ECE 3793
Test 1

Friday, March 6, 2015
7:00 PM - 10:00 PM

Spring 2015 — Name: SOLUTION

Dr. Havlicek — Student Num: ____________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. There are **four** problems. Work **all four**. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

____

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name: ____________ Date: ____________

1. **25 pts**. A discrete-time system H has input $x[n]$ and output $y[n]$ related by

$$y[n] = x[n] - x[n-1] + 1.$$

(a) **3 pts**. Is the system H memoryless? Justify your answer.

The value $y[1]$ of the output signal at time $n=1$ depends on the value $x[0]$ of the input signal from the previous time $n=0$.

NOT MEMORYLESS

(b) **3 pts**. Is the system H causal? Justify your answer.

The current value of the output signal $y[n]$ depends on the current value $x[n]$ of the input signal and on the previous value $x[n-1]$ of the input signal, but not on any future values of the input signal. So the system IS CAUSAL.

Problem 1, cont...

(c) **5 pts**. Is the system H linear? Justify your answer.

Let $x_1[n]$ and $x_2[n]$ be arbitrary input signals and let $c_1, c_2 \in \mathbb{C}$ be constants. Then

$$y_1[n] = H\{x_1[n]\} = x_1[n] - x_1[n-1] + 1 \quad \text{and}$$

$$y_2[n] = H\{x_2[n]\} = x_2[n] - x_2[n-1] + 1.$$

So $c_1 y_1[n] + c_2 y_2[n] = c_1\left(x_1[n] - x_1[n-1]\right) + c_2\left(x_2[n] - x_2[n-1]\right) + c_1 + c_2.$

Now let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then $y_3[n] = H\{x_3[n]\} = x_3[n] - x_3[n-1] + 1$

$$= c_1 x_1[n] + c_2 x_2[n] - c_1 x_1[n-1] - c_2 x_2[n-1] + 1$$

$$= c_1\left(x_1[n] - x_1[n-1]\right) + c_2\left(x_2[n] - x_2[n-1]\right) + 1 \neq c_1 y_1[n] + c_2 y_2[n].$$

THE SYSTEM IS <u>NOT</u> <u>LINEAR</u>.

(d) **5 pts**. Is the system H time invariant? Justify your answer.

Let the input be $x_1[n]$ and let $n_0 \in \mathbb{Z}$. Then

$$y_1[n] = H\{x_1[n]\} = x_1[n] - x_1[n-1] + 1,$$

So $y_1[n-n_0] = x_1[n-n_0] - x_1[n-n_0-1] + 1.$

Now let $x_2[n] = x_1[n-n_0]$. Then

$$y_2[n] = x_2[n] - x_2[n-1] + 1 = x_1[n-n_0] - x_1[n-n_0-1] + 1$$

$$= y_1[n-n_0] \checkmark$$

The system is <u>TIME INVARIANT</u>

Problem 1, cont...

(e) **5 pts**. Is the system H BIBO stable? Justify your answer.

Let $x[n]$ be a bounded input signal. Then $\exists B \in \mathbb{R}$, $B \geq 0$, s.t. $|x[n]| \leq B \ \forall n \in \mathbb{Z}$. We have

$$|y[n]| = |x[n] - x[n-1] + 1| \leq |x[n]| + |x[n-1]| + 1$$
$$\leq B + B + 1 = 2B + 1.$$

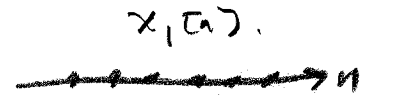
So $y[n]$ is bounded by $2B+1$.

Since every bounded input signal produces a bounded output signal, the system <u>IS STABLE</u>.

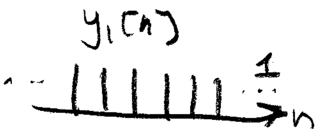
(f) **4 pts**. Is the system H invertible? Justify your answer.

- If you answer *yes*, then give the input-output relation for the inverse system.
- If you answer *no*, then give two distinct input signals that both result in the same output signal.

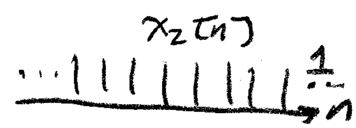
Let the input signal be $x_1[n] = 0 \ \forall n$.



