ECE 3793
Test 1

Wednesday, March 23, 2016
7:00 PM - 10:00 PM

Spring 2016

Dr. Havlicek

Name: SOLUTION

Student Num:

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:

Date:

1. **25 pts.** The input $x(t)$ and output $y(t)$ of a continuous-time system are related by

$$y(t) = \mathcal{E}v\{x(t)\}. = \frac{1}{2}\{x(t) + x(-t)\}$$

(a) **4 pts.** Is the system memoryless? Justify your answer. $= \frac{1}{2}x(t) + \frac{1}{2}x(-t)$

When $t = -10$, we have

$$y(-10) = \frac{1}{2}x(-10) + \frac{1}{2}x(10).$$

Since the value of the output signal at $t = -10$ depends on the value of the input signal at $t = 10$, which is a different time,

the system is not memoryless.

(b) **4 pts.** Is the system causal? Justify your answer.

As shown in part (a), the value of the output signal at $t = -10$ depends on $x(10)$, the value of the input signal from a future time.

Therefore, the system is not causal.

Problem 1, cont...

(c) 4 pts. Is the system linear? Justify your answer.

Let the input signal be $x_1(t)$. Then the output signal is $y_1(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_1(-t)$.
Let the input signal be $x_2(t)$. Then the output signal is $y_2(t) = \frac{1}{2}x_2(t) + \frac{1}{2}x_2(-t)$.
Now let $a, b \in \mathbb{C}$ be constants and let the input signal be $x_3(t) = a x_1(t) + b x_2(t)$.
Then the output signal is

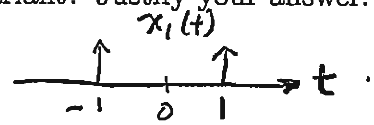
$$y_3(t) = \tfrac{1}{2}x_3(t) + \tfrac{1}{2}x_3(-t) = \tfrac{1}{2}\{a x_1(t) + b x_2(t)\} + \tfrac{1}{2}\{a x_1(-t) + b x_2(-t)\}$$
$$= \tfrac{1}{2}a x_1(t) + \tfrac{1}{2}a x_1(-t) + \tfrac{1}{2}b x_2(t) + \tfrac{1}{2}b x_2(-t)$$
$$= a\{\tfrac{1}{2}x_1(t) + \tfrac{1}{2}x_1(-t)\} + b\{\tfrac{1}{2}x_2(t) + \tfrac{1}{2}x_2(-t)\}$$
$$= a y_1(t) + b y_2(t) \ \checkmark$$

$\longrightarrow$ Therefore, the system <u>is</u> <u>linear</u>.

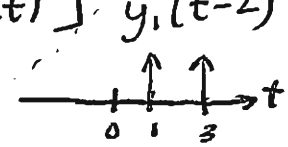
NOTES page 1.67:
$\delta(at) = \frac{1}{|a|}\delta(t)$
so
$\delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$

(d) 4 pts. Is the system time invariant? Justify your answer.

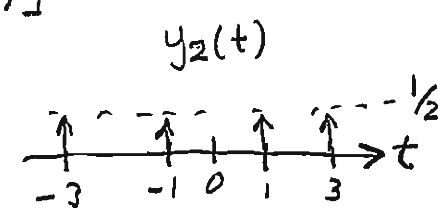
Let $x_1(t) = \delta(t+1) + \delta(t-1)$



Then $y_1(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_1(-t) = \frac{1}{2}\delta(t+1) + \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(-t+1) + \frac{1}{2}\delta(-t-1)$
$$= \tfrac{1}{2}\delta(t+1) + \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta[-(t-1)] + \tfrac{1}{2}\delta[-(t+1)]$$
$$= \tfrac{1}{2}\delta(t+1) + \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t+1) = \delta(t+1) + \delta(t-1).$$
[since $x_1(t)$ was already even, we get $y_1(t) = x_1(t)$]



