ECE 3793
Test 1

Wednesday, March 3, 1999
12:30 PM - 1:20 PM

SP99
~~Fall 1998~~

Name: SOLUTION

Dr. Havlicek

Student Num: __________

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 50 minutes to complete the test. All work must be your own. You may use one 8.5 $\times$ 11 inch two-sided note sheet as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. **2.** **3.** **4.** **5.**

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts.** The input $x(t)$ and output $y(t)$ of linear time invariant system are related by

$$y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau - 1)\, d\tau.$$

(a) **8 pts.** Find the system impulse response $h(t)$.

h(t) is the output when $\delta(t)$ is the input:

$$h(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} \delta(\tau - 1)\, d\tau = \begin{cases} 0, & t < 1 \\ e^{-2(t-1)}, & t \geq 1 \end{cases}$$

$$= e^{-2(t-1)} u(t-1)$$

(b) **6 pts.** Is the system causal? Justify your answer.

The system is causal, because $h(t) = 0 \ \forall \ t < 0$.

Problem 1, cont...

(c) **6 pts.** Is the system stable? Justify your answer.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1)\, dt = \int_{1}^{\infty} e^{-2(t-1)}\, dt$$

$$= \int_{1}^{\infty} e^{-2t} e^{2} dt = e^{2} \int_{1}^{\infty} e^{-2t} dt = e^{2}\left(-\tfrac{1}{2}\right)\left[e^{-2t}\right]_{1}^{\infty}$$

$$= -\frac{e^{2}}{2}\left[0 - e^{-2}\right] = \frac{1}{2}.$$

The system <u>is</u> stable, because $\int_{-\infty}^{\infty} |h(t)|dt < \infty$.

(d) **5 pts.** Is the system memoryless? Justify your answer.

The system is <u>not</u> memoryless, because $y(t)$ depends on $x(t-1)$ and all values of the input before time $t-1$.

2. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system are related by

$$y[n] = \mathcal{O}d\{x[n-1]\}.$$

(a) **5 pts.** Is the system memoryless? Justify your answer.

$$y[n] = \frac{1}{2}\{x[n-1] - x[-n-1]\}.$$

The system is <u>not</u> memoryless, because $y[n]$ depends on $x[n-1]$ and $x[-n-1]$.

(b) **5 pts.** Is the system time invariant? Justify your answer.

Let $x_1[n]$ be an arbitrary input. Then $y_1[n] = \frac{1}{2}\{x_1[n-1] - x_1[-n-1]\}$.

Then $y_1[n-n_0] = \frac{1}{2}\{x_1[n-1-n_0] - x_1[-n-1-n_0]\}$.

Let $x_2[n] = x_1[n-n_0]$.

Then
$$\begin{aligned} y_2[n] &= \frac{1}{2}\{x_2[n-1] - x_2[-n-1]\} \\ &= \frac{1}{2}\{x_1[n-n_0-1] - x_1[-n+n_0-1]\} \\ &\neq y_1[n-n_0]. \end{aligned}$$

Therefore, the system is <u>not</u> shift invariant.

(c) **5 pts.** Is the system causal? Justify your answer.

The system is <u>not</u> causal, because, for example,

$$y[-1] = \frac{1}{2}\{x[-2] - x[0]\},$$

which depends on the <u>future</u> input $x[0]$.

Problem 2, cont...

(d) **5 pts.** Is the system linear? Justify your answer.

Let $x_1[n]$ and $x_2[n]$ be two arbitrary inputs.

Then $y_1[n] = \frac{1}{2}\{x_1[n-1] - x_1[-n-1]\}$ and $y_2[n] = \frac{1}{2}\{x_2[n-1] - x_2[-n-1]\}$.

Let $x_3[n] = a x_1[n] + b x_2[n]$, where a and b are constants.

$$
\begin{aligned}
\text{Then } y_3[n] &= \tfrac{1}{2}\{x_3[n-1] - x_3[-n-1]\} \\
&= \tfrac{1}{2}\{a x_1[n-1] + b x_2[n-1] - a x_1[-n-1] - b x_2[-n-1]\} \\
&= a\tfrac{1}{2}\{x_1[n-1] - x_1[-n-1]\} + b\tfrac{1}{2}\{x_2[n-1] - x_2[-n-1]\} \\
&= a y_1[n] + b y_2[n].
\end{aligned}
$$

Therefore, the system <u>is</u> linear.

(e) **5 pts.** Is the system Stable? Justify your answer.

Let $x[n]$ be a bounded input. Then $\exists B \in \mathbb{R}$, $B \geq 0$, such that

$|x[n]| \leq B \;\; \forall n \in \mathbb{Z}$.

$$
\begin{aligned}
\text{Then } |y[n]| &= \left|\tfrac{1}{2}\{x[n-1] - x[-n-1]\right| \\
&= \tfrac{1}{2}|x[n-1] - x[-n-1]| \\
&\leq \tfrac{1}{2}(|x[n-1]| + |x[-n-1]|) \\
&\leq \tfrac{1}{2}(B + B) \\
&= B.
\end{aligned}
$$

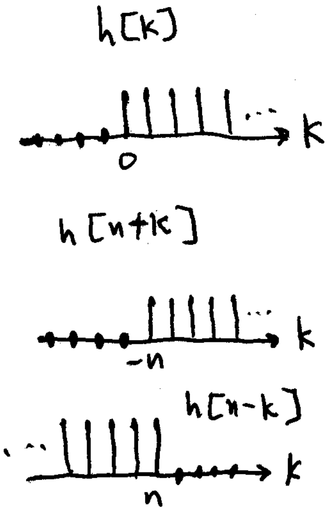
Therefore, $y[n]$ is bounded by B and the system <u>is</u> stable.