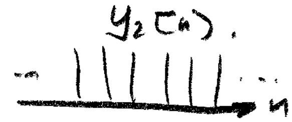
Then the output is $y_1[n] = x_1[n] - x_1[n-1] + 1$
$= 0 - 0 + 1 = 1 \ \forall n.$



Now let the input signal be $x_2[n] = 1 \ \forall n$



Then the output is $y_2[n] = x_2[n] - x_2[n-1] + 1$
$= 1 - 1 + 1 = 1 \ \forall n$



So $y_1[n] = y_2[n] \ \forall n \in \mathbb{Z}$.

Since two different input signals both produced the same output signal, the system is <u>NOT INVERTIBLE</u>.

Here is another way to do 1(f):

- To find the I/O relation for the inverse system, we must solve the original I/O relation for the input $x[n]$ and then switch the names of $x[n]$ & $y[n]$.
- So let's try to solve the original I/O relation for the input signal $x[n]$:

we have $\quad y[n] = x[n] - x[n-1] + 1$

So $\quad x[n] = y[n] + \underbrace{x[n-1]}_{= y[n-1] + x[n-2] - 1} - 1$

$$= y[n] + y[n-1] + x[n-2] - 2$$

$$= y[n] + y[n-1] + y[n-2] + x[n-3] - 3$$

$$\vdots$$

$$= y[n] + \cdots + y[n-(N-1)] + x[n-N] - N$$

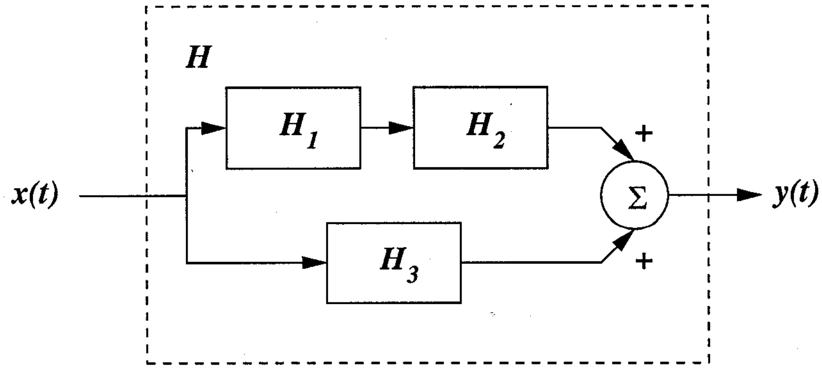
$$= \left(\sum_{k=0}^{N-1} y[n-k]\right) + x[n-N] - N$$

- But we can't continue this recursion indefinitely, because the last term will blow up.

→ So the inverse system diverges.

→ **NOT INVERTIBLE**

2. **25 pts.** A continuous-time LTI system H is formed by connecting three continuous-time LTI systems H_1, H_2, and H_3 together as shown in the figure below:



The impulse response of system H_1 is $h_1(t) = e^{-2t}u(t)$.
The impulse response of system H_2 is $h_2(t) = \delta(t+2) + \delta(t-2)$.
The impulse response of system H_3 is $h_3(t) = e^{2t}u(t)$.

(a) **10 pts.** Find the impulse response $h(t)$ for the overall system H.

$$h(t) = h_1(t) * h_2(t) + h_3(t) = h_1(t) * [\delta(t+2) + \delta(t-2)] + h_3(t)$$

$$= h_1(t) * \delta(t+2) + h_1(t) * \delta(t-2) + h_3(t)$$

$$= h_1(t+2) + h_1(t-2) + h_3(t)$$

$$h(t) = e^{-2(t+2)}u(t+2) + e^{-2(t-2)}u(t-2) + e^{2t}u(t)$$

(b) **5 pts.** Is the system H memoryless? Justify your answer.

A continuous-time LTI system is memoryless iff $h(t) = K\delta(t)$ for some constant K.

Since that's <u>not</u> the case here, the system is

<u>NOT MEMORYLESS</u>

Problem 2, cont...

A continuous-time LTI system is causal iff $h(t)=0 \ \forall\ t<0$.

