ECE 3793
Test 1

Wednesday, March 12, 2008
6:00 PM - 9:00 PM

Spring 2008 — Name: SOLUTION

Dr. Havlicek — Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. Formulas appear at the **end** of the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________________ Date: ____________________

1. **25 pts**. A discrete-time system H has input $x[n]$ and output $y[n]$ related by

$$y[n] = \sin\big(n\pi x[n+1]\big).$$

(a) **3 pts**. Is the system H memoryless? Justify your answer.

When $n=2$, we have $y[2] = \sin\big(2\pi x[3]\big)$, which depends on the input $x[3]$ from a <u>different</u> time.

Therefore, H is <u>NOT MEMORYLESS</u>.

(b) **3 pts**. Is the system H causal? Justify your answer.

As shown in part (a), when $n=2$ we have that $y[2]$ depends on $x[3]$, which is a <u>future</u> input.

Therefore, H is <u>NOT CAUSAL</u>.

Problem 1, cont...

(c) **5 pts.** Is the system H linear? Justify your answer.

Let $y_1[n] = H\{x_1[n]\} = \sin(2\pi x_1[n+1])$.

Let $y_2[n] = H\{x_2[n]\} = \sin(2\pi x_2[n+1])$.

Let $a_1, a_2 \in \mathbb{C}$ be constants.

Then $a_1 y_1[n] + a_2 y_2[n] = a_1 \sin(n\pi x_1[n+1]) + a_2 \sin(n\pi x_2[n+1])$.

Now let $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$.

Then $y_3[n] = H\{x_3[n]\} = \sin(n\pi x_3[n+1])$

$= \sin(a_1 n\pi x_1[n+1] + a_2 n\pi x_2[n+1])$

$\neq a_1 y_1[n] + a_2 y_2[n]$ in general.

$\boxed{\text{NOT LINEAR}}$

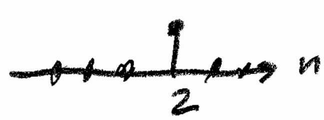
(d) **5 pts.** Is the system H time invariant? Justify your answer.

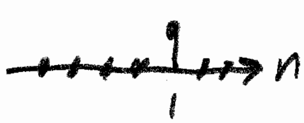
Let $x_1[n] = \delta[n-1] =$  . Then

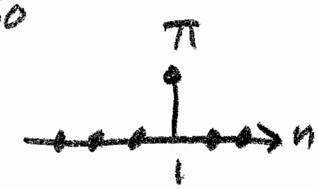
$x_1[n+1] = \delta[n] =$  and

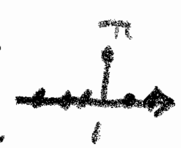
$n\pi x_1[n+1] = 0 \;\forall n =$  , so

$y_1[n] = H\{x_1[n]\} = \sin(n\pi x_1[n+1]) = \sin(0) = 0 =$  .

Now let $x_2[n] = x_1[n-1] = \delta[n-2] =$  .

Then $x_2[n+1] = \delta[n-1] =$  , so

$n\pi x_2[n+1] = n\pi\delta[n-1] = \pi\delta[n-1] =$  .

So $y_2[n] = \sin(n\pi x_2[n+1]) = \sin($  $) = \begin{cases} 0, & n \neq 1 \\ 1, & n = 1 \end{cases}$

$=$  $= \delta[n-1] \neq y_1[n-1]$.

NOT TIME INVARIANT

Problem 1, cont...

(e) **5 pts.** Is the system H BIBO stable? Justify your answer.

Let $x[n]$ be a bounded input. Then $\exists B \in \mathbb{R}, B \geq 0,$ s.t. $|x[n]| \leq B \ \forall n \in \mathbb{Z}$.

Now, $|y[n]| = |\sin(n\pi x[n+1])| \leq 1$.

So $y[n]$ is a bounded signal bounded by $B = 1$.

So every bounded input produces a bounded output.

Therefore, H IS BIBO STABLE.

(f) **4 pts.** Is the system H invertible? Justify your answer.

