ECE 3793
Test 1

Wednesday, March 8, 2017
6:30 PM - 9:30 PM

Spring 2017 Name: SOLUTION

Dr. Havlicek Student Num:__________________

Directions: This test is closed book and closed notes. Calculators are not allowed. All work must be your own. You have 180 minutes to complete the test. You may use the formula sheet provided with the test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________

TOTAL (100):

____

On my honor, I affirm that I have neither given nor received inappropriate aid in the completion of this test.

Name:__________________ Date:__________________

1. **25 pts.** The input $x(t)$ and output $y(t)$ of a continuous-time system are related by

$$y(t) = \begin{cases} tx(t+1), & t \geq 0, \\ x(t), & t < 0. \end{cases}$$

(a) **4 pts.** Is the system memoryless? Justify your answer.

When $t=2$, the value of the output signal is

$y(2) = 2x(3)$

which depends on the value of the input signal from a <u>different</u> time $(t=3)$.

Therefore, the system is <u>not</u> memoryless.

(b) **4 pts.** Is the system causal? Justify your answer.

As shown in part (a), $y(2)$ depends on $x(3)$, which is a <u>future</u> value of the input signal.

Therefore, the system is <u>not</u> causal.

Problem 1, cont...

(c) 4 pts. Is the system linear? Justify your answer.

There does not seem to be anything like squaring or addition of a constant that could interfere with linear combinations. So I guess "yes!"

Let the input be $x_1(t)$. Then the output is $y_1(t) = \begin{cases} t x_1(t+1), & t \geq 0 \\ x_1(t), & t < 0 \end{cases}$.

Let the input be $x_2(t)$. Then the output is $y_2(t) = \begin{cases} t x_2(t+1), & t \geq 0 \\ x_2(t), & t < 0 \end{cases}$

Let $a, b \in \mathbb{C}$ be constants and let
$x_3(t) = a x_1(t) + b x_2(t)$.

Then
$$y_3(t) = \begin{cases} t x_3(t+1), & t \geq 0 \\ x_3(t), & t < 0 \end{cases} = \begin{cases} t[a x_1(t+1) + b x_2(t+1)], & t \geq 0 \\ a x_1(t) + b x_2(t), & t < 0 \end{cases}$$

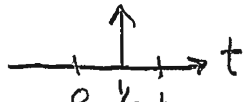
$$= \begin{cases} a t x_1(t+1) + b t x_2(t+1), & t \geq 0 \\ a x_1(t) + b x_2(t), & t < 0 \end{cases} = a y_1(t) + b y_2(t) \checkmark$$

$\Rightarrow$ The system <u>is</u> <u>linear</u>.

(d) 4 pts. Is the system time invariant? Justify your answer.

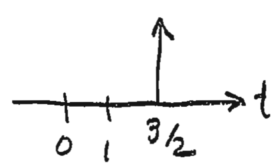
For $t \geq 0$, the input signal gets multiplied by "t"! This will not commute with time shifts. So I guess that the system is not time invariant. Also note that the value of the input signal for $0 \leq t < 1$ does not make it to the output... which will also cause time invariance to fail. So I guess "no"!

Let $x_1(t) = \delta(t - ½)$



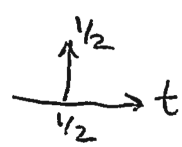
Then $y_1(t) = \begin{cases} 0, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow y_1(t) = 0 \ \forall t \in \mathbb{R}$

Now let $t_0 = 1$ and $x_2(t) = x_1(t-1) = \delta(t - 3/2)$



Then
$$y_2(t) = \begin{cases} t x_2(t+1), & t \geq 0 \\ x_2(t), & t < 0 \end{cases} = \begin{cases} t\delta(t+1-3/2), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \begin{cases} t\delta(t-\tfrac{1}{2}), & t \geq 0 \\ 0, & t < 0 \end{cases} = t\delta(t-\tfrac{1}{2}) = \tfrac{1}{2}\delta(t-½)$$



$y_2(t) \neq y_1(t-t_0) \mapsto$ <u>NOT TIME INVARIANT</u>

Problem 1, cont...