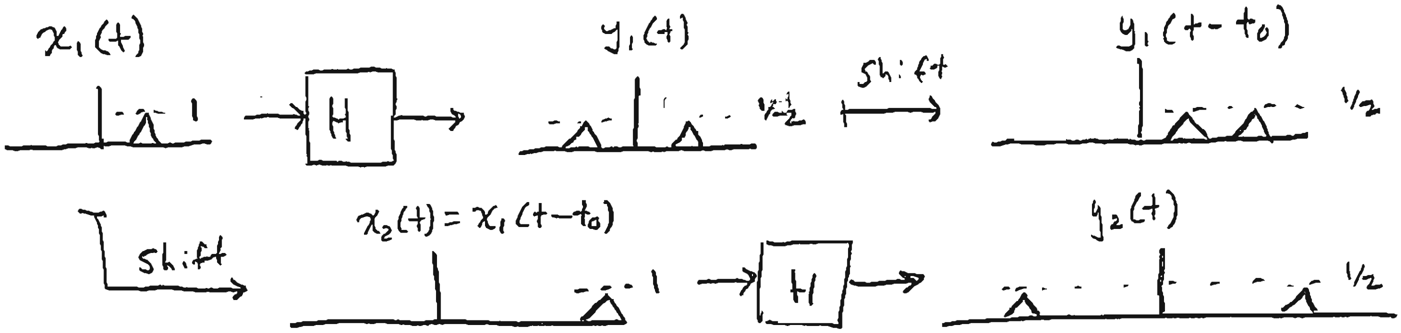
Now let $t_0 = 2$. Then $y_1(t - t_0) = y_1(t-2) = \delta(t-1) + \delta(t-3)$
Next, let $x_2(t) = x_1(t-2) = \delta(t-1) + \delta(t-3)$. Then
$$y_2(t) = \tfrac{1}{2}x_2(t) + \tfrac{1}{2}x_2(-t) = \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-3) + \tfrac{1}{2}\delta(-t-1) + \tfrac{1}{2}\delta(-t-3)$$
$$= \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-3) + \tfrac{1}{2}\delta[-(t+1)] + \tfrac{1}{2}\delta[-(t+3)]$$
$$= \tfrac{1}{2}\delta(t-1) + \tfrac{1}{2}\delta(t-3) + \tfrac{1}{2}\delta(t+1) + \tfrac{1}{2}\delta(t+3)$$



Since $y_2(t) \neq y_1(t-2)$,

the system is <u>not</u> <u>time</u> <u>invariant</u>.

Here is another way to do 1(d) using pictures:



$$y_2(t) \neq y_1(t-t_0)$$

$\longrightarrow$ not time invariant.

And here is a "general" solution using the rule of thumb on page 1.86 of the notes:

Case I) input is $x_1(t)$. Put it through the system, then shift.

$$x_1(t) \longrightarrow \boxed{H} \longrightarrow y_1(t) \xmapsto{\text{shift}} y_1(t-t_0)$$

First xform: through the system $t \mapsto \begin{cases} t \\ -t \end{cases}$

Second xform: shift $t \mapsto t-t_0$

Rule of thumb: $t \mapsto \begin{cases} t-t_0 \\ -(t-t_0) = -t+t_0 \end{cases}$

$$y_1(t-t_0) = \tfrac{1}{2}x_1(t-t_0) + \tfrac{1}{2}x(-t+t_0)$$

Case II) input is $x_2(t) = x_1(t-t_0)$. Put it thrugh the system.

$$x_1(t) \xmapsto{\text{Shift}} x_2(t) = x_1(t-t_0) \longrightarrow \boxed{H} \longrightarrow y_2(t)$$

First xform: shift $t \mapsto t-t_0$, independent variable is "t"

Second xform: through the system $t \mapsto \begin{cases} t \\ -t \end{cases}$

Rule of thumb: $t \mapsto \begin{cases} t-t_0 \\ -t-t_0 \end{cases}$

$$y_2(t) = \tfrac{1}{2}x(t-t_0) + \tfrac{1}{2}x(-t-t_0)$$

$\Rightarrow y_2(t) \neq y_1(t-t_0) \longrightarrow$ Not time invariant.

Problem 1, cont...

(e) 4 pts. Is the system Stable? Justify your answer.