(c) **5 pts**. Is the system H causal? Justify your answer.

At $t=-1$, we have

$$h(-1) = e^{-2}u(1) + e^{-2(-3)}\cancel{u(-3)}^{0} + e^{-2}\cancel{u(-1)}^{0} = e^{-2} \neq 0.$$

$\Rightarrow$ NOT CAUSAL

(d) **5 pts**. Is the system H BIBO stable? Justify your answer.

A continuous-time LTI system is stable iff $h(t)$ is absolutely integrable. For this system, we have

$$\int_{-\infty}^{\infty}|h(t)|dt = \int_{-\infty}^{\infty}\left|e^{-2(t+2)}u(t+2) + e^{-2(t-2)}u(t-2) + e^{2t}u(t)\right|dt$$

$$= \int_{-\infty}^{\infty} e^{-2(t+2)}u(t+2) + e^{-2(t-2)}u(t-2) + e^{2t}u(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-2(t+2)}u(t+2)dt + \int_{-\infty}^{\infty} e^{-2(t-2)}u(t-2)dt + \int_{-\infty}^{\infty} e^{2t}u(t)\,dt$$

$$= \int_{-2}^{\infty} e^{-2t}e^{-4}dt + \int_{2}^{\infty} e^{-2t}e^{4}dt + \int_{0}^{\infty} e^{2t}dt$$

$$= \lim_{A\to\infty} e^{-4}\int_{-2}^{A} e^{-2t}dt + \lim_{A\to\infty} e^{4}\int_{2}^{A} e^{-2t}dt + \lim_{A\to\infty}\int_{0}^{A} e^{2t}dt$$

$$= e^{-4}\lim_{A\to\infty}\left[-\tfrac{1}{2}e^{-2t}\right]_{t=-2}^{A} + e^{4}\lim_{A\to\infty}\left[-\tfrac{1}{2}e^{-2t}\right]_{t=2}^{A} + \lim_{A\to\infty}\tfrac{1}{2}e^{2t}\Big|_{t=0}^{A}$$

$$= -\tfrac{1}{2}e^{-4}\lim_{A\to\infty}\left[e^{-2A}-e^{4}\right] - \tfrac{1}{2}e^{4}\lim_{A\to\infty}\left[e^{-2A}-e^{-4}\right] + \tfrac{1}{2}\lim_{A\to\infty}\left[e^{2A}-1\right]$$

$$= -\tfrac{1}{2}e^{-4}\left[0-e^{4}\right] - \tfrac{1}{2}e^{4}\left[0-e^{-4}\right] + \tfrac{1}{2}\lim_{A\to\infty}e^{2A} - \tfrac{1}{2}$$

$$= \tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}\lim_{A\to\infty}e^{2A} = \tfrac{1}{2}+\tfrac{1}{2}\lim_{A\to\infty}e^{2A} \longrightarrow \infty$$

NOT BIBO STABLE

3. **25 pts**. A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-4(t-2)}u(t-1).$$

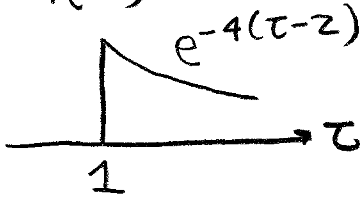
The system input is given by

$$x(t) = e^{-2t}u(t-2).$$

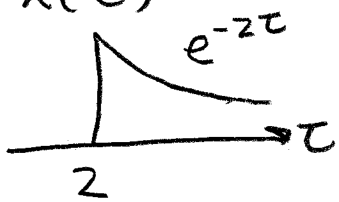
Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