(e) **4 pts.** Is the system BIBO stable? Justify your answer.

For $t \geq 0$, the input is multiplied by "t" which will grow without bound as $t \to \infty$. I guess "no".

Let $x(t) = u(t)$. Then $|x(t)| \leq 1 \ \forall \ t \in \mathbb{R}$. So $x(t)$ is bounded by $B = 1$. The output is given by

$$y(t) = \begin{cases} t u(t+1), & t \geq 0 \\ 0, & t < 0 \end{cases} = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = t u(t)$$

For any $B \in \mathbb{R}$ such that $B > 0$, let $t_0 = B + 1$.

Then $|y(t_0)| = |y(B+1)| = (B+1) u(B+1) = B + 1 > B$.

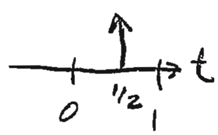
Therefore, there is no positive number $B \in \mathbb{R}$ that bounds $y(t)$. So $y(t)$ is unbounded. Since a bounded input signal produced an unbounded output signal, the system is UNSTABLE.

(f) **5 pts.** Is the system H invertible? Justify your answer.

- If you answer *yes*, then give the input-output relation for the inverse system.
- If you answer *no*, then give two distinct input signals that both result in the same output signal.

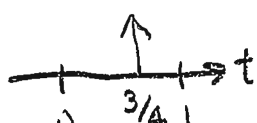
The values of the input signal for $0 \leq t < 1$ never make it to the output. So two input signals that differ only on $0 \leq t < 1$ will both make the same output signal. I guess "no".

Let $x_1(t) = \delta(t - 1/2)$



Then $y_1(t) = \begin{cases} 0, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow y_1(t) = 0 \quad \forall \ t \in \mathbb{R}$.

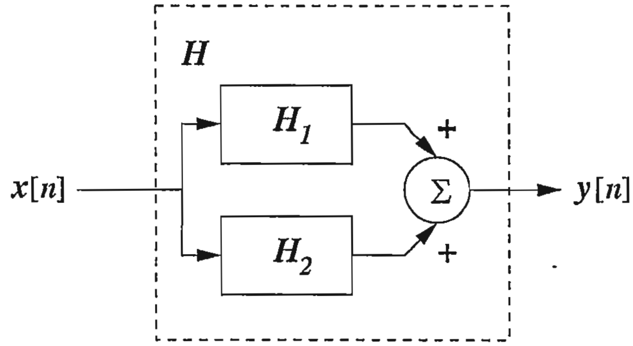
Let $x_2(t) = \delta(t - 3/4)$



Then $y_2(t) = 0 = y_1(t)$.

Since two different input signals both made the same output signal, the system is NOT INVERTIBLE.

2. **25 pts.** The discrete-time system H is formed by connecting two discrete-time LTI systems H_1 and H_2 in parallel as shown in the figure below.



Hint: we proved in class that the parallel connection of two LTI systems is LTI.
The impulse response of H_1 is given by $h_1[n] = \left(\frac{1}{2}\right)^n u[n]$.
The impulse response of H_2 is given by $h_2[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1]$.

(a) **7 pts.** Is the system H memoryless? Justify your answer.

First find the impulse response of H. Note that

$$h_2[n] = \left(\tfrac{1}{2}\right)^{n+1} u[n+1] = \underbrace{\left(\tfrac{1}{2}\right)^{-1+1} \delta[n+1]}_{n=-1} + \underbrace{\left(\tfrac{1}{2}\right)^{n+1} u[n]}_{n\geq 0}$$

$$= \delta[n+1] + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n]$$

So $h[n] = h_1[n] + h_2[n]$

$$= \left(\tfrac{1}{2}\right)^n u[n] + \delta[n+1] + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^n u[n]$$

$$= \delta[n+1] + \left(1+\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^n u[n] = \delta[n+1] + \tfrac{3}{2}\left(\tfrac{1}{2}\right)^n u[n].$$

$\Rightarrow$ Since $h[n]$ is <u>not</u> a constant times $\delta[n]$,

the system is <u>not memoryless</u>.

