ECE 3793
Test 2A

Wednesday, April 19, 2000
6:00 PM - 9:00 PM

Spring 2000 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 180 minutes to complete the test. All work must be your own. You may use two 8.5 $\times$ 11 inch two-sided note sheets as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

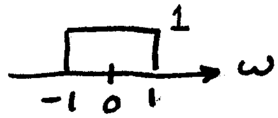
1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________
5. (25) ________

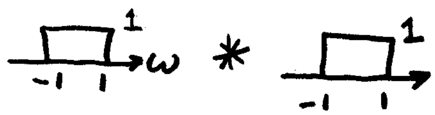
TOTAL (100):

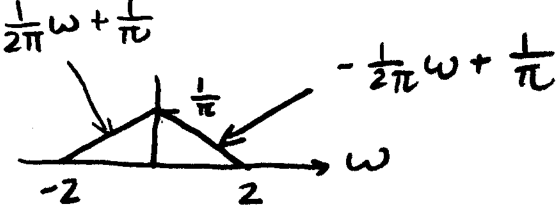
1. **25 pts.** The signal $x(t)$ is given by

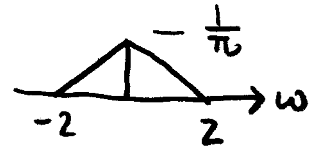
$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2.$$

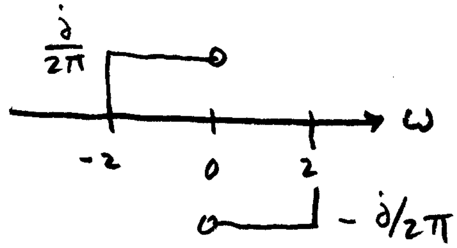
Find the Fourier transform $X(\omega)$.

Table: $\dfrac{\sin t}{\pi t} \xleftrightarrow{\mathcal{F}}$ 

$\left(\dfrac{\sin t}{\pi t}\right)^2 \xleftrightarrow{\mathcal{F}} \dfrac{1}{2\pi}\Big($  $\Big)$

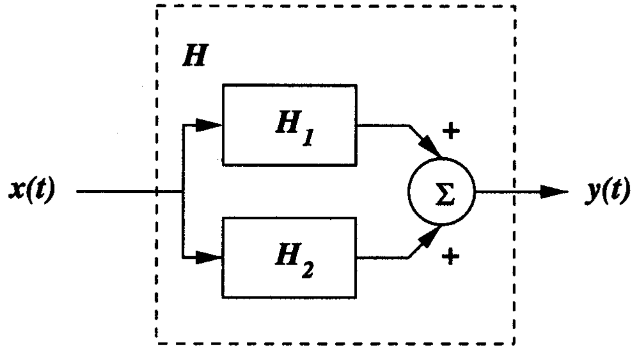
$=$ 

$x(t) = t\left(\dfrac{\sin t}{\pi t}\right)^2 \xleftrightarrow{\mathcal{F}} j\dfrac{d}{d\omega}\Big[$  $\Big]$

$=$ 

$$X(\omega) = \begin{cases} -j/2\pi, & 0 < \omega \leq 2 \\ +j/2\pi, & -2 \leq \omega < 0 \\ 0, & \text{otherwise} \end{cases}$$

2. **25 pts.** Consider a continuous-time system H formed by connecting two systems H_1 and H_2 in parallel as shown in the figure below.



The impulse response of system H_1 is given by

$$h_1(t) = -e^{-3t}u(t). \qquad H_1(\omega) = \frac{-1}{3+j\omega}$$

When the overall system input is

$$x(t) = 4e^{-4t}u(t), \qquad X(\omega) = \frac{4}{4+j\omega}$$

the system output is observed to have Fourier transform

$$Y(\omega) = \frac{4}{6+5j\omega-\omega^2}.$$

(a) **15 pts.** Find the impulse response $h_2(t)$ of the system H_2.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4}{6+5j\omega+(j\omega)^2} \quad \frac{4+j\omega}{4} = \frac{4+j\omega}{(3+j\omega)(2+j\omega)}$$

$$H_2(\omega) = H(\omega) - H_1(\omega) = \frac{4+j\omega}{(3+j\omega)(2+j\omega)} + \frac{1}{3+j\omega}$$

$$= \frac{6+2j\omega}{(3+j\omega)(2+j\omega)} = \frac{2(3+j\omega)}{(3+j\omega)(2+j\omega)} = \frac{2}{2+j\omega}$$

$$\boxed{h_2(t) = 2e^{-2t}u(t)}$$

Problem 2, cont...

(b) **10 pts.** Find the differential equation relating the input $x(t)$ and output $y(t)$ of the overall cascade system.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4 + j\omega}{6 + 5j\omega + (j\omega)^2}$$

$$6Y(\omega) + 5j\omega Y(\omega) + (j\omega)^2 Y(\omega) = 4X(\omega) + j\omega X(\omega)$$

$$6y(t) + 5y'(t) + y''(t) = 4x(t) + x'(t)$$

3. **25 pts.** Consider a causal LTI system described by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

(a) **10 pts.** Find the system frequency response $H(e^{j\omega})$ **and** impulse response $h[n]$.

$$Y(e^{j\omega})\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\right] = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}} = \underline{\underline{\frac{2}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}}}$$

$$= \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \left.\frac{2}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2}{1-2} = -2\ ; \quad B = \left.\frac{2}{1-\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2}{1-\frac{1}{2}} = 4$$

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$$

(b) **5 pts.** Is the system stable?

$$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{\infty}\left|4\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n\right| \le 4\sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^n + 2\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= 4\frac{1}{1-\frac{1}{2}} + 2\frac{1}{1-\frac{1}{4}} = 8 + \frac{8}{3}$$

$$= \frac{32}{3} < \infty.$$

The system <u>is</u> STABLE.