- If you answer *yes*, then give the input-output relation for the inverse system.
- If you answer *no*, then give two distinct input signals that both result in the same output signal.

Let $x_1[n] = \delta[n-1]$ and let $x_2[n] = 2\delta[n-1]$.

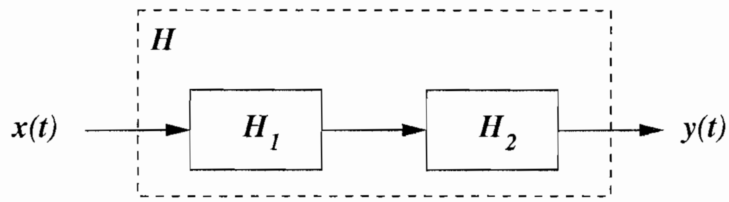
As shown in part (d),

$$y_1[n] = H\{x_1[n]\} = y_2[n] = H\{x_2[n]\} = 0.$$

So two different input signals both produced the same output signal. Therefore,

H is NOT INVERTIBLE

2. **25 pts.** A continuous-time system H is formed by cascading two continuous-time systems H_1 and H_2 as shown in the figure below:



System H_1 is LTI and has impulse response $h_1(t) = e^{-3t}u(t)$. System H_2 is linear and has input $x(t)$ and output $y(t)$ related by $y(t) = x(t) - x(t-2)$.

(a) **10 pts.** Is the overall system H an LTI system? Justify your answer.

→ H_1 is given as LTI. Therefore, if we can show that H_2 is also LTI then it follows immediately from results proven in class that H is also LTI.

→ Moreover, H_2 is given as linear, so it is sufficient to show that H_2 is also time invariant.

→ Consider H_2: $x(t) \longrightarrow \boxed{H_2} \longrightarrow y(t) = x(t) - x(t-2)$.

Let the input to H_2 be $x_1(t)$. Then the output of H_2 is $y_1(t) = H_2\{x_1(t)\} = x_1(t) - x_1(t-2)$.

Now shift by t_0: $y_1(t-t_0) = x_1(t-t_0) - x_1(t-t_0-2)$.

Let $x_2(t) = x_1(t-t_0)$. Then

$$y_2(t) = H_2\{x_2(t)\} = x_2(t) - x_2(t-2)$$
$$= x_1(t-t_0) - x_1(t-t_0-2) = y_1(t-t_0) \checkmark$$

Therefore H_2 is <u>LTI</u>, which implies that

<u>**H IS LTI**</u>.

Problem 2, cont...

(b) **5 pts**. If you answered *yes* in part (a), then find the impulse response $h(t)$ of the overall system. If instead you answered *no* in part (a), give the input-output relation for H.

Let the input be $\delta(t)$. Then the output is $h(t)$.

→ In this case, the input to H_1 is $\delta(t)$ and the output of H_1 is $h_1(t) * \delta(t) = h_1(t)$. This is the input to H_2. So the output of H_2 is

$$y(t) = h_1(t) - h_1(t-2) = \boxed{e^{-3t}u(t) - e^{-3(t-2)}u(t-2) = h(t).}$$

(c) **5 pts**. Is the system H causal? Justify your answer.

We have from (a) that $h(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2)$.

The first term "turns on" at $t=0$ and the second term "turns on" at $t=2$.

Therefore, $h(t) = 0 \ \forall\ t<0$ and the system <u>IS CAUSAL</u>.

(d) **5 pts**. Is the system H BIBO stable? Justify your answer.