Problem 2, cont...

(b) **8 pts.** Is the system H causal? Justify your answer.

From part (a), $h[n] = \delta[n+1] + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n]$.

$h[-1] = 1 \neq 0$.

Since $h[n]$ is not zero $\forall\, n<0$,

the system is <u>not</u> <u>causal</u>.

(c) **10 pts.** Is the system H BIBO stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left| \delta[n+1] + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n] \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |\delta[n+1]| + \sum_{n=-\infty}^{\infty} \left| \frac{3}{2}\left(\frac{1}{2}\right)^n u[n] \right|$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{3}{2}\lim_{A\to\infty}\sum_{n=0}^{A}\left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{3}{2}\lim_{A\to\infty}\frac{\left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^{A+1}}{1-1/2} = 1 + \frac{3}{2}\,\frac{1}{1-1/2}$$

$$= 1 + \frac{3}{2}\,\frac{1}{1/2} = 1 + \frac{3}{2}\cdot 2 = 1+3 = 4 < \infty.$$

Since the impulse response is absolutely summable,

the system <u>IS</u> <u>BIBO</u> <u>STABLE</u>.

3. **25 pts.** A continuous-time LTI system H has impulse response $h(t)$ given by

$$h(t) = e^{-6t}u(t-1).$$

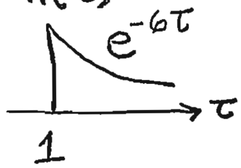
The system input is given by

$$x(t) = e^{-5t}u(t+1).$$

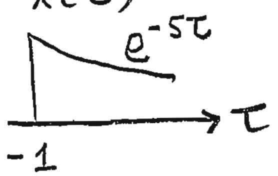
Find the system output $y(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

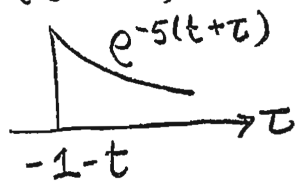
$h(\tau)$



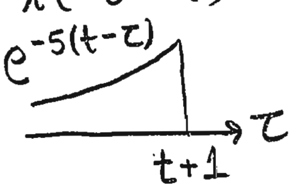
$x(\tau)$



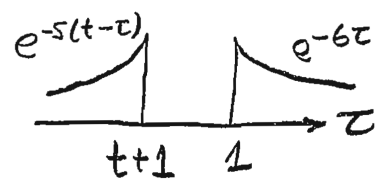
$x(\tau - -t) = x(t+\tau)$



$x(-\tau - -t) = x(t-\tau)$

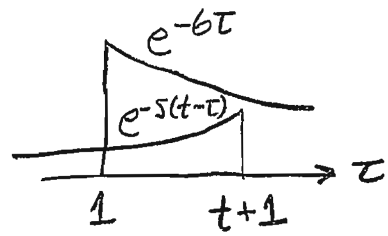


case I) $t+1 < 1$; $t < 0$



$$y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

case II) $t+1 \geq 1$: $t \geq 0$



$$y(t) = \int_{1}^{t+1} e^{-6\tau} e^{-5(t-\tau)}\, d\tau$$

$$= \int_{1}^{t+1} e^{-6\tau} e^{-5t} e^{5\tau}\, d\tau$$

$$= e^{-5t}\int_{1}^{t+1} e^{-\tau}\, d\tau = e^{-5t}\frac{1}{-1}e^{-\tau}\Big|_{\tau=1}^{t+1}$$

$$= -e^{-5t}\left[e^{-(t+1)} - e^{-1}\right]$$

$$= e^{-5t}e^{-1} - e^{-5t}e^{-t}e^{-1}$$

$$= e^{-5t-1} - e^{-6t-1}$$

All Together:

$$y(t) = \begin{cases} e^{-5t-1} - e^{-6t-1} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

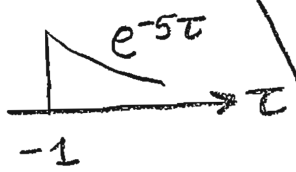
$$\boxed{y(t) = e^{-5t-1}u(t) - e^{-6t-1}u(t)}$$

OTHER WAY

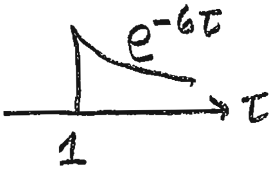
More Workspace for Problem 3...

