ECE 3793  
Test 1  

Wednesday, March 3, 1999  
12:30 PM - 1:20 PM  

Name: SOLUTION  
Student Num:  

Dr. Havlicek  

Directions: There are five problems on this test. Work any four of them. Only four problems will be graded. You have 50 minutes to complete the test. All work must be your own. You may use one 8.5 × 11 inch two-sided note sheet as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the four problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):
1. **25 pts.** The input $x(t)$ and output $y(t)$ of linear time invariant system are related by

$$y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}x(\tau - 1) \, d\tau.$$ 

(a) **8 pts.** Find the system impulse response $h(t)$.

$h(t)$ is the output when $\delta(t)$ is the input:

$$h(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} \delta(\tau - 1) \, d\tau = \begin{cases} 
0, & t < 1 \\
e^{-2(t-1)}, & t \geq 1 
\end{cases}$$

$$= e^{-2(t-1)}u(t-1)$$

(b) **6 pts.** Is the system causal? Justify your answer.

The system is causal, because $h(t) = 0 \forall t < 0$. 


Problem 1, cont...

(c) 6 pts. Is the system stable? Justify your answer.

\[
\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1) dt = \int_{1}^{\infty} e^{-2(t-1)} dt = \int_{1}^{\infty} e^{-2t} dt = e^2 \int_{1}^{\infty} e^{-2t} dt = e^2 \left( -\frac{1}{2} \right) \left[ e^{-2t} \right]_{1}^{\infty} = -\frac{e^2}{2} \left[ 0 - e^{-2} \right] = \frac{1}{2}.
\]

The system is stable, because \( \int_{-\infty}^{\infty} |h(t)| dt < \infty. \)

(d) 5 pts. Is the system memoryless? Justify your answer.

The system is not memoryless, because \( y(t) \) depends on \( x(t-1) \) and all values of the input before time \( t-1 \).
2. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time system are related by

$$y[n] = 0d\{x[n-1]\}.$$

(a) **5 pts.** Is the system memoryless? Justify your answer.

$$y[n] = \frac{1}{2} \{ x[n-1] - x[n-1] \}.$$ 

The system is **not** memoryless, because $y[n]$ depends on $x[n-1]$ and $x[n-1].$

(b) **5 pts.** Is the system time invariant? Justify your answer.

Let $x_1[n]$ be an arbitrary input. Then $y_1[n] = \frac{1}{2} \{ x_1[n-1] - x_1[n-1] \}.$

Then $y_1[n-n_0] = \frac{1}{2} \{ x_1[n-1-n_0] - x_1[n-1-n_0] \}.$

Let $x_2[n] = x_1[n-n_0].$

Then $y_2[n] = \frac{1}{2} \{ x_2[n-1] - x_2[n-1] \}$

$$= \frac{1}{2} \{ x_1[n-n_0-1] - x_1[n+n_0-1] \}$$

$$\neq y_1[n-n_0].$$

Therefore, the system is **not** shift invariant.

(c) **5 pts.** Is the system causal? Justify your answer.

The system is **not** causal, because, for example,

$$y[\cdot-1] = \frac{1}{2} \{ x[\cdot-2] - x[\cdot] \},$$

which depends on the **future** input $x[\cdot].$
Problem 2, cont...

(d) 5 pts. Is the system linear? Justify your answer.

Let \( x_1[n] \) and \( x_2[n] \) be two arbitrary inputs.

Then \( y_1[n] = \frac{1}{2} \left\{ x_1[n-1] - x_1[n-2] \right\} \) and \( y_2[n] = \frac{1}{2} \left\{ x_2[n-1] - x_2[n-2] \right\} \).

Let \( x_3[n] = a \cdot x_1[n] + b \cdot x_2[n] \), where \( a \) and \( b \) are constants.

Then \( y_3[n] = \frac{1}{2} \left\{ x_3[n-1] - x_3[n-2] \right\} \)

\[ = \frac{1}{2} \left\{ ax_1[n-1] + bx_2[n-1] - ax_1[n-2] - bx_2[n-2] \right\} \]

\[ = \frac{1}{2} \left\{ ax_1[n-1] - x_1[n-1] \right\} + b \cdot \frac{1}{2} \left\{ bx_2[n-1] - x_2[n-1] \right\} \]

\[ = ay_1[n] + by_2[n]. \]

Therefore, the system is linear.

(e) 5 pts. Is the system Stable? Justify your answer.