Let $x(t)$ be a bounded input signal. Then $\exists B \in \mathbb{R}$, $B > 0$, such that $|x(t)| \leq B \ \forall t \in \mathbb{R}$.

Moreover, $|y(t)| = |\frac{1}{2}x(t) + \frac{1}{2}x(-t)| = \frac{1}{2}|x(t) + x(-t)|$

$$\leq \frac{1}{2}\{|x(t)| + |x(-t)|\}$$

$$\leq \frac{1}{2}(B + B) = \frac{1}{2} \cdot 2B = B.$$

Therefore, $|y(t)| \leq B \ \forall t \in \mathbb{R}$ and the output signal $y(t)$ is bounded.

→ Since every bounded input signal produces a bounded output signal, the system is stable.

(f) 5 pts. Is the system H invertible? Justify your answer.

- If you answer *yes*, then give the input-output relation for the inverse system.
- If you answer *no*, then give two distinct input signals that both result in the same output signal.

- Any odd signal has an even part that is everywhere zero. So it will produce the same output (all zero signal) as the input $x(t) = 0$.

→ I guess that the system is not invertible.

Let $x_1(t) = \sin(t)$ [an odd signal].

Then $y_1(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_1(-t) = \frac{1}{2}\sin(t) + \frac{1}{2}\sin(-t)$

$$= \frac{1}{2}\sin(t) + \frac{1}{2}\sin(-t) = \frac{1}{2}\sin(t) - \frac{1}{2}\sin(t) = 0 \quad \forall t.$$

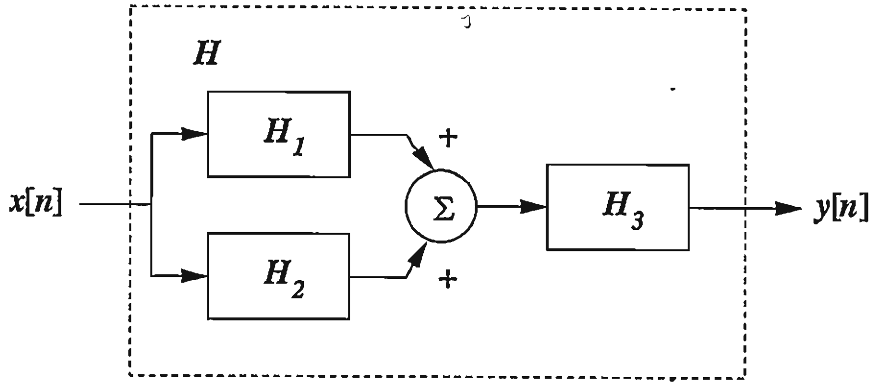
Now let $x_2(t) = 0 \ \forall t$ (the signal that is everywhere zero).

Then $y_2(t) = \frac{1}{2}x_2(t) + \frac{1}{2}x_2(-t) = \frac{1}{2}0 + \frac{1}{2}0 = 0 \quad \forall t.$

So $y_1(t) = y_2(t)$, but $x_1(t)$ and $x_2(t)$ are different.

Since two distinct input signals both produced the same output signal, the system is not invertible.

2. **25 pts.** The discrete-time LTI system H is formed by connecting three LTI systems H_1, H_2, and H_3 as shown in the figure below.



It follows immediately from results proven in class that the overall system H is LTI. The impulse responses of the three LTI systems H_1 through H_3 are given by

$$\begin{aligned} h_1[n] &= 2^n u[-n], \\ h_2[n] &= \left(\frac{1}{4}\right)^n u[n], \\ h_3[n] &= \delta[n-3]. \end{aligned}$$

(a) **10 pts.** Find the impulse response $h[n]$.