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

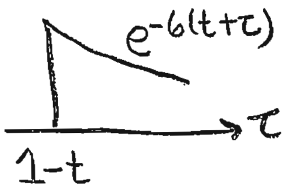
$x(\tau)$



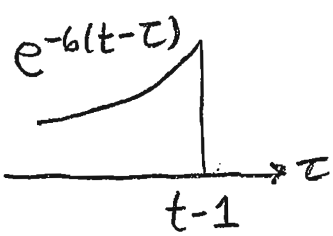
$h(\tau)$



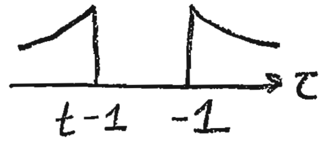
$h(\tau - -t) = h(t+\tau)$



$h(-\tau - -t) = h(t-\tau)$

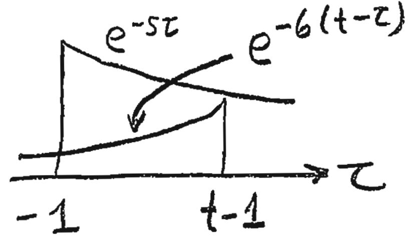


case I) $t-1 < -1 \;:\; t < 0$



$$y(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0.$$

case II) $t-1 \geq -1 \;:\; t \geq 0$



$$y(t) = \int_{-1}^{t-1} e^{-5\tau} e^{-6(t-\tau)}\,d\tau$$

$$= \int_{-1}^{t-1} e^{-5\tau} e^{-6t} e^{6\tau}\,d\tau = e^{-6t}\int_{-1}^{t-1} e^{\tau}\,d\tau$$

$$= e^{-6t} e^{\tau}\Big|_{\tau=-1}^{t-1} = e^{-6t}\left[e^{t-1} - e^{-1}\right]$$

$$= e^{-6t}e^{t}e^{-1} - e^{-6t}e^{-1}$$

$$= e^{-5t-1} - e^{-6t-1}$$

All Together:
$$y(t) = \begin{cases} e^{-5t-1} - e^{-6t-1}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{y(t) = e^{-5t-1}u(t) - e^{-6t-1}u(t)}$$

4. **25 pts.** A discrete-time LTI system H has impulse response $h[n]$ given by

$$h[n] = \begin{cases} \left(\frac{2}{5}\right)^n, & -3 \le n \le 4, \\ 0, & \text{otherwise} \end{cases} \quad = \quad \left(\frac{2}{5}\right)^n \{u[n+3] - u[n-5]\}.$$

The system input is given by

$$x[n] = 24\left(-\frac{1}{5}\right)^n u[n-1].$$

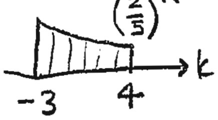
$y[n] = x[n] * h[n]$

$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

Find the system output $y[n]$.