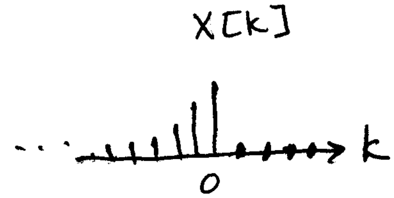
3. **25 pts.** The unit pulse response of a discrete-time LTI system is given by

$$h[n] = u[n].$$

Find the system output when the input is

$$x[n] = 2^n u[-n].$$





For $n<0$, $\displaystyle y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$= \sum_{k=-\infty}^{n} 2^k \qquad m = n-k, \quad k = n-m$$

$$= \sum_{m=\infty}^{0} 2^{n-m} = 2^n \sum_{m=0}^{\infty} 2^{-m}$$

$$= 2^n \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m = 2^n \frac{1}{1-1/2} = 2 \cdot 2^n = 2^{n+1}$$

For $n \geq 0$,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k \qquad m=-k, \quad k=-m$$

$$= \sum_{m=\infty}^{0} 2^{-m} = \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m = \frac{1}{1-1/2} = 2$$

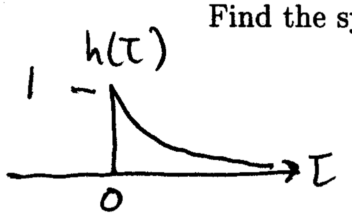
$$y[n] = \begin{cases} 2^{n+1}, & n<0 \\ 2, & n \geq 0 \end{cases}$$

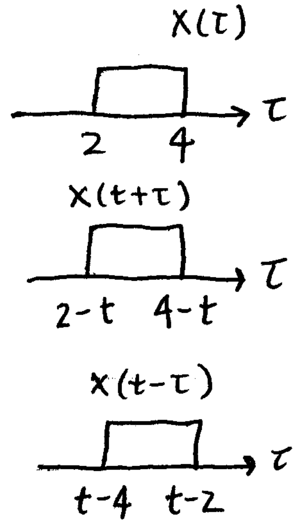
4. **25 pts.** The impulse response of a continuous-time LTI system is given by

$$h(t) = e^{-3t}u(t).$$

Find the system output when the input is

$$x(t) = u(t-2) - u(t-4).$$





For $t-2<0$, or $t<2$, $y(t)=0$.

For $t-2\geq 0$, or $t\geq 2$, and $t-4<0$, or $t<4$,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_0^{t-2} e^{-3\tau}\,d\tau = -\frac{1}{3}\Big[e^{-3\tau}\Big]_0^{t-2}$$

$$= -\frac{1}{3}\left[e^{-3(t-2)}-1\right] = \frac{1}{3}\left[1-e^{-3(t-2)}\right]$$

For $t-4\geq 0$, or $t\geq 4$,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{t-4}^{t-2} e^{-3\tau}\,d\tau = -\frac{1}{3}\Big[e^{-3\tau}\Big]_{t-4}^{t-2}$$

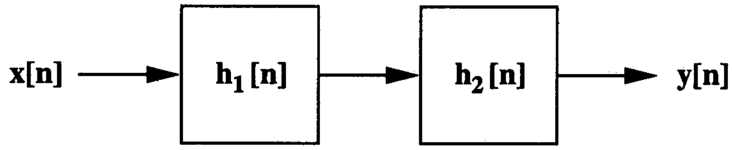
$$= -\frac{1}{3}\left[e^{-3(t-2)}-e^{-3(t-4)}\right] = \frac{1}{3}\left[e^{-3(t-4)}-e^{-3(t-2)}\right]$$

$$= \frac{1}{3}\left[e^{-3t}e^{12}-e^{-3t}e^{6}\right] = \frac{1}{3}\left[e^{-3t}e^{12}-e^{-3t}e^{12}e^{-6}\right]$$

$$= \frac{1}{3}(1-e^{-6})e^{-3t}e^{12} = \frac{1}{3}(1-e^{-6})e^{-3(t-4)}.$$

$$y(t) = \begin{cases} 0, & t<2 \\ \frac{1}{3}\left[1-e^{-3(t-2)}\right], & 2\leq t<4 \\ \frac{1}{3}(1-e^{-6})e^{-3(t-4)}, & t\geq 4 \end{cases}$$

5. **25 pts**. A discrete-time system H is formed by cascading two discrete-time LTI systems H_1 and H_2, as shown in the figure below:



The unit pulse response $h_1[n]$ is given by

$$h_1[n] = \sin 8n,$$

while the unit pulse response $h_2[n]$ is given by

$$h_2[n] = \alpha^n u[n],$$

where $|\alpha| < 1$. Find the system output $y[n]$ when the input is

$$x[n] = \delta[n] - \alpha\delta[n-1].$$

$$y[n] = x[n] * h_1[n] * h_2[n]$$
$$= h_1[n] * x[n] * h_2[n].$$

Now, $x[n] * h_2[n] = \left(\delta[n] - \alpha\delta[n-1]\right) * \left(\alpha^n u[n]\right)$

$$= \delta[n] * \alpha^n u[n] - \alpha\left(\delta[n-1] * \alpha^n u[n]\right)$$
$$= \alpha^n u[n] - \alpha\left(\alpha^{n-1} u[n-1]\right) = \alpha^n u[n] - \alpha^n u[n-1]$$
$$= \alpha^n \left(u[n] - u[n-1]\right) = \alpha^n \delta[n] = \delta[n]$$

So,

$$y[n] = h_1[n] * \delta[n] = h_1[n] = \underline{\underline{\sin 8n}}$$