$$h[n] = \left(h_1[n] + h_2[n]\right) * h_3[n] = h_1[n] * h_3[n] + h_2[n] * h_3[n]$$

$$= h_1[n] * \delta[n-3] + h_2[n] * \delta[n-3]$$

$$= h_1[n-3] + h_2[n-3]$$

$$= 2^{n-3} u[-(n-3)] + \left(\tfrac{1}{4}\right)^{n-3} u[n-3]$$

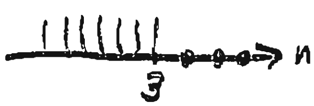
$$= 2^n 2^{-3} u[-n+3] + \left(\tfrac{1}{4}\right)^n \left(\tfrac{1}{4}\right)^{-3} u[n-3]$$

$$= \left(\tfrac{1}{2}\right)^3 2^n u[-n+3] + 4^3 \left(\tfrac{1}{4}\right)^n u[n-3]$$

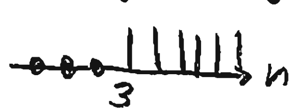
$$= \tfrac{1}{8} 2^n u[-n+3] + 64 \left(\tfrac{1}{4}\right)^n u[n-3]$$

$$\boxed{h[n] = \tfrac{1}{8} 2^n u[-n+3] + 64 \left(\tfrac{1}{4}\right)^n u[n-3]}$$

$u[-n+3]$: on at $n=3$ goes left



$u[n-3]$: on at $n=3$ goes right



(b) 4 pts. Is the system H memoryless? Justify your answer.

A discrete-time LTI system is memoryless iff the impulse response is a number times $\delta[n]$.
As shown in part (a), that is <u>not</u> the case for this system.

$\longrightarrow$ The system is <u>not</u> <u>memoryless.</u>

(c) 4 pts. Is the system H causal? Justify your answer.

A discrete-time LTI system is causal iff $h[n] = 0 \ \forall \ n < 0$.

But for this system, we have, e.g.,

$$h[-1] = \frac{1}{8} 2^{-1} = \frac{1}{16} \neq 0.$$

$\longrightarrow$ The system is <u>not</u> <u>causal</u>.

Problem 2, cont...

(d) 7 pts. Is the system H BIBO stable? Justify your answer.

A discrete-time LTI system is stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

For this system, we have

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left| \tfrac{1}{8} 2^n u[-n+3] + 64 \left(\tfrac{1}{4}\right)^n u[n-3] \right|$$

$$= \sum_{n=-\infty}^{\infty} \tfrac{1}{8} 2^n u[-n+3] + 64 \left(\tfrac{1}{4}\right)^n u[n-3]$$

$$= \tfrac{1}{8} \sum_{n=-\infty}^{3} 2^n + 64 \sum_{n=3}^{\infty} \left(\tfrac{1}{4}\right)^n = \tfrac{1}{8} \lim_{A\to\infty} \sum_{n=-A}^{3} 2^n + 64 \lim_{A\to\infty} \sum_{n=3}^{A} \left(\tfrac{1}{4}\right)^n$$

$$= \tfrac{1}{8} \lim_{A\to\infty} \frac{\cancel{2^{-A}}^{\to 0} - 2^4}{1-2} + 64 \lim_{A\to\infty} \frac{\left(\tfrac{1}{4}\right)^3 - \cancel{\left(\tfrac{1}{4}\right)^{A+1}}^{\to 0}}{1 - 1/4}$$

$$= \tfrac{1}{8} \frac{-16}{-1} + 64 \frac{\tfrac{1}{64}}{3/4} = \frac{16}{8} + \frac{1}{3/4}$$

$$= 2 + \frac{4}{3} = \frac{6}{3} + \frac{4}{3} = \frac{10}{3} < \infty \quad //$$

$\longrightarrow$ The system <u>is</u> <u>BIBO</u> <u>stable</u>.

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = \begin{cases} e^{2t}, & -1 \le t \le 2, \\ 0, & \text{otherwise} \end{cases}$$

$$= e^{2t}\{u(t+1) - u(t-2)\}.$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

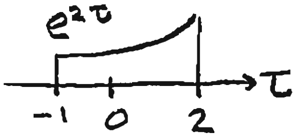
The system input is given by

$$x(t) = e^{-4t}u(-t+1).$$

Find the system output $y(t)$.