The system is LTI, so it is BIBO stable iff $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} |e^{-3t}u(t) - e^{-3(t-2)}u(t-2)|\,dt$$

$$\leq \int_{-\infty}^{\infty} e^{-3t}u(t)\,dt + \int_{-\infty}^{\infty} e^{-3t+6}u(t-2)\,dt$$

$$= \int_0^{\infty} e^{-3t}\,dt + \int_2^{\infty} e^{-3t+6}\,dt = \int_0^{\infty} e^{-3t}\,dt + e^6 \int_2^{\infty} e^{-3t}\,dt$$

$$= -\frac{1}{3}e^{-3t}\Big|_{t=0}^{\infty} + e^6\left[-\frac{1}{3}e^{-3t}\right]_{t=2}^{\infty} = -\frac{1}{3}[0-1] - \frac{e^6}{3}[0-e^{-6}]$$

$$= \frac{1}{3} + \frac{e^6 e^{-6}}{3} = \frac{1}{3} + \frac{1}{3} = 2/3 < \infty \ \checkmark$$

Therefore <u>H IS BIBO STABLE</u>.

3. **25 pts**. A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \begin{cases} 4^n, & -2 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

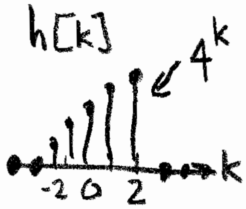
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The system input is given by

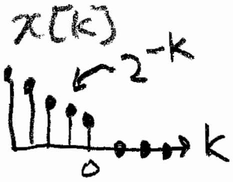
$$x[n] = 2^{-n}u[-n].$$

Find the system output $y[n]$.

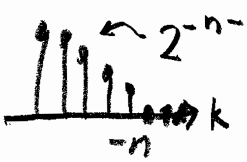
$h[k]$



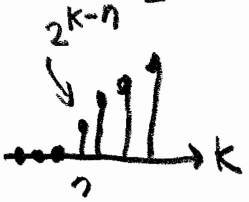
$x[k]$



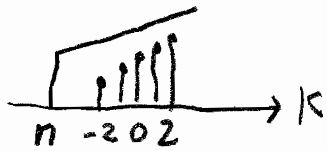
$x[n+k] = x[k--n]$



$x[n-k]$



Case I: $n \le -2$

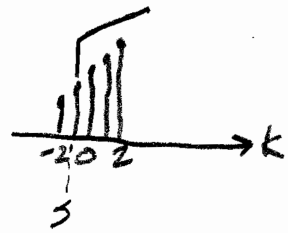


$$y[n] = \sum_{k=-2}^{2} 4^k 2^{k-n} = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{2} 4^k 2^k = \left(\tfrac{1}{2}\right)^n \sum_{k=-2}^{2} 8^k$$

$$= \left(\tfrac{1}{2}\right)^n \left[\frac{8^{-2} - 8^3}{1-8}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{\frac{1}{64} - 512}{-7}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{1 - 32768}{-7 \cdot 64}\right]$$

$$= \left(\tfrac{1}{2}\right)^n \frac{32767}{7 \cdot 64} = \frac{4681}{64}\left(\tfrac{1}{2}\right)^n$$

case II: $n > -2$ and $n \le 2$; $-2 < n \le 2$

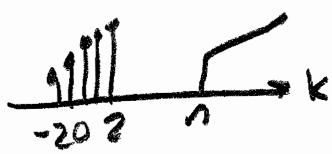


$$y[n] = \sum_{k=n}^{2} 4^k 2^{k-n} = \left(\tfrac{1}{2}\right)^n \sum_{k=n}^{2} (2 \cdot 4)^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{k=n}^{2} 8^k = \left(\tfrac{1}{2}\right)^n \left[\frac{8^n - 8^3}{1-8}\right] = \left(\tfrac{1}{2}\right)^n \left[\frac{512 - 8^n}{7}\right]$$

$$= \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{1}{7}\left(\tfrac{1}{2} \cdot 8\right)^n = \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{1}{7} \cdot 4^n$$

case III: $n > 2$



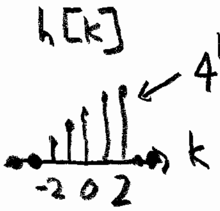
$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0.$$

