ECE 3793
Test 2

Tuesday, December 3, 2002
7:00 PM - 10:00 PM

Fall 2002 Name: SOLUTION

Dr. Havlicek Student Num:__________________

Directions: This test is closed book and closed notes. You have 180 minutes to complete the test. All work must be your own. There are **four** problems. Work all **four**.

▶ You are allowed to use the **separate** formula sheet provided with this test.

SHOW ALL OF YOUR WORK for maximum partial credit!

GOOD LUCK!

SCORE:

1. (25) ________

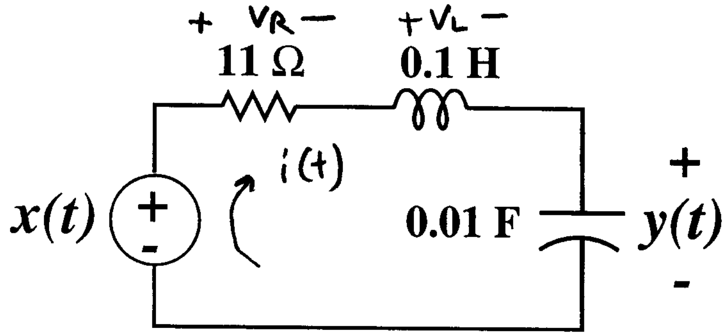
2. (25) ________

3. (25) ________

4. (25) ________

TOTAL (100):

1. **25 pts.** Consider the causal, stable LTI system H shown below. Voltage $x(t)$ is the system input and voltage $y(t)$ is the system output.



(a) **15 pts.** Use the Fourier transform to find the system frequency response $H(\omega)$.

$V_R(t) = 11\,i(t)$ {Resistor} $\qquad V_L(t) = 0.1\, i'(t)$ {inductor}

$i(t) = 0.01\, y'(t)$ { capacitor }

$\Rightarrow V_R(t) = 0.11\, y'(t), \qquad V_L(t) = 0.001\, y''(t)$

KVL: $V_R(t) + V_L(t) + y(t) = x(t)$

$$0.001\, y''(t) + 0.11\, y'(t) + y(t) = x(t)$$

$$y''(t) + 110\, y'(t) + 1000\, y(t) = 1000\, x(t)$$

$$(jw)^2 Y(w) + 110 jw\, Y(w) + 1000\, Y(w) = 1000\, X(w)$$

$$[(jw)^2 + 110 jw + 1000]\, Y(w) = 1000\, X(w)$$

$$\boxed{H(w) = \frac{Y(w)}{X(w)} = \frac{1000}{(jw)^2 + 110 jw + 1000}}$$

(b) **10 pts**. Find the system impulse response $h(t)$.

$$H(\omega) = \frac{1000}{(j\omega)^2 + 110 j\omega + 1000} = \frac{A}{j\omega + 100} + \frac{B}{j\omega + 10}$$

$$A = \left.\frac{1000}{\theta + 10}\right|_{\theta = -100} = \frac{1000}{-90} = -\frac{100}{9}$$

$$B = \left.\frac{1000}{\theta + 100}\right|_{\theta = -10} = \frac{1000}{90} = \frac{100}{9}$$

$$H(\omega) = \frac{100}{9}\frac{1}{j\omega + 10} - \frac{100}{9}\frac{1}{j\omega + 100}$$

$$\boxed{h(t) = \frac{100}{9}\left[e^{-10t} - e^{-100t}\right]u(t)}$$

2. **25 pts.**

An LTI system H_1 with impulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in parallel with another causal LTI system H_2 with impulse response $h_2[n]$. For the resulting parallel interconnection system H, the input $x[n]$ and output $y[n]$ are related by

$$12y[n] - 7y[n-1] + y[n-2] = 5x[n-1] - 12x[n].$$

(a) **10 pts.** Determine $h_2[n]$. $H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{3}{3-e^{-j\omega}}$

$$12Y(e^{j\omega}) - 7e^{-j\omega}Y(e^{j\omega}) + e^{-j2\omega}Y(e^{j\omega}) = 5e^{-j\omega}X(e^{j\omega}) - 12X(e^{j\omega})$$