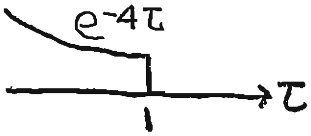
$h(\tau)$

$e^{2\tau}$



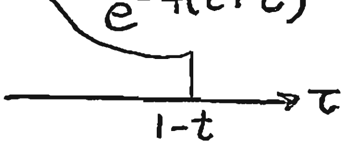
$x(\tau)$

$e^{-4\tau}$



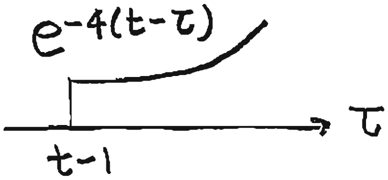
$x(\tau - -t) = x(t+\tau)$

$e^{-4(t+\tau)}$

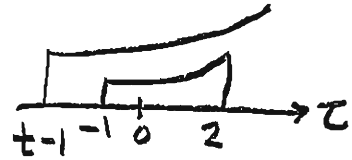


$x(-\tau - -t) = x(t-\tau)$

$e^{-4(t-\tau)}$



case I) $t-1 < -1$; $t < 0$:



- The product graph is nonzero from $\tau = -1$ to $\tau = 2$

$$y(t) = \int_{-1}^{2} e^{2\tau} e^{-4(t-\tau)}\,d\tau = e^{-4t}\int_{-1}^{2} e^{2\tau}e^{4\tau}\,d\tau$$

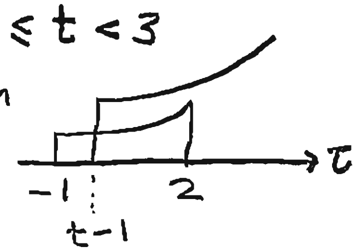
$$= e^{-4t}\int_{-1}^{2} e^{6\tau}\,d\tau = e^{-4t}\,\frac{1}{6}\left[e^{6\tau}\right]_{\tau=-1}^{2}$$

$$= \frac{1}{6}e^{-4t}\left[e^{12} - e^{-6}\right] = \frac{e^{12} - e^{-6}}{6}e^{-4t}$$

Case II) $t-1 \geq -1$ and $t-1 < 2$

$t \geq 0$ and $t < 3$: $0 \leq t < 3$

- The product graph is nonzero from $\tau = t-1$ to $\tau = 2$.



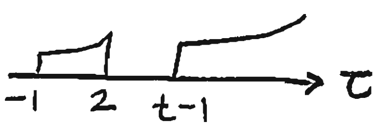
$$y(t) = \int_{t-1}^{2} e^{2\tau} e^{-4(t-\tau)}\,d\tau$$

$$= e^{-4t}\int_{t-1}^{2} e^{6\tau}\,d\tau = \frac{1}{6}e^{-4t}\left[e^{6\tau}\right]_{\tau=t-1}^{2}$$

$$= \frac{1}{6}e^{-4t}\left[e^{12} - e^{6(t-1)}\right]$$

$$= \frac{1}{6}e^{-4t}\left[e^{12} - e^{-6}e^{6t}\right] = \frac{e^{12}}{6}e^{-4t} - \frac{e^{-6}}{6}e^{2t}$$

Case III) $t \geq 3$



: The product graph is everywhere zero

$$y(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0$$

All Together:

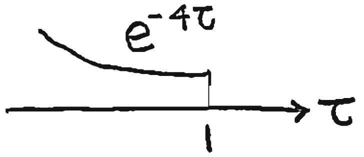
$$\boxed{y(t) = \begin{cases} \dfrac{e^{12} - e^{-6}}{6}e^{-4t}, & t < 0 \\ \dfrac{e^{12}}{6}e^{-4t} - \dfrac{e^{-6}}{6}e^{2t}, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}}$$

8

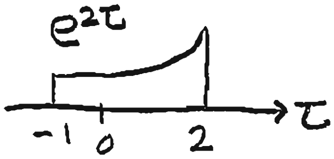
OTHER WAY

More Workspace for Problem 3...

