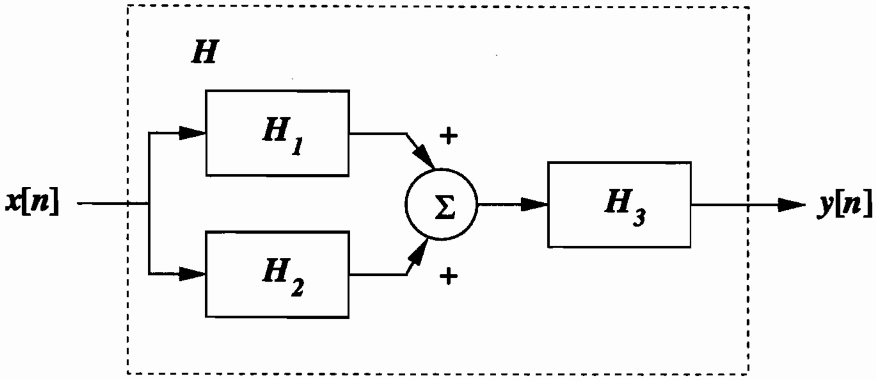
Then $t_0 > B$, so

$$|y(t_0)| = |t_0| = B + 1 > B.$$

Therefore, there does not exist any $B \in \mathbb{R}$ that bounds $y(t)$. In other words, $y(t)$ is unbounded. QED.

→ The system is <u>UNSTABLE</u>, since a bounded input produced an unbounded output.

2. **25 pts.** The discrete-time LTI system H is formed by connecting three LTI systems H_1, H_2, and H_3 as shown in the figure below.



It follows immediately from results proven in class that the overall system H is LTI. The impulse responses of the three LTI systems H_1 through H_3 are given by

$$\begin{aligned} h_1[n] &= \left(-\frac{1}{4}\right)^n u[n], \\ h_2[n] &= \left(-\frac{1}{4}\right)^{-n} u[-n-1], \\ h_3[n] &= \delta[n+2] + \delta[n-1]. \end{aligned}$$

(a) **10 pts.** Find the impulse response $h[n]$.

$$h[n] = \left(h_1[n] + h_2[n]\right) * h_3[n]$$

$$= \left[\left(-\tfrac{1}{4}\right)^n u[n] + \left(-\tfrac{1}{4}\right)^{-n} u[-n-1]\right] * h_3[n]$$

$$= \left(-\tfrac{1}{4}\right)^{|n|} * h_3[n] = \left(-\tfrac{1}{4}\right)^{|n|} * \left(\delta[n+2] + \delta[n-1]\right)$$

$$= \left(-\tfrac{1}{4}\right)^{|n+2|} + \left(-\tfrac{1}{4}\right)^{|n-1|}.$$

$$\boxed{h[n] = \left(-\tfrac{1}{4}\right)^{|n+2|} + \left(-\tfrac{1}{4}\right)^{|n-1|}}$$

Problem 2, cont...

(b) **4 pts**. Is the system H memoryless? Justify your answer.

A discrete-time LTI system is memoryless iff the impulse response is a constant times $\delta[n]$.

As shown in part (a), thats not the case here. So this system is <u>NOT</u> <u>MEMORYLESS</u>.

(c) **4 pts**. Is the system H causal? Justify your answer.

It is <u>not</u> <u>causal</u> because it is <u>not</u> <u>true</u> that $h[n] = 0 \ \forall n < 0$.

For example, when $n = -2$,

$$h[-2] = \left(-\tfrac{1}{4}\right)^0 + \left(-\tfrac{1}{4}\right)^{|-3|} = 1 + \left(-\tfrac{1}{4}\right)^3$$

$$= 1 - \frac{1}{64} = \frac{63}{64} \neq 0.$$

Problem 2, cont...

(d) **7 pts.** Is the system H BIBO stable? Justify your answer.