$$\left[e^{-j2\omega} - 7e^{-j\omega} + 12\right]Y(e^{j\omega}) = \left[5e^{-j\omega} - 12\right]X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{5e^{-j\omega}-12}{e^{-j2\omega}-7e^{-j\omega}+12} = \frac{5e^{-j\omega}-12}{(e^{-j\omega}-4)(e^{-j\omega}-3)}$$

$$= \frac{5e^{-j\omega}-12}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

$$H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega}) = \frac{5e^{-j\omega}-12}{(4-e^{-j\omega})(3-e^{-j\omega})} - \frac{3}{3-e^{-j\omega}}$$

$$= \frac{5e^{-j\omega}-12-3(4-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{5e^{-j\omega}-12-12+3e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{8e^{-j\omega}-24}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8(3-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8}{4-e^{-j\omega}} = -\frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

(b) **5 pts.** Is the overall system H system stable?

$$h[n] = h_1[n] + h_2[n] = (\tfrac{1}{3})^n u[n] - 2(\tfrac{1}{4})^n u[n]$$

$$\boxed{\Rightarrow h_2[n] = -2(\tfrac{1}{4})^n u[n]}$$

$$\sum_{n=-\infty}^{\infty}|h[n]| = \sum_{n=0}^{\infty}\left|(\tfrac{1}{3})^n - 2(\tfrac{1}{4})^n\right|$$

$$\leq \sum_{n=0}^{\infty}\left[(\tfrac{1}{3})^n + 2(\tfrac{1}{4})^n\right] = \sum_{n=0}^{\infty}(\tfrac{1}{3})^n + 2\sum_{n=0}^{\infty}(\tfrac{1}{4})^n = \frac{1}{1-\frac{1}{3}} + 2\frac{1}{1-\frac{1}{4}}$$

$$= \frac{1}{2/3} + \frac{2}{3/4} = \frac{3}{2} + \frac{8}{3} = \frac{9}{6} + \frac{16}{6} = \frac{25}{6} < \infty$$

$\longrightarrow$ Therefore H is stable.

Problem 2, cont...

(c) **10 pts.** Find the system response $y[n]$ of the overall system H when the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}} = \frac{4}{4-e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{4}{4-e^{-j\omega}} \; \frac{5e^{-j\omega}-12}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{20e^{-j\omega}-48}{(4-e^{-j\omega})^2(3-e^{-j\omega})} = \frac{A}{4-e^{-j\omega}} + \frac{B}{(4-e^{-j\omega})^2} + \frac{C}{3-e^{-j\omega}}$$

$$C = \left.\frac{20\theta-48}{(4-\theta)^2}\right|_{\theta=3} = \frac{60-48}{1} = 12$$

$$B = \left.\frac{20\theta-48}{3-\theta}\right|_{\theta=4} = \frac{80-48}{-1} = -32$$

$$\frac{d}{d\theta}\left[(20\theta-48)(3-\theta)^{-1}\right]_{\theta=4} = \frac{d}{d\theta}\left[(4-\theta)A\right]_{\theta=4}$$

$$\left[20(3-\theta)^{-1} + (20\theta-48)(3-\theta)^{-2}\right]_{\theta=4} = -A\Big|_{\theta=4}$$

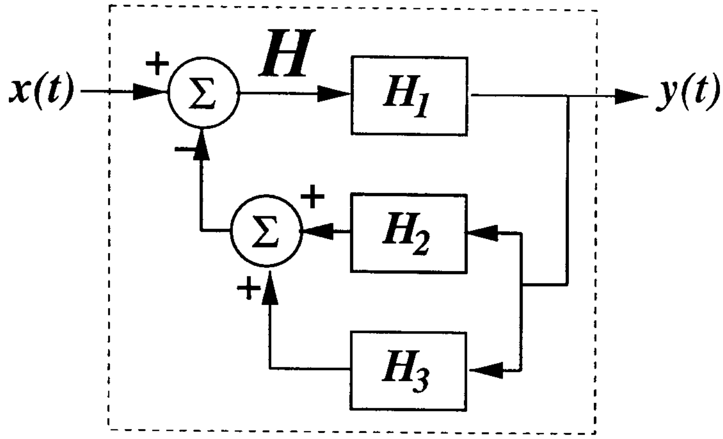
$$-20 + (80-48) = -A \Rightarrow 12 = -A \Rightarrow A = -12$$

$$Y(e^{j\omega}) = \frac{12}{3-e^{-j\omega}} - \frac{32}{(4-e^{-j\omega})^2} - \frac{12}{4-e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{3}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} - \frac{3}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{y[n] = 4\left(\tfrac{1}{3}\right)^n u[n] - 2(n+1)\left(\tfrac{1}{4}\right)^n u[n] - 3\left(\tfrac{1}{4}\right)^n u[n].}$$