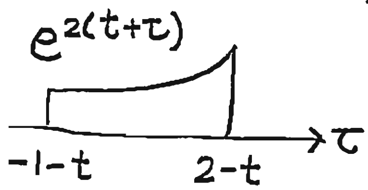
$x(\tau)$



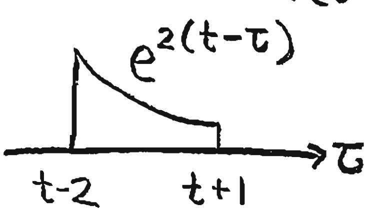
$h(\tau)$



$h(\tau - -t) = h(t+\tau)$



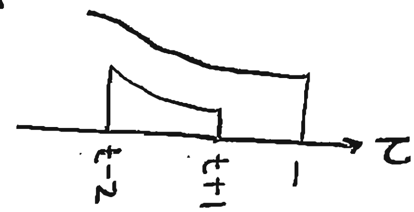
$h(-\tau - -t) = h(t-\tau)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

case I) $t+1 < 1$: $t<0$

The product graph is nonzero from $\tau = t-2$ to $\tau = t+1$.



$$y(t) = \int_{t-2}^{t+1} e^{-4\tau} e^{2(t-\tau)} d\tau = e^{2t} \int_{t-2}^{t+1} e^{-4\tau} e^{-2\tau} d\tau$$

$$= e^{2t} \int_{t-2}^{t+1} e^{-6\tau} d\tau = -\frac{1}{6} e^{2t} \left[e^{-6\tau} \right]_{\tau=t-2}^{t+1}$$

$$= -\frac{1}{6} e^{2t} \left[e^{-6(t+1)} - e^{-6(t-2)} \right]$$

$$= -\frac{1}{6} e^{2t} \left[e^{-6t} e^{-6} - e^{-6t} e^{12} \right]$$

$$= -\frac{1}{6} e^{2t} e^{-6t} \left[e^{-6} - e^{12} \right] = -\frac{1}{6} e^{-4t} \left[e^{-6} - e^{12} \right]$$

$$= \frac{1}{6} e^{-4t} \left[e^{12} - e^{-6} \right] = \frac{e^{12} - e^{-6}}{6} e^{-4t}$$

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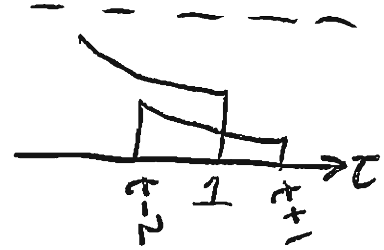
case II) $t+1 \geq 1$ and $t-2 < 1$

$t \geq 0$ and $t < 3$

$0 \leq t < 3$



The product graph is nonzero from $\tau = t-2$ to $\tau = 1$

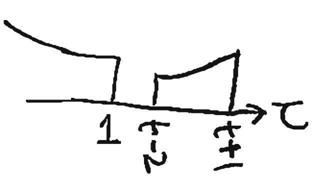
$$y(t) = \int_{t-2}^{1} e^{-4\tau} e^{2(t-\tau)} d\tau = e^{2t} \int_{t-2}^{1} e^{-4\tau} e^{-2\tau} d\tau = e^{2t} \int_{t-2}^{1} e^{-6\tau} d\tau$$

$$= -\frac{1}{6} e^{2t} \left[e^{-6\tau} \right]_{\tau=t-2}^{1} = -\frac{1}{6} e^{2t} \left[e^{-6} - e^{-6(t-2)} \right]$$

$$= -\frac{1}{6} e^{2t} \left[e^{-6} - e^{-6t} e^{12} \right] = \frac{1}{6} e^{2t} \left[e^{12} e^{-6t} - e^{-6} \right]$$

$$= \frac{e^{12}}{6} e^{-4t} - \frac{e^{-6}}{6} e^{2t}$$

Case III) $t \geq 3$



The product graph is everywhere zero.

$$y(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$$

All Together:

$$y(t) = \begin{cases} \frac{e^{12} - e^{-6}}{6} e^{-4t}, & t<0 \\ \frac{e^{12}}{6} e^{-4t} - \frac{e^{-6}}{6} e^{2t}, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

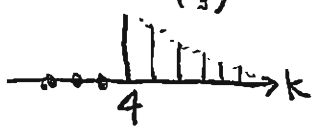
The system input is given by

$$x[n] = \left(\frac{1}{3}\right)^{n-3} u[n-4].$$

Find the system output $y[n]$.