A discrete-time LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty}|h[n]| < \infty$. Here, we have

$$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=-\infty}^{\infty}\left|\left(-\tfrac{1}{4}\right)^{|n+2|}+\left(-\tfrac{1}{4}\right)^{|n-1|}\right|$$

$$= \sum_{n=-\infty}^{-2}\left|\left(-\tfrac{1}{4}\right)^{-n-2}+\left(-\tfrac{1}{4}\right)^{-n+1}\right| + \sum_{n=-1}^{1}\left|\left(-\tfrac{1}{4}\right)^{n+2}+\left(-\tfrac{1}{4}\right)^{-n+1}\right| + \sum_{n=2}^{\infty}\left|\left(-\tfrac{1}{4}\right)^{n+2}+\left(-\tfrac{1}{4}\right)^{n-1}\right|$$

$$\leq \sum_{n=-\infty}^{-2}\left(\tfrac{1}{4}\right)^{-n-2} + \sum_{n=-\infty}^{-2}\left(\tfrac{1}{4}\right)^{-n+1} + \sum_{n=-1}^{1}\left(\tfrac{1}{4}\right)^{n+2} + \sum_{n=-1}^{1}\left(\tfrac{1}{4}\right)^{-n+1} + \sum_{n=2}^{\infty}\left(\tfrac{1}{4}\right)^{n+2} + \sum_{n=2}^{\infty}\left(\tfrac{1}{4}\right)^{n-1}$$

$$= \left(\tfrac{1}{4}\right)^{-2}\sum_{n=-\infty}^{-2}4^n + \tfrac{1}{4}\sum_{n=-\infty}^{-2}4^n + \left(\tfrac{1}{4}\right)^2\sum_{n=-1}^{1}\left(\tfrac{1}{4}\right)^n + \tfrac{1}{4}\sum_{n=-1}^{1}4^n + \left(\tfrac{1}{4}\right)^2\sum_{n=2}^{\infty}\left(\tfrac{1}{4}\right)^n + 4\sum_{n=2}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= \left(16+\tfrac{1}{4}\right)\sum_{n=-\infty}^{-2}4^n + \tfrac{1}{16}\left[4+1+\tfrac{1}{4}\right] + \tfrac{1}{4}\left[\tfrac{1}{4}+1+4\right] + \left[\tfrac{1}{16}+4\right]\sum_{n=2}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= \frac{65}{4}\lim_{A\to\infty}\frac{4^{-A}-4^{-1}}{1-4} + \frac{1}{16}\left[\frac{21}{4}\right] + \frac{1}{4}\left[\frac{21}{4}\right] + \frac{65}{4}\lim_{A\to\infty}\frac{\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^{A+1}}{1-\frac{1}{4}}$$

$$= \frac{65}{4}\,\frac{-\frac{1}{4}}{-3} + \frac{5}{16}\left[\frac{21}{4}\right] + \frac{65}{4}\,\frac{\frac{1}{16}-0}{3/4} = \frac{65}{4}\cdot\frac{1}{12} + \frac{3\cdot5\cdot7}{2^6} + \frac{65}{4}\cdot\frac{4}{3\cdot16}$$

$$= \frac{5\cdot13}{3\cdot2^4} + \frac{3\cdot5\cdot7}{2^6} + \frac{5\cdot13}{3\cdot2^4} = \frac{5\cdot13\cdot2^3+3^2\cdot5\cdot7}{3\cdot2^6} = \frac{5\left[13\cdot8+9\cdot7\right]}{3\cdot64}$$

$$= \frac{5\left[104+63\right]}{192} = \frac{520+315}{192} = \frac{835}{192} < \infty.$$

Therefore it **IS BIBO STABLE**

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-6t}u(t).$$

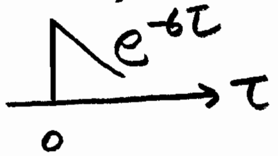
The system input is given by

$$x(t) = \begin{cases} e^{-3(t-1)}, & -1 \le t \le 1, \\ 0, & \text{otherwise} \end{cases} = e^{-3(t-1)}\left[u(t+1) - u(t-1)\right].$$

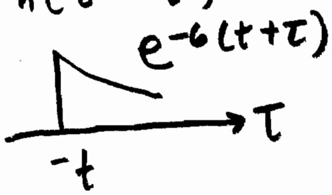
Find the system output $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$h(\tau)$

