ECE 3793
Test 2

Wednesday, November 29, 2000
7:00 PM - 10:00 PM

Fall 2000 Name: SOLUTION

Dr. Havlicek Student Num:

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **five** problems. Work any **four** of them. Only **four** problems will be graded. Below, you must circle the numbers of the **four** problems you wish to have graded.

▶ Formulas appear **after** problem five.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

Circle the numbers of the **four** problems you wish to have graded:

1. 2. 3. 4. 5.

SCORE:

1. (25) ________

2. (25) ________

3. (25) ________

4. (25) ________

5. (25) ________

TOTAL (100):

1. **25 pts.** The input $x(t)$ and output $y(t)$ of a continuous-time LTI system H are related by

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t).$$

(a) **6 pts.** Find the system frequency response $H(\omega)$.

$$[-\omega^2 + 4j\omega + 3]\,Y(\omega) = (j\omega + 2)\,X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega+2}{-\omega^2+4j\omega+3}$$

(b) **9 pts.** Find the system impulse response $h(t)$.

$$H(\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)} = \frac{A}{j\omega+3} + \frac{B}{j\omega+1}$$

$$A = \left.\frac{\theta+2}{\theta+1}\right|_{\theta=-3} = \frac{-1}{-2} = \frac{1}{2} \; ; \; B = \left.\frac{\theta+2}{\theta+3}\right|_{\theta=-1} = \frac{1}{2}$$

$$H(\omega) = \frac{1/2}{3+j\omega} + \frac{1/2}{1+j\omega}$$

$$h(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

Problem 1, cont...

(c) **10 pts.** Suppose that the system input is given by

$$x(t) = e^{-t}u(t).$$

Find the system output y(t). $\quad X(\omega) = \dfrac{1}{1+j\omega}$

$$Y(\omega) = H(\omega)X(\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)^2} = \frac{A}{j\omega+3} + \frac{B}{(j\omega+1)^2} + \frac{C}{j\omega+1}$$

$$A = \left.\frac{\theta+2}{(\theta+1)^2}\right|_{\theta=-3} = \frac{-1}{(-2)^2} = -\frac{1}{4}$$

$$B = \left.\frac{\theta+2}{\theta+3}\right|_{\theta=-1} = \frac{1}{2}$$

$$\frac{d}{d\theta}\left[(\theta+2)(\theta+3)^{-1}\right]_{\theta=-1} = \frac{d}{d\theta}\left[(\theta+1)^2(\theta+3)^{-1}\right]_{\theta=-1} A + \cancel{\frac{d}{d\theta}B}^{\,0} + \frac{d}{d\theta}[\theta+1]_{\theta=-1}\, C$$

$$\left[(\theta+2)(-1)(\theta+3)^{-2} + (\theta+3)^{-1}\right]_{\theta=-1} = \left[(\cancel{\theta+1})^{2}(-1)(\theta+3)^{-2} + 2(\cancel{\theta+1})(\theta+3)^{-1}\right]_{\theta=-1} A + C$$

$$(1)(-1)\left(\tfrac{1}{4}\right) + \tfrac{1}{2} = C \;\Rightarrow\; C = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$$

$$Y(\omega) = \frac{-1/4}{j\omega+3} + \frac{1/2}{(j\omega+1)^2} + \frac{1/4}{j\omega+1}$$

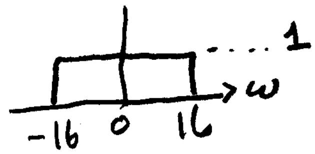
$$y(t) = \tfrac{1}{4}e^{-t}u(t) + \tfrac{1}{2}te^{-t}u(t) - \tfrac{1}{4}e^{-3t}u(t)$$

2. **25 pts.** Find the signal $x(t)$ that has a Fourier transform

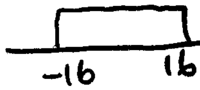
$$X(\omega) = \mathcal{F}[x(t)] = \begin{cases} \cos\frac{\pi\omega}{2}, & |\omega| \le 16 \\ 0 & |\omega| > 16. \end{cases}$$

