ECE 3793
Test 2B

Friday, April 21, 2000
6:00 PM - 9:00 PM

Spring 2000 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: There are **five** problems on this test. Work any **four** of them. Only **four** problems will be graded. You have 180 minutes to complete the test. All work must be your own. You may use two 8.5 $\times$ 11 inch two-sided note sheets as well as the summation formula sheet handed out in class.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

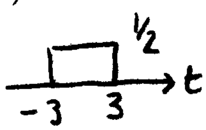
1. (25) ________
2. (25) ________
3. (25) ________
4. (25) ________
5. (25) ________

TOTAL (100):

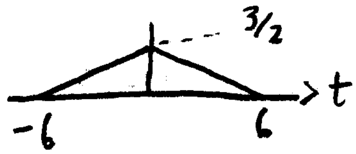
1. **25 pts.** The signal $x(t)$ has Fourier transform

$$X(\omega) = \frac{\sin^2(3\omega)\cos\omega}{\omega^2}.$$

Find the signal $x(t)$.

Let $x_1(t) =$ 

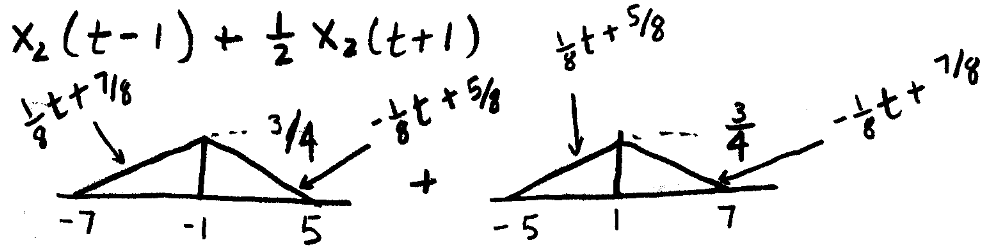
$\overset{\mathcal{F}}{\longleftrightarrow} \dfrac{\sin 3\omega}{\omega} = X_1(\omega)$

Then $x_2(t) = x_1(t) * x_1(t) =$ 

$\overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega) = \dfrac{\sin^2 3\omega}{\omega^2}$

$X(\omega) = X_2(\omega)\cos\omega = \frac{1}{2}e^{-j\omega}X_2(\omega) + \frac{1}{2}e^{j\omega}X_2(\omega)$

So $x(t) = \frac{1}{2}x_2(t-1) + \frac{1}{2}x_2(t+1)$

$=$ 

$t \in [-7,-5]$: $x(t) = \frac{1}{8}t + \frac{7}{8} = -\frac{1}{8}|t| + \frac{7}{8}$

$t \in [-5,-1]$: $x(t) = \frac{1}{8}t + \frac{7}{8} + \frac{1}{8}t + \frac{5}{8} = \frac{1}{4}t + \frac{3}{2} = -\frac{1}{4}|t| + \frac{3}{2}$

$t \in [-1,1]$: $x(t) = -\frac{1}{8}t + \frac{5}{8} + \frac{1}{8}t + \frac{5}{8} = \frac{5}{4}$

$t \in [1,5]$: $x(t) = -\frac{1}{8}t + \frac{5}{8} - \frac{1}{8}t + \frac{7}{8} = -\frac{1}{4}t + \frac{3}{2} = -\frac{1}{4}|t| + \frac{3}{2}$

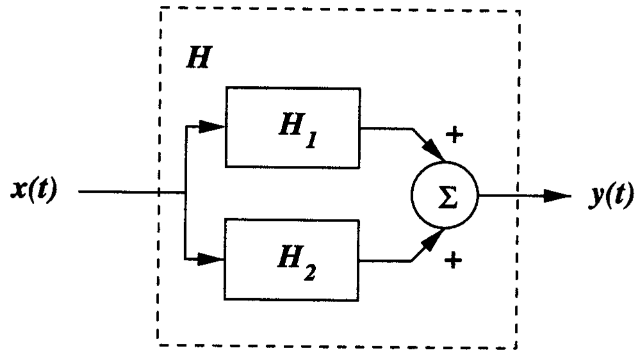
$t \in [5,7]$: $x(t) = -\frac{1}{8}t + \frac{7}{8} = -\frac{1}{8}|t| + \frac{7}{8}$

$$x(t) = \begin{cases} \frac{5}{4}, & |t| \le 1 \\ -\frac{1}{4}|t| + \frac{3}{2}, & 1 < |t| \le 5 \\ -\frac{1}{8}|t| + \frac{7}{8}, & 5 < |t| \le 7 \\ 0, & \text{otherwise} \end{cases}$$

More workspace for problem 1.

2. **25 pts.**

Consider a continuous-time system H formed by connecting two systems H_1 and H_2 in parallel as shown in the figure below.