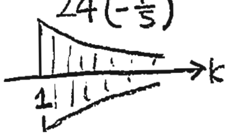
$h[k]$

$\left(\frac{2}{5}\right)^k$



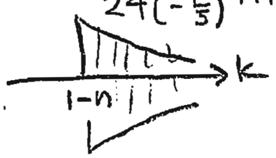
$x[k]$

$24\left(-\frac{1}{5}\right)^k$



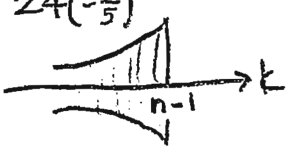
$x[k--n] = x[n+k]$

$24\left(-\frac{1}{5}\right)^{n+k}$

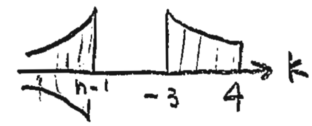


$x[-k--n] = x[n-k]$

$24\left(-\frac{1}{5}\right)^{n-k}$



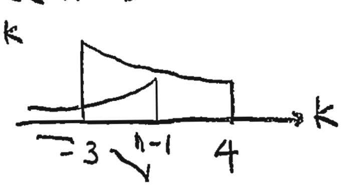
Case I) $n-1 < -3$: $n < -2$



$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$

Case II) $n-1 \geq -3$ and $n-1 < 4$: $n \geq -2$ and $n < 5$: $-2 \le n < 5$

$y[n] = \sum_{k=-3}^{n-1} \left(\frac{2}{5}\right)^k 24\left(-\frac{1}{5}\right)^{n-k} = 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{n-1} \left(\frac{2}{5}\right)^k \left(-\frac{1}{5}\right)^{-k}$

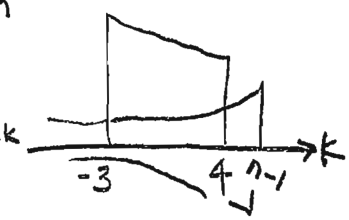


$= 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{n-1} \left(\frac{2}{5}\right)^k (-5)^k = 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{n-1} (-2)^k$

$= 24\left(-\frac{1}{5}\right)^n \frac{(-2)^{-3} - (-2)^n}{1-(-2)} = 24\left(-\frac{1}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^3 - (-2)^n}{3}$

$= 8\left(-\frac{1}{5}\right)^n\left(-\frac{1}{8}\right) - 8\left(-\frac{1}{5}\right)^n(-2)^n = -\left(-\frac{1}{5}\right)^n - 8\left(\frac{2}{5}\right)^n$

Case III) $n-1 \geq 4$: $n \geq 5$



$y[n] = \sum_{k=-3}^{4} \left(\frac{2}{5}\right)^k 24\left(-\frac{1}{5}\right)^{n-k} = 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{4} \left(\frac{2}{5}\right)^k\left(-\frac{1}{5}\right)^{-k}$

$= 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{4} \left(\frac{2}{5}\right)^k (-5)^k = 24\left(-\frac{1}{5}\right)^n \sum_{k=-3}^{4} (-2)^k$

$= 24\left(-\frac{1}{5}\right)^n \frac{(-2)^{-3} - (-2)^5}{1-(-2)} = 24\left(-\frac{1}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^3 + 32}{3}$

$= 8\left(-\frac{1}{5}\right)^n \left[-\frac{1}{8} + 32\right] = \left[-1 + 8\cdot 32\right]\left(-\frac{1}{5}\right)^n$

$= [256-1]\left(-\frac{1}{5}\right)^n = 255\left(-\frac{1}{5}\right)^n$

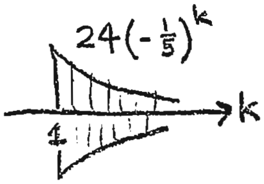
$$\boxed{\text{All Together:} \quad y[n] = \begin{cases} 0, & n < -2 \\ -\left(-\frac{1}{5}\right)^n - 8\left(\frac{2}{5}\right)^n, & -2 \le n < 5 \\ 255\left(-\frac{1}{5}\right)^n, & n \geq 5 \end{cases}}$$

OTHER WAY

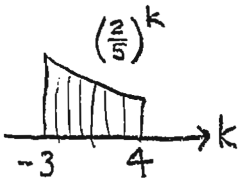
More Workspace for Problem 4...

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

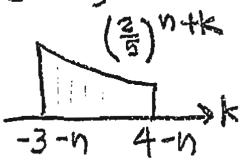
$x[k]$



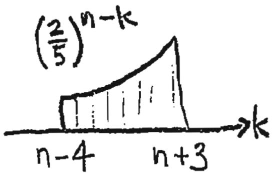
$h[k]$



$h[k--n] = h[n+k]$

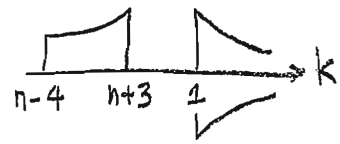


$h[-k--n] = h[n-k]$

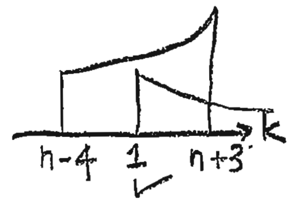


Case I) $n+3 < 1 \,:\, n < -2$

$$y[n] = \sum_{k=-\infty}^{\infty} 0 = 0$$



case II) $n+3 \geq 1$ and $n-4 < 1 \,:\, n \geq -2$ and $n < 5 \,:\, -2 \leq n < 5$



$$y[n] = \sum_{k=1}^{n+3} 24\left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{n-k} = 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{-k}$$