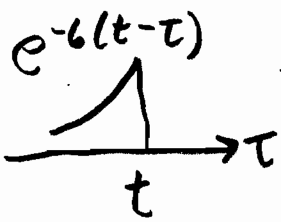


$h(\tau - -t)$

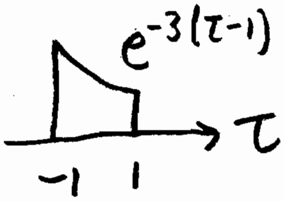


$h(t-\tau)$

$e^{-6(t-\tau)}$



$x(\tau)$



case I) $t < -1$:

$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

case II) $-1 \le t < 1$:

$$y(t) = \int_{-1}^{t} e^{-3(\tau-1)} e^{-6(t-\tau)}\,d\tau$$

$$= \int_{-1}^{t} e^{-3\tau}e^{3}e^{-6t}e^{6\tau}\,d\tau = e^{-6t}e^{3}\int_{-1}^{t} e^{3\tau}\,d\tau$$

$$= \tfrac{1}{3}e^{-6t}e^{3}\left[e^{3\tau}\right]_{\tau=-1}^{t} = \tfrac{1}{3}e^{-6t}e^{3}\left[e^{3t} - e^{-3}\right]$$

$$= \tfrac{1}{3}\left[e^{-3t}e^{3} - e^{-6t}\right] = \tfrac{1}{3}e^{-3(t-1)} - \tfrac{1}{3}e^{-6t}$$

case III) $t \geq 1$:

$$y(t) = \int_{-1}^{1} e^{-3(\tau-1)}e^{-6(t-\tau)}\,d\tau = e^{-6t}e^{3}\int_{-1}^{1} e^{3\tau}\,d\tau$$

$$= \tfrac{1}{3}e^{-6t}e^{3}\left[e^{3\tau}\right]_{\tau=-1}^{1} = \tfrac{1}{3}e^{-6t}e^{3}\left[e^{3} - e^{-3}\right]$$

$$= \tfrac{1}{3}e^{-6t}\left[e^{6} - 1\right] = \frac{e^{6}-1}{3}e^{-6t}$$

all Together:

$$y(t) = \begin{cases} 0, & t < -1 \\ \tfrac{1}{3}e^{-3(t-1)} - \tfrac{1}{3}e^{-6t}, & -1 \le t < 1 \\ \dfrac{e^{6}-1}{3}e^{-6t}, & t \geq 1 \end{cases}$$

Other Way: $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

More Workspace for Problem 3...

$x(\tau)$

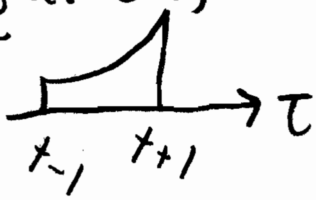
$e^{-3(\tau-1)}$

$x(\tau - -t)$

$e^{-3(t+\tau-1)}$

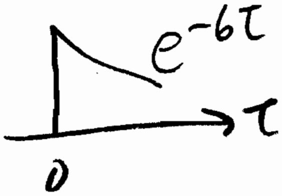
$x(t-\tau)$

$e^{-3(t-\tau-1)}$

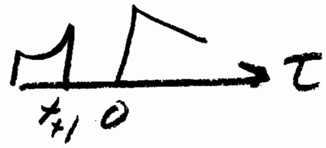


$h(\tau)$

$e^{-6\tau}$



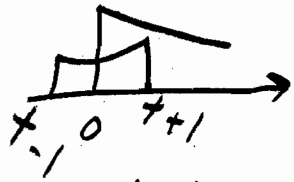
case I) $t+1<0$: $t<-1$;



$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

case II) $t+1 \geq 0$ and $t-1<0$:

$t \geq -1$ and $t<1$; $-1 \leq t < 1$



$$y(t) = \int_0^{t+1} e^{-6\tau} e^{-3(t-\tau-1)}\, d\tau$$

$$= \int_0^{t+1} e^{-6\tau} e^{-3t} e^{3\tau} e^{3}\, d\tau = e^{-3t} e^{3} \int_0^{t+1} e^{-3\tau}\, d\tau$$