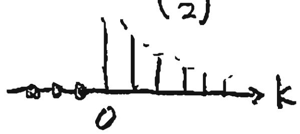
$x[k]$

$\left(\frac{1}{3}\right)^{k-3}$



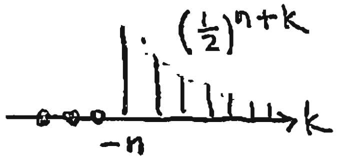
$h[k]$

$\left(\frac{1}{2}\right)^k$



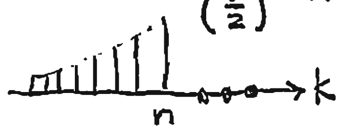
$h[k - -n] = h[n+k]$

$\left(\frac{1}{2}\right)^{n+k}$

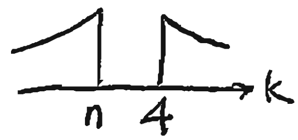


$h[-k - -n] = h[n-k]$

$\left(\frac{1}{2}\right)^{n-k}$



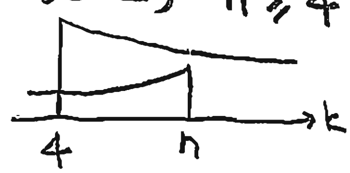
case I) $n < 4$



The product graph is everywhere zero

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

Case II) $n \geq 4$



The product graph is nonzero from $k=4$ to $k=n$.

$$y[n] = \sum_{k=4}^{n} \left(\tfrac{1}{3}\right)^{k-3}\left(\tfrac{1}{2}\right)^{n-k} = \sum_{k=4}^{n} \left(\tfrac{1}{3}\right)^{k}\left(\tfrac{1}{3}\right)^{-3}\left(\tfrac{1}{2}\right)^{n}\left(\tfrac{1}{2}\right)^{-k}$$

$$= \left(\tfrac{1}{3}\right)^{-3}\left(\tfrac{1}{2}\right)^{n} \sum_{k=4}^{n} \left(\tfrac{1}{3}\right)^{k}\left(\tfrac{1}{2}\right)^{-k} = 3^3\left(\tfrac{1}{2}\right)^{n} \sum_{k=4}^{n} \left(\tfrac{1}{3}\right)^{k} 2^k$$

$$= 27\left(\tfrac{1}{2}\right)^{n} \sum_{k=4}^{n} \left(\tfrac{2}{3}\right)^{k} = 27\left(\tfrac{1}{2}\right)^{n} \frac{\left(\tfrac{2}{3}\right)^4 - \left(\tfrac{2}{3}\right)^{n+1}}{1 - 2/3}$$

$$= 27\left(\tfrac{1}{2}\right)^{n} \frac{\tfrac{16}{81} - \tfrac{2}{3}\left(\tfrac{2}{3}\right)^n}{1/3} = 81\left(\tfrac{1}{2}\right)^{n}\left[\tfrac{16}{81} - \tfrac{2}{3}\left(\tfrac{2}{3}\right)^n\right]$$

$$= 16\left(\tfrac{1}{2}\right)^{n} - \frac{2 \cdot 81}{3}\left(\tfrac{2}{3}\right)^n\left(\tfrac{1}{2}\right)^n$$

$$= 16\left(\tfrac{1}{2}\right)^{n} - 2 \cdot 27\left(\tfrac{1}{3}\right)^n = 16\left(\tfrac{1}{2}\right)^{n} - 54\left(\tfrac{1}{3}\right)^n$$

All Together:

$$y[n] = \begin{cases} 0 & , \; n < 4 \\ 16\left(\tfrac{1}{2}\right)^{n} - 54\left(\tfrac{1}{3}\right)^n & , \; n \geq 4 \end{cases}$$

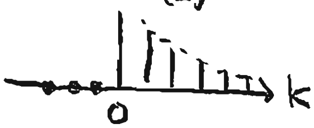
$$= \left\{16\left(\tfrac{1}{2}\right)^{n} - 54\left(\tfrac{1}{3}\right)^n\right\} u[n-4]$$

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ OTHER WAY

More Workspace for Problem 4...