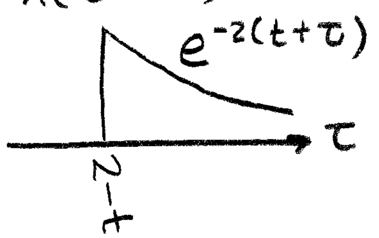
$h(\tau)$



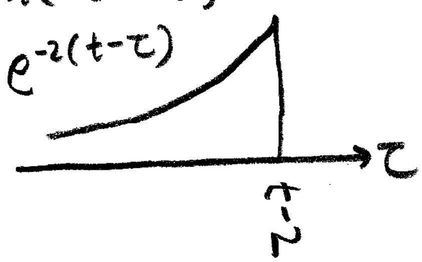
$x(\tau)$



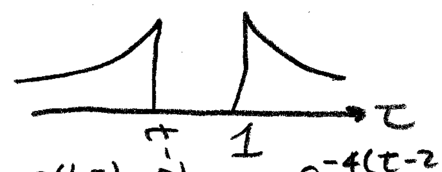
$x(\tau - -t) = x(t+\tau)$



$x(-\tau - -t) = x(t-\tau)$

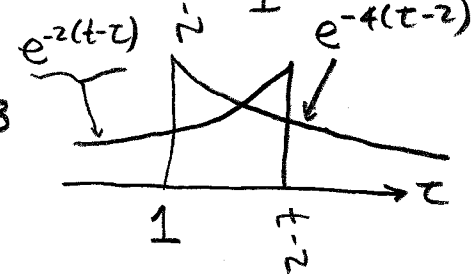


Case I) $t-2 < 1$: $t < 3$:



$$y(t) = \int_{-\infty}^{\infty} 0 \cdot 0 \, dt = 0$$

Case II) $t-2 \geq 1$; $t \geq 3$



$$y(t) = \int_{1}^{t-2} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{1}^{t-2} e^{-4(\tau-2)} e^{-2(t-\tau)} \, d\tau$$

$$= \int_{1}^{t-2} e^{-4\tau} e^{8} e^{-2t} e^{2\tau} d\tau$$

$$= e^{8} e^{-2t} \int_{1}^{t-2} e^{-4\tau} e^{2\tau} d\tau = e^{8} e^{-2t} \int_{1}^{t-2} e^{-2\tau} d\tau$$

$$= e^{8} e^{-2t} \left(-\tfrac{1}{2}\right) \left[e^{-2\tau} \right]_{\tau=1}^{t-2}$$

$$= -\tfrac{1}{2} e^{8} e^{-2t} \left[e^{-2(t-2)} - e^{-2} \right]$$

$$= -\tfrac{1}{2} e^{8} e^{-2t} \left[e^{-2t} e^{4} - e^{-2} \right]$$

$$= \tfrac{1}{2} e^{6} e^{-2t} - \tfrac{1}{2} e^{12} e^{-4t}$$

$$= \frac{e^{6}}{2} \left[e^{-2t} - e^{6} e^{-4t} \right]$$

All together :

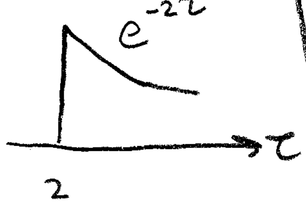
$$y(t) = \begin{cases} 0 & , \quad t < 3 \\ \frac{e^{6}}{2} \left[e^{-2t} - e^{6} e^{-4t} \right], & t \geq 3 \end{cases}$$

$$= \frac{e^{6}}{2} \left[e^{-2t} - e^{6} e^{-4t} \right] u(t-3) \quad //$$

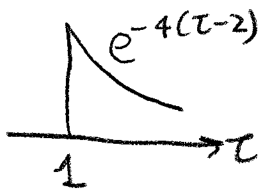
"OTHER WAY" $\quad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

More Workspace for Problem 3...