Let \( x[n] \) be a bounded input. Then \( \exists B \in \mathbb{R}, B > 0 \), such that

\[ |x[n]| \leq B \quad \forall n \in \mathbb{Z}. \]

Then \( |y[n]| = \frac{1}{2} |x[n-1] - x[n-2]| \)

\[ = \frac{1}{2} |x[n-1] - x[n-2]| \]

\[ \leq \frac{1}{2} (|x[n-1]| + |x[n-2]|) \]

\[ \leq \frac{1}{2} (B + B) \]

\[ = B. \]

Therefore, \( y[n] \) is bounded by \( B \) and the system is stable.
3. **25 pts.** The unit pulse response of a discrete-time LTI system is given by

\[ h[n] = u[n]. \]

Find the system output when the input is

\[ x[n] = 2^n u[-n]. \]

For \( n < 0 \),

\[ y[n] = \sum_{k=0}^{\infty} x[k] h[n-k] \]

\[ = \sum_{k=0}^{\infty} 2^k \quad m = n-k, \quad k = n-m \]

\[ = \sum_{m=0}^{\infty} 2^{n-m} = 2^n \sum_{m=0}^{\infty} 2^{-m} \]

\[ = 2^n \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^m = 2^n \frac{1}{1 - \frac{1}{2}} = 2 \cdot 2^n = 2^{n+1} \]

For \( n \geq 0 \),

\[ y[n] = \sum_{k=0}^{\infty} x[k] h[n-k] = \sum_{k=0}^{\infty} 2^k \quad m = -k, \quad k = -m \]

\[ = \sum_{m=0}^{\infty} 2^{-m} = \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^m = \frac{1}{1 - \frac{1}{2}} = 2 \]

\[ y[n] = \begin{cases} 
2^{n+1}, & n < 0 \\
2, & n \geq 0
\end{cases} \]
4. **25 pts.** The impulse response of a continuous-time LTI system is given by

\[ h(t) = e^{-3t}u(t). \]

Find the system output when the input is

\[ x(t) = u(t - 2) - u(t - 4). \]

For \( t < 0 \), or \( t < 2 \), \( y(t) = 0 \).

For \( t \geq 0 \), or \( t \geq 2 \), and \( t < 4 \), or \( t < 4 \),

\[ y(t) = \int_{-\infty}^{t} h(\tau) x(t - \tau) \, d\tau = \int_{0}^{t-2} e^{-3\tau} \, d\tau = -\frac{1}{3} \left[ e^{-3\tau} \right]_0^{t-2} = -\frac{1}{3} \left[ e^{-3(t-2)} - 1 \right] = \frac{1}{3} \left[ 1 - e^{-3(t-2)} \right] \]

For \( t > 4 \), or \( t > 4 \),

\[ y(t) = \int_{-\infty}^{t} h(\tau) x(t - \tau) \, d\tau = \int_{t-4}^{t-2} e^{-3\tau} \, d\tau = -\frac{1}{3} \left[ e^{-3\tau} \right]_{t-4}^{t-2} = -\frac{1}{3} \left[ e^{-3(t-4)} - e^{-3(t-2)} \right] = \frac{1}{3} \left[ e^{-3(t-4)} - e^{-3(t-2)} \right] = \frac{1}{3} \left[ e^{-3t} e^{12} - e^{-3t} e^{6} \right] = \frac{1}{3} \left[ e^{-3t} e^{12} - e^{-3t} e^{12} e^{-6} \right] = \frac{1}{3} \left( 1 - e^{-6} \right) e^{-3(t-4)} \]

\[
y(t) = \begin{cases} 
0, & t < 2 \\
\frac{1}{3} \left[ 1 - e^{-3(t-2)} \right], & 2 \leq t < 4 \\
\frac{1}{3} \left( 1 - e^{-6} \right) e^{-3(t-4)}, & t \geq 4
\end{cases}
\]
5. **25 pts.** A discrete-time system \( H \) is formed by cascading two discrete-time LTI systems \( H_1 \) and \( H_2 \), as shown in the figure below:

![Diagram of cascaded systems](image)

The unit pulse response \( h_1[n] \) is given by

\[ h_1[n] = \sin 8n, \]

while the unit pulse response \( h_2[n] \) is given by

\[ h_2[n] = \alpha^n u[n], \]

where \( |\alpha| < 1 \). Find the system output \( y[n] \) when the input is

\[ x[n] = \delta[n] - \alpha \delta[n - 1]. \]

\[ y[n] = x[n] * h_1[n] * h_2[n] \]

\[ = h_1[n] * x[n] * h_2[n]. \]

Now, \( x[n] * h_2[n] = (\delta[n] - \alpha \delta[n - 1]) * (\alpha^n u[n]) \)

\[ = \delta[n] * \alpha^n u[n] - \alpha (\delta[n - 1] * \alpha^n u[n]) \]

\[ = \alpha^n u[n] - \alpha (\alpha^{n-1} u[n-1]) = \alpha^n u[n] - \alpha^n u[n-1] \]

\[ = \alpha^n (u[n] - u[n-1]) = \alpha^n \delta[n] = \delta[n] \]

So,

\[ y[n] = h_1[n] * \delta[n] = h_1[n] = \sin 8n \]