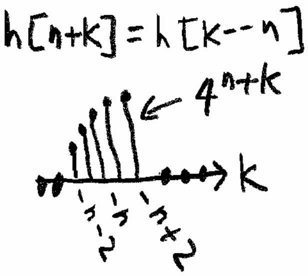
$$\boxed{\text{All Together: } y[n] = \begin{cases} \frac{4681}{64}\left(\frac{1}{2}\right)^n, & n \le -2 \\ \frac{512}{7}\left(\frac{1}{2}\right)^n - \frac{1}{7} \cdot 4^n, & -2 < n \le 2 \\ 0, & n > 2 \end{cases}}$$

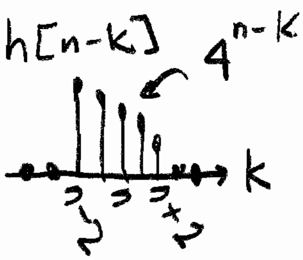
More Workspace for Problem 3...

OTHER WAY...

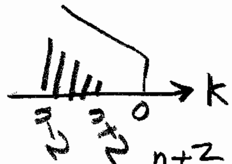
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$







case I) $n+2 < 0 \ : \ n < -2$



$$y[n] = \sum_{k=n-2}^{n+2} 2^{-k}4^{n-k} = 4^n \sum_{k=n-2}^{n+2} (2\cdot 4)^{-k}$$

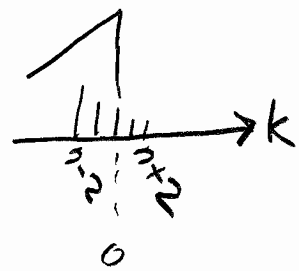
$$= 4^n \sum_{k=n-2}^{n+2} \left(\tfrac{1}{8}\right)^k = 4^n \left[\frac{\left(\frac{1}{8}\right)^{n-2} - \left(\frac{1}{8}\right)^{n+3}}{1-\frac{1}{8}}\right]$$

$$= 4^n \left[\frac{\left(\frac{1}{8}\right)^n\left(\frac{1}{8}\right)^{-2} - \left(\frac{1}{8}\right)^n\left(\frac{1}{8}\right)^{3}}{7/8}\right] = \frac{8}{7}\left[\left(\tfrac{1}{2}\right)^n 64 - \left(\tfrac{1}{2}\right)^n \tfrac{1}{512}\right]$$

$$= \frac{8}{7}\left(\tfrac{1}{2}\right)^n \left[64 - \tfrac{1}{512}\right] = \frac{8}{7}\cdot\frac{32767}{512}\left(\tfrac{1}{2}\right)^n = \frac{2^3\cdot 32767}{7\cdot 2^9}\left(\tfrac{1}{2}\right)^n$$

$$= \frac{32767}{7\cdot 2^6}\left(\tfrac{1}{2}\right)^n = \frac{32767}{7\cdot 64}\left(\tfrac{1}{2}\right)^n = \frac{4681}{64}\left(\tfrac{1}{2}\right)^n$$

case II) $n+2 > 0$ and $n-2 \le 0$; $-2 < n \le 2$.

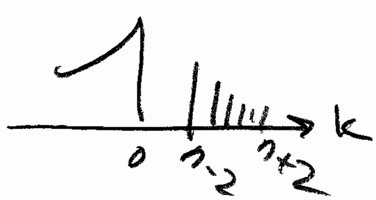


$$y[n] = \sum_{k=n-2}^{0} 2^{-k}4^{n-k} = 4^n \sum_{k=n-2}^{0} 8^{-k} = 4^n \sum_{k=n-2}^{0} \left(\tfrac{1}{8}\right)^k$$

$$= 4^n \left[\frac{\left(\frac{1}{8}\right)^{n-2} - \frac{1}{8}}{1-1/8}\right] = \frac{\left(4\cdot\frac{1}{8}\right)^n 8^2 - \frac{1}{8}4^n}{7/8}$$

$$= \left[64\left(\tfrac{1}{2}\right)^n - \tfrac{1}{8}4^n\right]\frac{8}{7} = \frac{512}{7}\left(\tfrac{1}{2}\right)^n - \frac{1}{7}4^n.$$

case III) $n > 2$.