Hint: $X(\omega)$ is a boxcar times a sum of two complex exponentials. Use the time shift property.

$$\frac{\sin 16t}{\pi t} \longleftrightarrow$$



$$\cos\frac{\pi}{2}\omega = \frac{1}{2}e^{-j\frac{\pi}{2}\omega} + \frac{1}{2}e^{j\frac{\pi}{2}\omega}$$

$X(\omega) = \frac{1}{2}e^{-j\pi/2\,\omega} \times$  $+ \frac{1}{2}e^{j\pi/2\,\omega} \times$ 

$$\frac{\sin 16(t-\frac{\pi}{2})}{2\pi(t-\frac{\pi}{2})} \qquad \frac{\sin 16(t+\frac{\pi}{2})}{2\pi(t+\frac{\pi}{2})}$$

$$x(t) = \frac{\sin 16(t-\frac{\pi}{2})}{2\pi(t-\frac{\pi}{2})} + \frac{\sin 16(t+\frac{\pi}{2})}{2\pi(t+\pi/2)}$$

3. **25 pts**. The signal $y(t)$ is given by

$$y(t) = x_1(t-2) * x_2(-t+3),$$

where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$. Find $Y(s)$, the Laplace transform of $y(t)$. Be sure to specify the ROC.

$$x_1(t) \longleftrightarrow X_1(s) = \frac{1}{s+2}, \quad Re[s] > -2$$

$$x_1(t-2) \longleftrightarrow \frac{e^{-2s}}{s+2}, \quad Re[s] > -2$$

$$x_2(t) \longleftrightarrow X_2(s) = \frac{1}{s+3}, \quad Re[s] > -3$$

$$x_2(t+3) \longleftrightarrow \frac{e^{3s}}{s+3}, \quad Re[s] > -3$$

$$x_2(-t+3) \longleftrightarrow \frac{e^{-3s}}{3-s}, \quad Re[s] < 3$$

$$Y(s) = \frac{e^{-2s}}{s+2} \frac{e^{-3s}}{-s+3} = \frac{-e^{-5s}}{(s+2)(s-3)}$$

$$= \frac{-e^{-5s}}{s^2-s-6}, \quad -2 < Re[s] < 3$$

4. **25 pts.** Consider a causal LTI system described by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

(a) **10 pts.** Find the system frequency response $H(e^{j\omega})$ and impulse response $h[n]$.

$$Y(e^{j\omega})\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right] = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j\omega}} = \frac{2}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \left.\frac{2}{1-\frac{1}{2}\theta}\right|_{\theta=4} = \frac{2}{-1} = -2 \; ; \; B = \left.\frac{2}{1-\frac{1}{4}\theta}\right|_{\theta=2} = \frac{2}{1-\frac{1}{2}} = 4$$

$$\boxed{H(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}}$$

$$\boxed{h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]}$$

(b) **5 pts.** Is the system stable? Justify your answer.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{4} \left|4\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n\right| \leq 4\sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^n + 2\sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n$$

$$= 4\frac{1}{1-\frac{1}{2}} + 2\frac{1}{1-\frac{1}{4}} = 8 + 8/3 = \frac{32}{3} < \infty.$$

The system is stable

Problem 4, cont...

(c) **10 pts.** Find the system response $y[n]$ when the input is

$$x[n] = \left(\frac{1}{3}\right)^n u[n].$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \frac{2}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

$$= \frac{A}{1-\frac{1}{3}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}} + \frac{C}{1-\frac{1}{2}e^{-j\omega}}$$

$$A = \left.\frac{2}{(1-\frac{1}{4}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=3} = \frac{2}{(1-\frac{3}{4})(1-\frac{3}{2})} = \frac{2}{(\frac{1}{4})(-\frac{1}{2})}$$