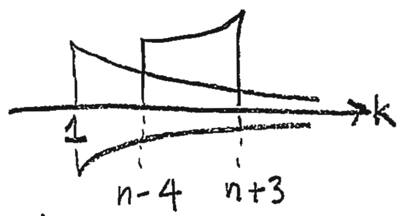
$$= 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{5}{2}\right)^k = 24\left(\tfrac{2}{5}\right)^n \sum_{k=1}^{n+3} \left(-\tfrac{1}{2}\right)^k$$

$$= 24\left(\tfrac{2}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^1 - \left(-\frac{1}{2}\right)^{n+4}}{1-\left(-\frac{1}{2}\right)} = 24\left(\tfrac{2}{5}\right)^n \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)^n \left(-\frac{1}{2}\right)^4}{3/2}$$

$$= \tfrac{2}{3}\cdot 24\left(\tfrac{2}{5}\right)^n \left[-\tfrac{1}{2} - \tfrac{1}{16}\left(-\tfrac{1}{2}\right)^n\right] = 16\left(\tfrac{2}{5}\right)^n \left[-\tfrac{1}{2} - \tfrac{1}{16}\left(-\tfrac{1}{2}\right)^n\right]$$

$$= -8\left(\tfrac{2}{5}\right)^n - \left(\tfrac{2}{5}\right)^n\left(-\tfrac{1}{2}\right)^n = -\left(-\tfrac{1}{5}\right)^n - 8\left(\tfrac{2}{5}\right)^n$$

case III) $n-4 \geq 1 \,:\, n \geq 5$



$$y[n] = \sum_{k=n-4}^{n+3} 24\left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{n-k} = 24\left(\tfrac{2}{5}\right)^n \sum_{k=n-4} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{2}{5}\right)^{-k}$$

$$= 24\left(\tfrac{2}{5}\right)^n \sum_{k=n-4}^{n+3} \left(-\tfrac{1}{5}\right)^k \left(\tfrac{5}{2}\right)^k = 24\left(\tfrac{2}{5}\right)^n \sum_{k=n-4}^{n+3} \left(-\tfrac{1}{2}\right)^k$$

$$= 24\left(\tfrac{2}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^{n-4} - \left(-\frac{1}{2}\right)^{n+4}}{1-\left(-1/2\right)} = 24\left(\tfrac{2}{5}\right)^n \frac{\left(-\frac{1}{2}\right)^n\left(-\frac{1}{2}\right)^{-4} - \left(-\frac{1}{2}\right)^n\left(-\frac{1}{2}\right)^4}{3/2}$$

$$= \tfrac{2}{3}\cdot 24\left(\tfrac{2}{5}\right)^n \left[\left(-\tfrac{1}{2}\right)^n(-2)^4 - \left(-\tfrac{1}{2}\right)^n\left(-\tfrac{1}{2}\right)^4\right] = 16\left(\tfrac{2}{5}\right)^n\left(-\tfrac{1}{2}\right)^n\left[16 - \tfrac{1}{16}\right]$$

$$= \left(-\tfrac{1}{5}\right)^n\left[256 - 1\right] = 255\left(-\tfrac{1}{5}\right)^n$$

All Together:

$$y[n] = \begin{cases} 0, & n < -2 \\ -\left(-\frac{1}{5}\right)^n - 8\left(\frac{2}{5}\right)^n, & -2 \leq n < 5 \\ 255\left(-\frac{1}{5}\right)^n, & n \geq 5 \end{cases}$$