The impulse response of system H_1 is given by

$$h_1(t) = -e^{-3t}u(t). \longleftrightarrow H_1(\omega) = \frac{-1}{3+j\omega}$$

When the overall system input is

$$x(t) = 4e^{-4t}u(t), \longleftrightarrow X(\omega) = \frac{4}{4+j\omega}$$

the system output is observed to have Fourier transform

$$Y(\omega) = \frac{4}{6 + 5j\omega - \omega^2}.$$

(a) **15 pts.** Find the impulse response $h_2(t)$ of the system H_2.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4}{6+5j\omega+(j\omega)^2} \quad \frac{4+j\omega}{4}$$

$$= \frac{4+j\omega}{(3+j\omega)(2+j\omega)}$$

$$H_2(\omega) = H(\omega) - H_1(\omega) = \frac{4+j\omega}{(3+j\omega)(2+j\omega)} + \frac{1}{3+j\omega}$$

$$= \frac{6+2j\omega}{(3+j\omega)(2+j\omega)} = \frac{2(3+j\omega)}{(3+j\omega)(2+j\omega)} = \frac{2}{2+j\omega}$$

$$\boxed{h_2(t) = 2e^{-2t}u(t)}$$

(b) **10 pts**. Find the differential equation relating the input $x(t)$ and output $y(t)$ of the overall cascade system.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{4 + j\omega}{6 + 5j\omega + (j\omega)^2}$$

$$6Y(\omega) + 5j\omega Y(\omega) + (j\omega)^2 Y(\omega) = 4X(\omega) + j\omega X(\omega)$$

$$6y(t) + 5y'(t) + y''(t) = 4x(t) + x'(t)$$

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{3}{3-e^{-j\omega}}$$

3. **25 pts.**

An LTI system H_1 with impulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in parallel with another causal LTI system H_2 with impulse response $h_2[n]$. For the resulting parallel interconnection system H, the input $x[n]$ and output $y[n]$ are related by

$$12y[n] - 7y[n-1] + y[n-2] = 5x[n-1] - 12x[n].$$

(a) **10 pts.** Determine $h_2[n]$. $12Y(e^{j\omega}) - 7e^{-j\omega}Y(e^{j\omega}) + e^{-j2\omega}Y(e^{j\omega})$

$$= -12X(e^{j\omega}) + 5e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left[12 - 7e^{-j\omega} + e^{-j2\omega}\right] = X(e^{j\omega})\left[-12 + 5e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} = \frac{-12+5e^{-j\omega}}{(-4+e^{-j\omega})(-3+e^{-j\omega})}$$

$$H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} - \frac{12-3e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{-24+8e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8(3-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8}{4-e^{-j\omega}} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{h_2[n] = -2\left(\tfrac{1}{4}\right)^n u[n]}$$

(b) **5 pts.** Is the overall system H system stable?

$$h[n] = h_1[n] + h_2[n] = \left(\tfrac{1}{3}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| \le \sum_{n=-\infty}^{\infty} |h_1[n]| + \sum_{n=-\infty}^{\infty} |h_2[n]| = \sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^n + 2\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= \frac{1}{1-\frac{1}{3}} + 2\,\frac{1}{1-\frac{1}{4}} = \frac{3}{2} + 2\cdot\frac{4}{3} = \frac{3}{2} + \frac{10}{3} = \frac{9}{6} + \frac{20}{6}$$

$$= \frac{29}{6} < \infty$$

$\Rightarrow$ H is stable.

Problem 3, cont...

(c) **10 pts.** Find the system response $y[n]$ of the overall system H when the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}} = \frac{4}{4-e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{4}{4-e^{-j\omega}} \cdot \frac{5e^{-j\omega}-12}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{20e^{-j\omega}-48}{(4-e^{-j\omega})^2(3-e^{-j\omega})} = \frac{A}{4-e^{-j\omega}} + \frac{B}{(4-e^{-j\omega})^2} + \frac{C}{3-e^{-j\omega}}$$

$$C = \left.\frac{20\theta-48}{(4-\theta)^2}\right|_{\theta=3} = \frac{60-48}{1} = 12$$

$$B = \left.\frac{20\theta-48}{3-\theta}\right|_{\theta=4} = \frac{80-48}{-1} = -32$$

$$\frac{d}{d\theta}\left[(20\theta-48)(3-\theta)^{-1}\right]_{\theta=4} = \frac{d}{d\theta}\left[(4-\theta)A\right]_{\theta=4}$$

$$\left[(20\theta-48)(-1)(3-\theta)^{-2}(-1) + 20(3-\theta)^{-1}\right]_{\theta=4} = -A\Big|_{\theta=4}$$

$$\frac{80-48}{(-1)^2} + \frac{20}{-1} = -A \Rightarrow 32-20 = -A \Rightarrow A = -12$$

$$Y(e^{j\omega}) = \frac{-12}{4-e^{-j\omega}} - \frac{32}{(4-e^{-j\omega})^2} + \frac{12}{3-e^{-j\omega}} = \frac{-3}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{4}{1-\frac{1}{3}e^{-j\omega}}$$

$$\boxed{y[n] = 4\left(\tfrac{1}{3}\right)^n u[n] - 3\left(\tfrac{1}{4}\right)^n u[n] - 2(n+1)\left(\tfrac{1}{4}\right)^n u[n]}$$