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0.$$

All Together:

$$y[n] = \begin{cases} \frac{4681}{64}\left(\frac{1}{2}\right)^n, & n < -2 \\ \frac{512}{7}\left(\frac{1}{2}\right)^n - \frac{1}{7}\cdot 4^n, & -2 < n \le 2 \\ 0, & n > 2 \end{cases}$$

4. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = t\{u(t) - u(t-2)\} = \begin{cases} t, & 0 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

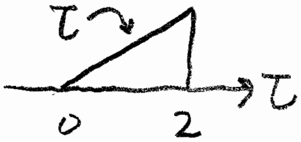
The system input is given by

$$x(t) = e^{-2t}u(t).$$

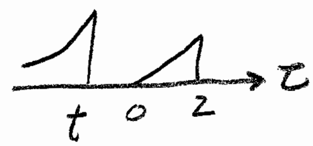
Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$h(\tau)$

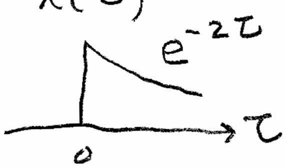


case I) $t < 0$:

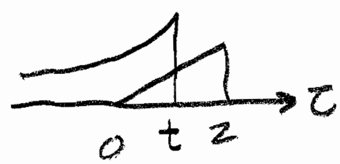


$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0.$$

$x(\tau)$

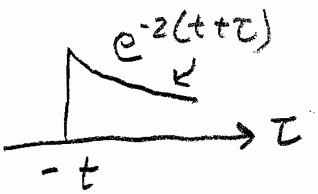


case II) $0 \le t$ and $t < 2$: $0 \le t < 2$:



$$y(t) = \int_0^t \tau e^{-2(t-\tau)}\, d\tau$$

$x(t+\tau) = x(\tau - -t)$



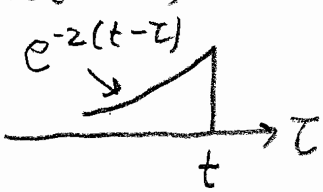
$$= e^{-2t}\int_0^t \tau e^{2\tau}\, d\tau$$

$$= e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \frac{1}{2}\right)\right]_{\tau=0}^{t}$$

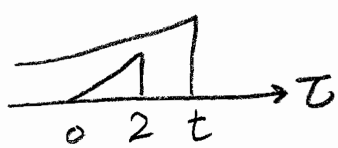
$$= e^{-2t}\left[\frac{1}{2}e^{2t}\left(t-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)\right] = e^{-2t}\left[\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}\right]$$

$$= \frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}$$

$x(t-\tau)$



case III) $t \ge 2$:



$$y(t) = \int_0^2 \tau e^{-2(t-\tau)}\, d\tau = e^{-2t}\int_0^2 \tau e^{2\tau}\, d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\left(\tau - \frac{1}{2}\right)\right]_{\tau=0}^{2}$$

$$= e^{-2t}\left[\frac{1}{2}e^4\left(2-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)\right] = e^{-2t}\left[e^4\left(1-\frac{1}{4}\right) + \frac{1}{4}\right]$$

$$= e^{-2t}\left[\frac{3}{4}e^4 + \frac{1}{4}\right] = \frac{3e^4+1}{4}e^{-2t}$$

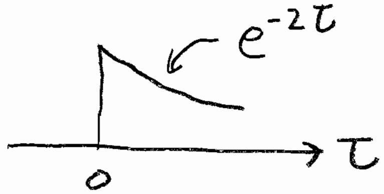
All Together:

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}, & 0 \le t < 2 \\ \frac{3e^4+1}{4}e^{-2t}, & t \ge 2 \end{cases}$$

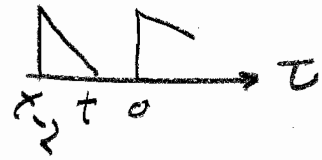
OTHER WAY: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

More Workspace for Problem 4…

$x(\tau)$

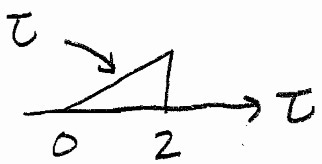


case I) $t<0$:



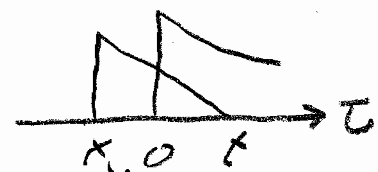
$$y(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0.$$

$h(\tau)$



case II) $t \geq 0$ and $t-2<0$: $0 \leq t < 2$

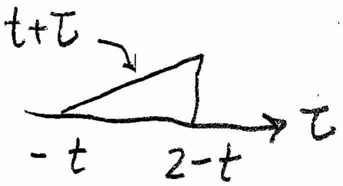
$$y(t) = \int_0^t e^{-2\tau}(t-\tau)\,d\tau$$



$$= t\int_0^t e^{-2\tau}d\tau - \int_0^t \tau e^{-2\tau}d\tau$$

$h(t+\tau) = h(\tau - -t)$

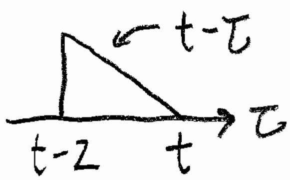
$$= t\left[-\tfrac{1}{2}e^{-2\tau}\right]_{\tau=0}^{t} - \left[\frac{e^{-2\tau}}{-2}\left(\tau+\tfrac{1}{2}\right)\right]_{\tau=0}^{t}$$



$$= t\left[-\tfrac{1}{2}e^{-2t}+\tfrac{1}{2}\right] + \left[\tfrac{1}{2}e^{-2t}\left(t+\tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\right]$$

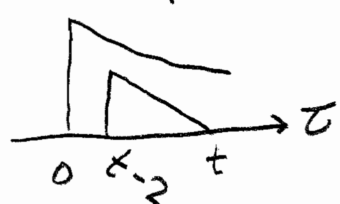
$h(t-\tau)$

$$= -\tfrac{1}{2}te^{-2t} + \tfrac{1}{2}t + \tfrac{1}{2}te^{-2t} + \tfrac{1}{4}e^{-2t} - \tfrac{1}{4}$$



$$= \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t} - \tfrac{1}{4}.$$

case III) $t \geq 2$



$$y(t) = \int_{t-2}^{t} e^{-2\tau}(t-\tau)\,d\tau = t\int_{t-2}^{t} e^{-2\tau}d\tau - \int_{t-2}^{t}\tau e^{-2\tau}d\tau$$

$$= t\left[-\tfrac{1}{2}e^{-2\tau}\right]_{\tau=t-2}^{t} - \left[\frac{e^{-2\tau}}{-2}\left(\tau+\tfrac{1}{2}\right)\right]_{\tau=t-2}^{t}$$

$$= t\left[-\tfrac{1}{2}e^{-2t} + \tfrac{1}{2}e^{-2(t-2)}\right] + \tfrac{1}{2}\left[e^{-2t}\left(t+\tfrac{1}{2}\right) - e^{-2(t-2)}\left(t-2+\tfrac{1}{2}\right)\right]$$

$$= -\tfrac{1}{2}te^{-2t} + \tfrac{1}{2}te^{-2t}e^{4} + \tfrac{1}{2}\left[te^{-2t} + \tfrac{1}{2}e^{-2t} - e^{-2t}e^{4}\left(t-\tfrac{3}{2}\right)\right]$$

$$= e^{-2t}\left[-\tfrac{1}{2}t + \tfrac{1}{2}te^{4} + \tfrac{1}{2}t + \tfrac{1}{4} - \tfrac{1}{2}e^{4}t + \tfrac{3}{4}e^{4}\right] = \frac{3e^4+1}{4}e^{-2t}$$

All Together: $y(t) = \begin{cases} 0 & , t<0 \\ \tfrac{1}{2}t + \tfrac{1}{4}e^{-2t} - \tfrac{1}{4} & , 0 \leq t < 2 \\ \frac{3e^4+1}{4}e^{-2t} & , t \geq 2 \end{cases}$