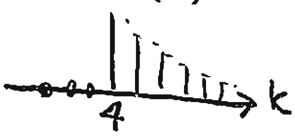
$h[k]$

$\left(\frac{1}{2}\right)^k$



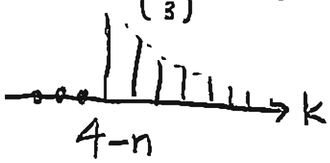
$x[k]$

$\left(\frac{1}{3}\right)^{k-3}$



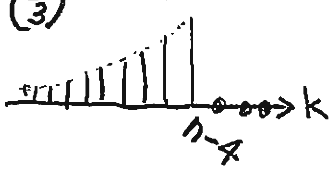
$x[k--n] = x[n+k]$

$\left(\frac{1}{3}\right)^{n+k-3}$

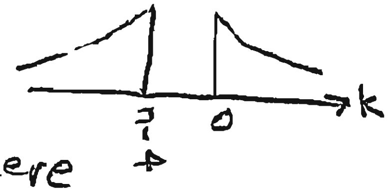


$x[-k--n] = x[n-k]$

$\left(\frac{1}{3}\right)^{n-k-3}$



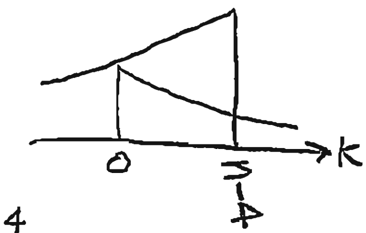
case I) $n-4 < 0$: $n < 4$



The product graph is everywhere zero.

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$

case II) $n-4 \geq 0$: $n \geq 4$



The product graph is nonzero from $k=0$ to $k=n-4$.

$$y[n] = \sum_{k=0}^{n-4} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{3}\right)^{n-k-3} = \sum_{k=0}^{n-4} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-k} \left(\tfrac{1}{3}\right)^{-3}$$

$$= \left(\tfrac{1}{3}\right)^n \left(\tfrac{1}{3}\right)^{-3} \sum_{k=0}^{n-4} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{3}\right)^{-k}$$

$$= 3^3 \left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-4} \left(\tfrac{1}{2}\right)^k 3^k = 27\left(\tfrac{1}{3}\right)^n \sum_{k=0}^{n-4} \left(\tfrac{3}{2}\right)^k$$

$$= 27\left(\tfrac{1}{3}\right)^n \frac{\left(\frac{3}{2}\right)^0 - \left(\frac{3}{2}\right)^{n-3}}{1 - 3/2}$$

$$= 27\left(\tfrac{1}{3}\right)^n \frac{1 - \left(\frac{3}{2}\right)^n \left(\frac{3}{2}\right)^{-3}}{-1/2} = 54\left(\tfrac{1}{3}\right)^n \left[\left(\tfrac{2}{3}\right)^3 \left(\tfrac{3}{2}\right)^n - 1\right]$$

$$= 54\left(\tfrac{1}{3}\right)^n \left[\tfrac{8}{27}\left(\tfrac{3}{2}\right)^n - 1\right]$$

$$= \frac{54 \cdot 8}{27}\left(\tfrac{1}{3}\right)^n \left(\tfrac{3}{2}\right)^n - 54\left(\tfrac{1}{3}\right)^n = 2 \cdot 8\left(\tfrac{1}{2}\right)^n - 54\left(\tfrac{1}{3}\right)^n$$

$$= 16\left(\tfrac{1}{2}\right)^n - 54\left(\tfrac{1}{3}\right)^n$$

All Together: $y[n] = \begin{cases} 0 & , n < 4 \\ 16\left(\frac{1}{2}\right)^n - 54\left(\frac{1}{3}\right)^n & , n \geq 4 \end{cases}$

$$= \left\{16\left(\tfrac{1}{2}\right)^n - 54\left(\tfrac{1}{3}\right)^n\right\} u[n-4]$$