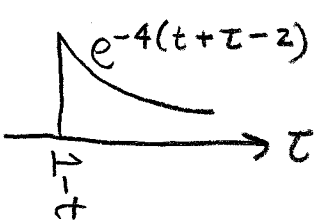
$x(\tau)$



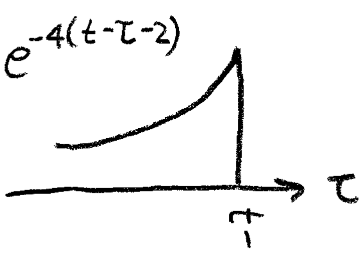
$h(\tau)$



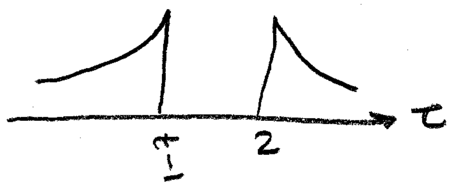
$h(\tau - -t) = h(t+\tau)$



$h(-\tau - -t) = h(t-\tau)$

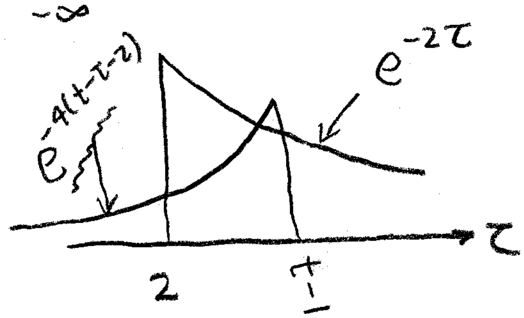


case I) $t-1 < 2 \; : \; t < 3$:



$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0.$$

case II) $t-1 \geq 2 \; : \; t \geq 3$



$$y(t) = \int_{2}^{t-1} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{2}^{t-1} e^{-2\tau} e^{-4(t-\tau-2)}\, d\tau = \int_{2}^{t-1} e^{-2\tau} e^{-4t} e^{4\tau} e^{8}\, d\tau$$

$$= e^{8} e^{-4t} \int_{2}^{t-1} e^{2\tau}\, d\tau = \tfrac{1}{2} e^{8} e^{-4t} \left[e^{2\tau} \right]_{\tau=2}^{t-1}$$

$$= \tfrac{1}{2} e^{8} e^{-4t} \left[e^{2(t-1)} - e^{4} \right]$$

$$= \tfrac{1}{2} e^{8} e^{-4t} \left[e^{2t} e^{-2} - e^{4} \right]$$

$$= \tfrac{1}{2} e^{8} \left[e^{-2} e^{-2t} - e^{4} e^{-4t} \right]$$

$$= \tfrac{1}{2} \left[e^{6} e^{-2t} - e^{12} e^{-4t} \right]$$

$$= \frac{e^{6}}{2} \left[e^{-2t} - e^{6} e^{-4t} \right].$$

All Together: $\quad y(t) = \begin{cases} 0, & t < 3 \\ \frac{e^{6}}{2}\left[e^{-2t} - e^{6} e^{-4t}\right], & t \geq 3 \end{cases}$

$$= \frac{e^{6}}{2} \left[e^{-2t} - e^{6} e^{-4t} \right] u(t-3)$$

4. **25 pts**. A discrete-time LTI system H has impulse response $h[n]$ given by

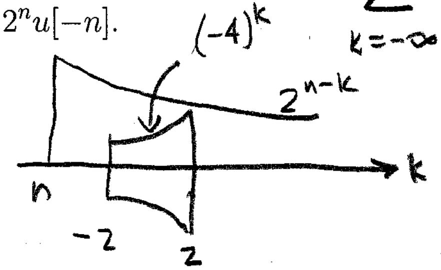
$$h[n] = \begin{cases} (-4)^n, & -2 \le n \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

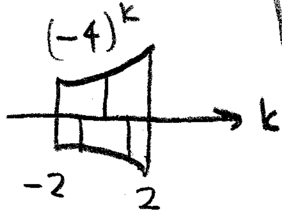
The system input is given by

$$x[n] = 2^n u[-n].$$

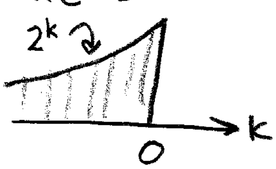
Find the system output $y[n]$.