(c) **10 pts.** Find the system response $y[n]$ when the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n]. \quad X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{2}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2(1-\frac{1}{2}e^{-j\omega})} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{C}{1-\frac{1}{4}e^{-j\omega}}$$

$$A = \left.\frac{2}{(1-\frac{1}{4}\theta)^2}\right|_{\theta=2} = \frac{2}{(1-\frac{1}{2})^2} = \frac{2}{1/4} = 8$$

$$B = \left.\frac{2}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2}{1-2} = -2$$

$$\frac{d}{d\theta}\left[2(1-\tfrac{1}{2}\theta)^{-1}\right]_{\theta=4} = \left.\frac{d}{d\theta}\left[1-\tfrac{1}{4}\theta\right]C\right|_{\theta=4}$$

$$\left.2(-1)(1-\tfrac{1}{2}\theta)^{-2}(-\tfrac{1}{2})\right|_{\theta=4} = -\tfrac{1}{4}C$$

$$1 = -\tfrac{1}{4}C \,; \quad C = -4$$

$$Y(e^{j\omega}) = \frac{8}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} - \frac{4}{1-\frac{1}{4}e^{-j\omega}}$$

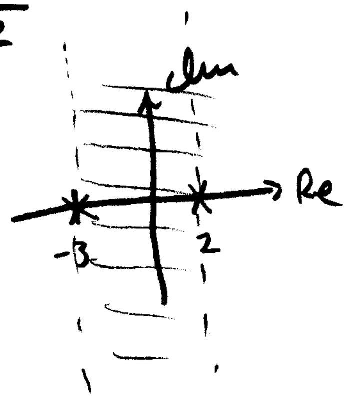
$$\boxed{y[n] = 8\left(\tfrac{1}{2}\right)^n u[n] - 2(n+1)\left(\tfrac{1}{4}\right)^n u[n] - 4\left(\tfrac{1}{4}\right)^n u[n]}$$

4. **25 pts.** An LTI system H has transfer function

$$H(s) = \frac{3s-1}{s^2+s-6}.$$

(a) **10 pts.** Assuming that the system is **stable**, find the impulse response $h[n]$.

$$H(s) = \frac{3s-1}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2}$$



$$A = \left.\frac{3s-1}{s-2}\right|_{s=-3} = \frac{-9-1}{-5} = 2$$

$$B = \left.\frac{3s-1}{s+3}\right|_{s=2} = \frac{5}{5} = 1$$

$$H(s) = \frac{2}{s+3} + \frac{1}{s-2}.$$

For stability, ROC must include the $j\omega$-axis: $-3 < \mathrm{Re}[s] < 2$.

ROC for $\frac{2}{s+3}$ must be $\mathrm{Re}[s] > -3 \overset{\mathcal{L}}{\longleftrightarrow} 2e^{-3t}u(t)$

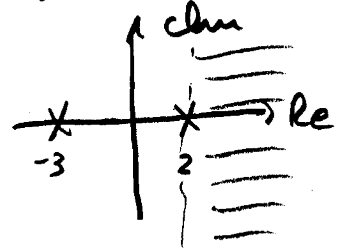
ROC for $\frac{1}{s-2}$ must be $\mathrm{Re}[s] < 2 \overset{\mathcal{L}}{\longleftrightarrow} -e^{2t}u(-t)$

$$\boxed{h(t) = 2e^{-3t}u(t) - e^{2t}u(-t)}$$

Problem 4, cont...

(b) **10 pts.** Now assume that the system is **causal**, but not necessarily stable. Find the impulse response $h[n]$.

For causality, ROC must be a right half-plane: $\mathrm{Re}[s] > 2$.



ROC for $\frac{2}{s+3}$ must be $\mathrm{Re}[s] > -3 \overset{\mathcal{L}}{\longleftrightarrow} 2e^{-3t}u(t)$

ROC for $\frac{1}{s-2}$ must be $\mathrm{Re}[s] > 2 \overset{\mathcal{L}}{\longleftrightarrow} e^{2t}u(t)$

$$\boxed{h(t) = 2e^{-3t}u(t) + e^{2t}u(t)}$$

(c) **5 pts.** Does a system with transfer function $H(s)$ exist that is both stable and causal?

For the system to be both causal and stable, all poles must lie in the left half-plane.

Since the pole at $s=2$ is in the right half-plane, no system with this $H(s)$ exists that is both stable and causal.

NO.

5. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system are related by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

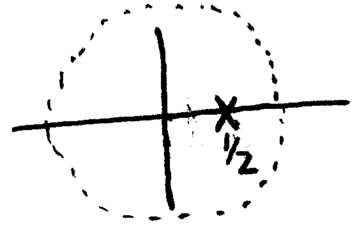
The system is **neither** stable **nor** causal. $\longrightarrow$ ROC is not exterior
ROC does not include unit circle.

(a) **10 pts.** Find the system transfer function $H(z)$.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



ROC: $|z| < \frac{1}{2}$

(b) **15 pts.** Find the system impulse response $h[n]$.

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{3}z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}}, \ |z| < \frac{1}{2} \xleftrightarrow{\ Z\ } -\left(\frac{1}{2}\right)^n u[-n-1]$$

$$\frac{1}{3}z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}}, \ |z| < \frac{1}{2} \xleftrightarrow{\ Z\ } -\frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

$$\boxed{h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]}$$