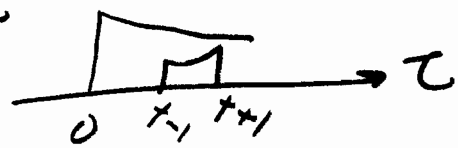
$$= -\frac{1}{3} e^{-3t} e^{3} \left[e^{-3\tau} \right]_{\tau=0}^{t+1}$$

$$= -\frac{1}{3} e^{-3t} e^{3} \left[e^{-3(t+1)} - 1 \right]$$

$$= -\frac{1}{3} e^{-3t} e^{3} e^{-3t} e^{-3} + \frac{1}{3} e^{-3t} e^{3}$$

$$= \frac{1}{3} e^{-3(t-1)} - \frac{1}{3} e^{-6t}$$

case III) $t \geq 1$



$$y(t) = \int_{t-1}^{t+1} e^{-6\tau} e^{-3t} e^{3\tau} e^{3}\, d\tau = e^{-3t} e^{3} \int_{t-1}^{t+1} e^{-3\tau}\, d\tau$$

$$= -\frac{1}{3} e^{-3t} e^{3} \left[e^{-3\tau} \right]_{\tau=t-1}^{t+1} = -\frac{1}{3} e^{-3t} e^{3} \left[e^{-3(t+1)} - e^{-3(t-1)} \right]$$

$$= \frac{1}{3} e^{-3t} e^{3} \left[e^{-3t} e^{3} - e^{-3t} e^{-3} \right] = \frac{1}{3} e^{-6t} \left[e^{6} - 1 \right] = \frac{e^{6}-1}{3} e^{-6t}$$

All Together:

$$y(t) = \begin{cases} 0, & t<-1 \\ \frac{1}{3} e^{-3(t-1)} - \frac{1}{3} e^{-6t}, & -1 \leq t < 1 \\ \frac{e^{6}-1}{3} e^{-6t}, & t \geq 1 \end{cases}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(-\frac{1}{4}\right)^n (u[n] - u[n-5]).$$

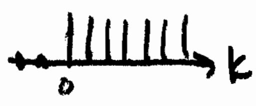
The system input is given by

$$x[n] = u[n].$$

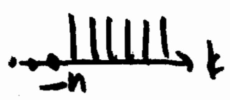
$$y[k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Find the system output $y[n]$.

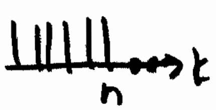
$x[k]$



$x[k - -n]$

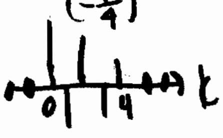


$x[n-k]$

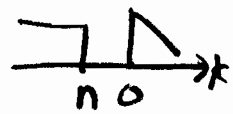


$h[k]$

$(-\frac{1}{4})^k$

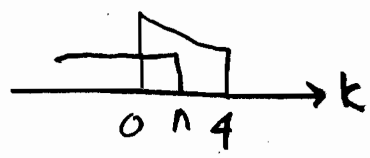


Case I) $n < 0$:



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

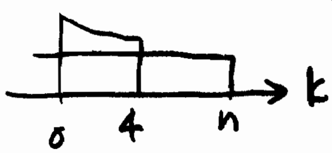
Case II) $n \geq 0$ and $n < 5$: $0 \leq n < 5$



$$y[n] = \sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k = \frac{(-\frac{1}{4})^0 - (-\frac{1}{4})^{n+1}}{1 - -\frac{1}{4}}$$

$$= \frac{1 - (-\frac{1}{4})^n(-\frac{1}{4})}{5/4} = \frac{4 + (-\frac{1}{4})^n}{5} = \frac{1}{5}\left(-\frac{1}{4}\right)^n + \frac{4}{5}$$

Case III) $n \geq 5$



$$y[n] = \sum_{k=0}^{4} \left(-\frac{1}{4}\right)^k = \frac{(-\frac{1}{4})^0 - (-\frac{1}{4})^5}{1 - -\frac{1}{4}} = \frac{1 + (\frac{1}{4})^5}{5/4}$$

$$= \frac{4}{5} + \frac{1}{5}\left(\frac{1}{4}\right)^4 = \frac{1}{5}\left[4 + \left(\frac{1}{4}\right)^4\right] = \frac{1}{5}\left[4 + \frac{1}{256}\right]$$

$$= \frac{1}{5}\,\frac{1025}{256} = \frac{205}{256}$$

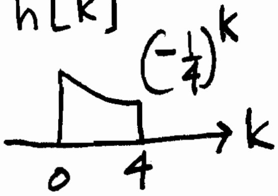
All Together:

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{5}(-\frac{1}{4})^n + \frac{4}{5}, & 0 \leq n < 5 \\ \frac{205}{256}, & n \geq 5 \end{cases}$$

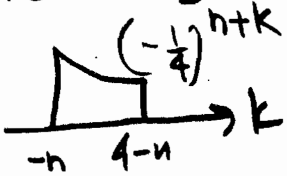
Other way: $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

More Workspace for Problem 4...

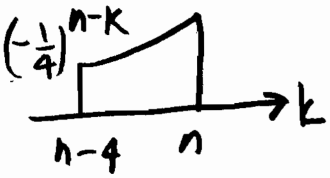
h[k]



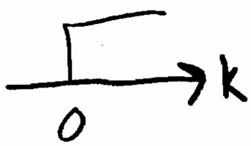
h[k--n]



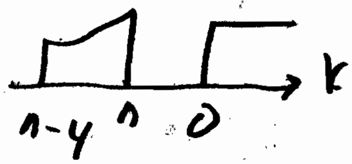
h[n-k]



x[k]



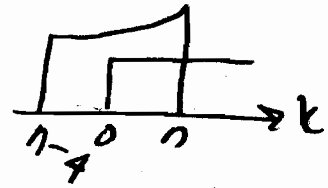
case I) $n<0$:



$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$

case II) $n \geq 0$ and $n-4 \leq 0$: $0 \leq n \leq 4$; $0 \leq n < 5$

$$y[n] = \sum_{k=0}^{n} \left(-\tfrac{1}{4}\right)^{n-k} = \left(-\tfrac{1}{4}\right)^n \sum_{k=0}^{n} (-4)^k$$

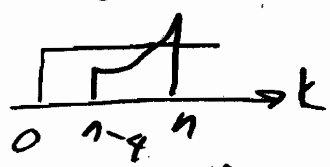


$$= \left(-\tfrac{1}{4}\right)^n \frac{-4^0 - (-4)^{n+1}}{1--4} = \left(-\tfrac{1}{4}\right)^n \frac{1+4(-4)^n}{5}$$

$$= \tfrac{1}{5}\left(-\tfrac{1}{4}\right)^n + \tfrac{4}{5}\left(-\tfrac{1}{4} \cdot -4\right)^n$$

$$= \tfrac{1}{5}\left(-\tfrac{1}{4}\right)^n + \tfrac{4}{5}1^n = \tfrac{1}{5}\left(-\tfrac{1}{4}\right)^n + \tfrac{4}{5}$$

case III) $n \geq 5$:



$$y[n] = \sum_{k=n-4}^{n} \left(-\tfrac{1}{4}\right)^{n-k} = \left(-\tfrac{1}{4}\right)^n \sum_{k=n-4}^{n} (-4)^k$$

$$= \left(-\tfrac{1}{4}\right)^n \frac{(-4)^{n-4} - (-4)^{n+1}}{1--4} = \tfrac{1}{5}\left[\left(-\tfrac{1}{4}\right)^n (-4)^n \left(-\tfrac{1}{4}\right)^4 + \left(-\tfrac{1}{4}\right)^n 4(-4)^n\right]$$

$$= \tfrac{1}{5}\left[\left(-\tfrac{1}{4}\right)^4 + 4\right] = \tfrac{1}{5}\left[\tfrac{1}{256} + 4\right] = \tfrac{1}{5}\left[\tfrac{1025}{256}\right] = \tfrac{205}{256}$$

All Together:
$$y[n] = \begin{cases} 0, & n<0 \\ \tfrac{1}{5}\left(-\tfrac{1}{4}\right)^n + \tfrac{4}{5}, & 0 \leq n < 5 \\ \tfrac{205}{256}, & n \geq 5 \end{cases}$$