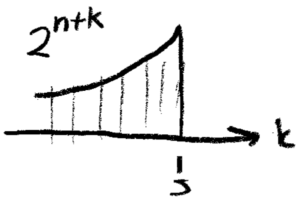
$h[k]$



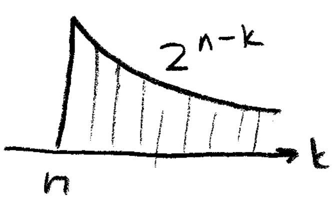
$x[k]$



$x[k - -n] = x[n+k]$



$x[-k - -n] = x[n-k]$



Case I) $n < -2$

$$y[n] = \sum_{k=-2}^{2} h[k]\, x[n-k]$$

$$= \sum_{k=-2}^{2} (-4)^k 2^{n-k} = 2^n \sum_{k=-2}^{2} (-4)^k 2^{-k} = 2^n \sum_{k=-2}^{2} (-4)^k \left(\tfrac{1}{2}\right)^k$$

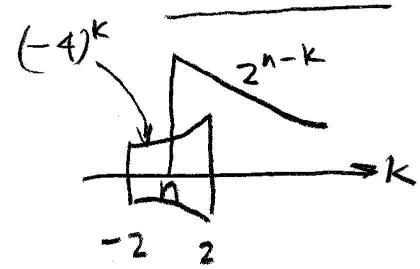
$$= 2^n \sum_{k=-2}^{2} (-2)^k = 2^n \frac{(-2)^{-2} - (-2)^3}{1-(-2)} = 2^n \frac{(-\frac{1}{2})^2 + 8}{3}$$

$$= \tfrac{1}{3} 2^n \left[\tfrac{1}{4} + 8\right] = \tfrac{1}{3} 2^n \left(\tfrac{1}{4} + \tfrac{32}{4}\right) = \tfrac{1}{3} 2^n \cdot \tfrac{33}{4}$$

$$= \tfrac{11}{4} 2^n.$$

Case II) $n \geq -2$ and $n \le 2$: $-2 \le n \le 2$: $\underline{-2 \le n < 3}$

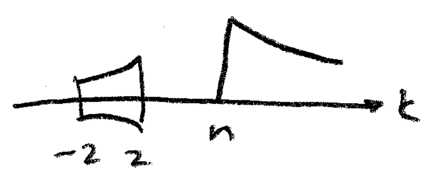
$$y[n] = \sum_{k=n}^{2} (-4)^k 2^{n-k}$$



$$= 2^n \sum_{k=n}^{2} (-4)^k 2^{-k} = 2^n \sum_{k=n}^{2} (-2)^k$$

$$= 2^n \frac{(-2)^n - (-2)^3}{1-(-2)} = 2^n \frac{(-2)^n + 8}{3} = \tfrac{1}{3}(-4)^n + \tfrac{8}{3} 2^n$$

Case III) $n \geq 3$



$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

All Together:

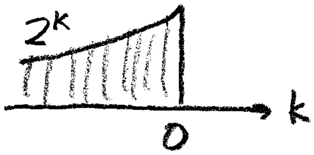
$$y[n] = \begin{cases} \frac{11}{4} 2^n, & n < -2 \\ \frac{1}{3}(-4)^n + \frac{8}{3} 2^n, & -2 \le n < 3 \\ 0, & n \geq 3 \end{cases}$$

"OTHER WAY"

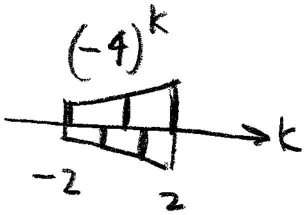
More Workspace for Problem 4...