4. **25 pts.** The input $x(t)$ and output $y(t)$ of an LTI system G are related by

$$y'(t) + 2y(t) = x''(t) + 2x'(t) - 3x(t).$$

Let LTI system H be the **inverse** system of G.

(a) **10 pts.** Assuming that H is **stable**, find the impulse response $h(t)$.

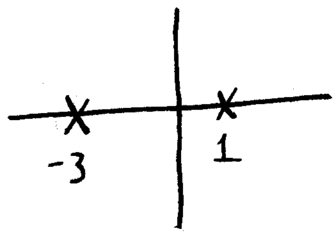
$$(s+2)Y(s) = (s^2+2s-3)X(s)$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s^2+2s-3}{s+2}$$

$$H(s) = \frac{1}{G(s)} = \frac{s+2}{s^2+2s-3} = \frac{s+2}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$A = \left.\frac{s+2}{s-1}\right|_{s=-3} = \frac{-1}{-4} = \frac{1}{4} ; \quad B = \left.\frac{s+2}{s+3}\right|_{s=1} = \frac{3}{4}$$

$$H(s) = \frac{1/4}{s+3} + \frac{3/4}{s-1}.$$



poles at $s = 1, -3$.

For H to be stable, ROC must include the $j\omega$-axis.

ROC $= -3 < \mathrm{Re}[s] < 1$.

ROC for $\frac{3/4}{s-1}$ must be $\mathrm{Re}[s] < 1 \overset{\mathcal{L}}{\longleftrightarrow} -\frac{3}{4}e^{t}u(-t)$

ROC for $\frac{1/4}{s+3}$ must be $\mathrm{Re}[s] > -3 \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{4}e^{-3t}u(t)$

$$\boxed{h(t) = \frac{1}{4}e^{-3t}u(t) - \frac{3}{4}e^{t}u(-t).}$$

Problem 4, cont...

(b) **10 pts.** Now assume that H is **causal**, but not necessarily stable. Find the impulse response $h(t)$.

For Causal, ROC = right half-plane = $\mathrm{Re}[s] > 1$.

ROC for $\frac{3/4}{s-1}$ must be $\mathrm{Re}[s] > 1 \xleftrightarrow{\mathcal{L}} \frac{3}{4} e^{t} u(t)$

ROC for $\frac{1/4}{s+3}$ must be $\mathrm{Re}[s] > -3 \xleftrightarrow{\mathcal{L}} \frac{1}{4} e^{-3t} u(t)$

$$\boxed{h(t) = \frac{3}{4} e^{t} u(t) + \frac{1}{4} e^{-3t} u(t)}$$

(c) **5 pts.** Does an inverse system with transfer function $H(s)$ exist that is both stable and causal?

NO. For stability, the ROC of $H(s)$ must include the line $\mathrm{Re}[s] = 0$. For causality, the ROC must be the right half-plane to the right of the rightmost pole.

For STABLE AND CAUSAL, all poles must be in the left half-plane.

But $H(s)$ has a right half-plane pole at $s = 1$.

5. **25 pts**. Consider a discrete-time LTI system H. When the system input is the unit step sequence $u[n]$, the output is

$$y[n] = \left[2 - \left(\tfrac{1}{2}\right)^n\right] u[n].$$

Find the system transfer function $H(z)$. Is the system causal? Is it BIBO stable? *(justify your answers)*

$$x[n] = u[n] \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$y[n] = 2u[n] - \left(\tfrac{1}{2}\right)^n u[n]$$

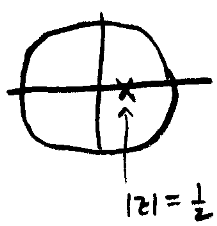
$$2u[n] \overset{z}{\longleftrightarrow} \frac{2}{1-z^{-1}}, \quad |z| > 1$$

$$-\left(\tfrac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} -\frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$

$$Y(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2 - z^{-1} - 1 + z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \cdot \frac{1-z^{-1}}{1}, \quad |z| > 1$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \tfrac{1}{2}$$



The system is causal and stable because the ROC of $H(z)$ is exterior and includes the unit circle.