$$= \frac{2}{-1/8} = -16$$

$$B = \left.\frac{2}{(1-\frac{1}{3}\theta)(1-\frac{1}{2}\theta)}\right|_{\theta=4} = \frac{2}{(1-\frac{4}{3})(1-2)} = \frac{2}{(-\frac{1}{3})(-1)} = \frac{2}{1/3} = 6$$

$$C = \left.\frac{2}{(1-\frac{1}{3}\theta)(1-\frac{1}{4}\theta)}\right|_{\theta=2} = \frac{2}{(1-\frac{2}{3})(1-\frac{1}{2})} = \frac{2}{(\frac{1}{3})(\frac{1}{2})} = \frac{2}{1/6} = 12$$

$$Y(e^{j\omega}) = \frac{6}{1-\frac{1}{4}e^{-j\omega}} + \frac{12}{1-\frac{1}{2}e^{-j\omega}} - \frac{16}{1-\frac{1}{3}e^{-j\omega}}$$

$$\boxed{y[n] = 6\left(\frac{1}{4}\right)^n u[n] + 12\left(\frac{1}{2}\right)^n u[n] - 16\left(\frac{1}{3}\right)^n u[n].}$$

5. **25 pts.** The input $x[n]$ and output $y[n]$ of a discrete-time LTI system H are related by the difference equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n] - 4x[n-1].$$

(a) **10 pts.** Assuming that the system is **stable**, find the impulse response $h[n]$.

$$Y(z)\left[1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}\right] = X(z)\left[1-4z^{-1}\right]$$

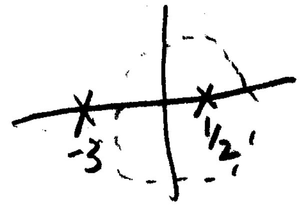
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-4z^{-1}}{1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}} = \frac{1-4z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+3z^{-1}}$$

$$A = \left.\frac{1-4\theta}{1+3\theta}\right|_{\theta=2} = \frac{-7}{7} = -1$$

$$B = \left.\frac{1-4\theta}{1-\frac{1}{2}\theta}\right|_{\theta=-\frac{1}{3}} = \frac{1+\frac{4}{3}}{1+\frac{1}{6}} = \frac{7/3}{7/6} = \frac{6}{3} = 2$$

$$X(z) = \frac{2}{1+3z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$



There are three possible ROC's:

$|z|>3$, $\frac{1}{2}<|z|<3$, and $|z|<\frac{1}{2}$.

For the system to be stable, the ROC must include the unit circle: $\frac{1}{2}<|z|<3$.

$$\frac{2}{1+3z^{-1}},\ |z|<3 \longleftrightarrow -2(-3)^n u[-n-1]$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}},\ |z|>\frac{1}{2} \longleftrightarrow -\left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = -\left(\frac{1}{2}\right)^n u[n] - 2(-3)^n u[-n-1]$$

Problem 5, cont...

(b) **10 pts**. Now assume that the system is **causal**, but not necessarily stable. Find the impulse response $h[n]$.

For the system to be causal, the ROC must be exterior: $|z|>3$.

$$\frac{2}{1+3z^{-1}}, \ |z|>3 \longleftrightarrow 2(-3)^n u[n]$$

$$\frac{-1}{1-\frac{1}{2}z^{-1}}, \ |z|>\frac{1}{2} \longleftrightarrow -\left(\frac{1}{2}\right)^n u[n]$$

$$\underline{\underline{h[n] = 2(-3)^n u[n] - \left(\tfrac{1}{2}\right)^n u[n]}}$$

(c) **5 pts**. Does a system exist that is causal **and** stable **and** has its input and output related by this difference equation? Justify your answer.

<u>NO</u>. The $h[n]$ in part (a) is stable but <u>not</u> causal. The $h[n]$ in part (b) is causal but <u>not</u> stable. The third possible $h[n]$ corresponds to the interior ROC $|z|<\frac{1}{2}$ and is <u>neither</u> stable nor causal.