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

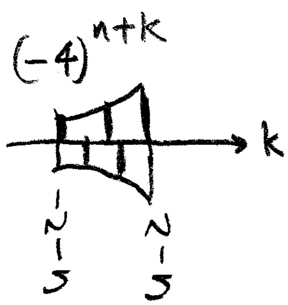
$x[k]$



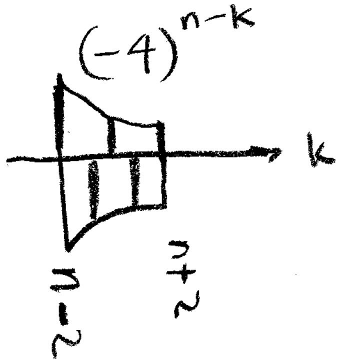
$h[k]$



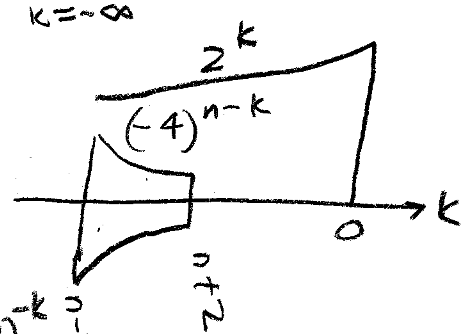
$h[k--n] = h[n+k]$



$h[-k--n] = h[n-k]$



case I) $n+2<0$; $n<-2$



$$y[n] = \sum_{k=n-2}^{n+2} x[k]\,h[n-k]$$

$$= \sum_{k=n-2}^{n+2} 2^k(-4)^{n-k} = \sum_{k=n-2}^{n+2} 2^k(-4)^n(-4)^{-k}$$

$$= (-4)^n \sum_{k=n-2}^{n+2} 2^k\left(-\tfrac{1}{4}\right)^k = (-4)^n \sum_{k=n-2}^{n+2}\left(-\tfrac{1}{2}\right)^k$$

$$= (-4)^n \frac{\left(-\frac{1}{2}\right)^{n-2} - \left(-\frac{1}{2}\right)^{n+3}}{1-\left(-\frac{1}{2}\right)}$$

$$= (-4)^n \frac{\left(-\frac{1}{2}\right)^n(-2)^2 - \left(-\frac{1}{2}\right)^n\left(-\frac{1}{2}\right)^3}{3/2}$$

$$= \frac{2}{3}(-4)^n\left[4\left(-\tfrac{1}{2}\right)^n + \tfrac{1}{8}\left(-\tfrac{1}{2}\right)^n\right]$$

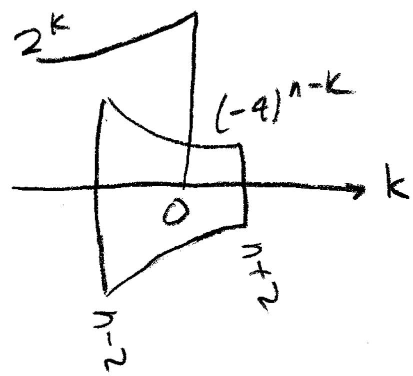
$$= \frac{2}{3}(-4)^n\left(-\tfrac{1}{2}\right)^n\left[4+\tfrac{1}{8}\right] = \frac{2}{3}2^n\left[\frac{32+1}{8}\right]$$

$$= \frac{2\cdot 33}{3\cdot 8}2^n = \frac{11}{4}2^n.$$

Case II) $n+2 \geq 0$ and $n-2 \leq 0$: $n \geq -2$ and $n \leq 2$

: $-2 \leq n \leq 2$

: $-2 \leq n < 3$



$$y[n] = \sum_{k=n-2}^{0} 2^k(-4)^{n-k}$$

$$= \sum_{k=n-2}^{0} 2^k(-4)^n(-4)^{-k}$$

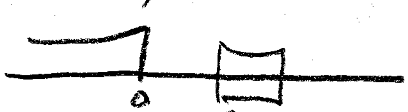
$$= (-4)^n \sum_{k=n-2}^{0} 2^k\left(-\tfrac{1}{4}\right)^k$$

$$= (-4)^n \sum_{k=n-2}^{0}\left(-\tfrac{1}{2}\right)^k$$

$$= (-4)^n \frac{\left(-\frac{1}{2}\right)^{n-2} - \left(-\frac{1}{2}\right)^1}{1-(-1/2)} = (-4)^n\frac{\left(-\frac{1}{2}\right)^n(-2)^2 + \frac{1}{2}}{3/2} = \frac{2}{3}(-4)^n\left[4\left(-\tfrac{1}{2}\right)^n + \tfrac{1}{2}\right]$$

$$= \frac{8}{3}(-4)^n\left(-\tfrac{1}{2}\right)^n + \frac{1}{3}(-4)^n = \frac{1}{3}(-4)^n + \frac{8}{3}2^n$$

Case III) $n-2 \geq 1$: $n \geq 3$:



$y[n] = 0$

ALL TOGETHER:

$$y[n] = \begin{cases} \frac{11}{4}2^n & , n<-2 \\ \frac{1}{3}(-4)^n + \frac{8}{3}2^n & , -2 \leq n < 3 \\ 0 & , n \geq 3 \end{cases}$$