3. **25 pts.** Consider the continuous-time LTI system H shown below. Note that the feedback path consists of a parallel connection of systems H_2 and H_3. Systems H_1, H_2, and H_3 are all LTI.



The system input is given by $x(t) = te^{-t}u(t)$. The output is $y(t) = e^{-t}u(t)$. The impulse response of system H_1 is $h_1(t) = 6\delta(t)$. The transfer function of system H_2 is

$$H_2(s) = \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2.$$

For system H_3, find the transfer function $H_3(s)$ and the impulse response $h_3(t)$. Don't forget to specify the ROC of $H_3(s)$!

$x(t) = te^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{1}{(s+1)^2}$, $\mathrm{Re}\{s\} > -1$. Table 9.2 # 8 $n=2, \alpha=1$.

$y(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{1}{s+1}$, $\mathrm{Re}\{s\} > -1$. Table 9.2 #6, $\alpha = 1$.

$h_1(t) = 6\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} H_1(s) = 6$, all s. Table 9.2 #1.

So:

$$H(s) = \frac{Y(s)}{X(s)} = \left[\frac{1}{s+1}\right]\left[(s+1)^2\right] = s+1, \text{ all } s.$$

$H_1(s) = 6$, all s.

$H_2(s) = \frac{1}{s+2}$, $\mathrm{Re}\{s\} > -2$

$H_3(s) = ?$ $\longrightarrow$

For the overall system H, the feedback path transfer function is $H_2(s) + H_3(s)$.

$$\text{So } H(s) = \frac{H_1(s)}{1 + H_1(s)\left[H_2(s) + H_3(s)\right]} = \frac{H_1(s)}{1 + H_1(s)H_2(s) + H_1(s)H_3(s)}$$

Cross Multiply:

$$H(s)\left[1 + H_1(s)H_2(s) + H_1(s)H_3(s)\right] = H_1(s)$$

$$H(s) + H(s)H_1(s)H_2(s) + H(s)H_1(s)H_3(s) = H_1(s)$$

$$H(s)H_1(s)H_3(s) = H_1(s) - H(s) - H(s)H_1(s)H_2(s)$$

$$H_3(s) = \frac{H_1(s) - H(s) - H(s)H_1(s)H_2(s)}{H(s)H_1(s)} = \frac{1}{H(s)} - \frac{1}{H_1(s)} - H_2(s)$$

$$= \underbrace{\frac{1}{s+1}}_{Re\{s\}>-1} - \underbrace{\frac{1}{6}}_{\text{all } s} - \underbrace{\frac{1}{s+2}}_{Re\{s\}>-2}, \quad Re\{s\} > -1.$$

$$h_3(t) = e^{-t}u(t) - \frac{1}{6}\delta(t) - e^{-2t}u(t).$$

4. 25 pts. The input $x(t)$ and output $y(t)$ of an LTI system H are related by

$$y''(t) + 3y'(t) + 2y(t) = x(t).$$

The initial conditions on the output are $y(0^-) = 3$ and $y'(0^-) = -5$. Use the unilateral Laplace transform to find the system output for $t > 0$ when the input is $x(t) = 2u(t)$.

$$s^2 Y_u(s) - s y(0^-) - y'(0^-) + 3\left[s Y_u(s) - y(0^-)\right] + 2Y_u(s) = X_u(s)$$

$$s^2 Y_u(s) - 3s + 5 + 3\left[s Y_u(s) - 3\right] + 2Y_u(s) = X_u(s)$$

$$s^2 Y_u(s) - 3s + 5 + 3s Y_u(s) - 9 + 2Y_u(s) = X_u(s)$$

$$\left[s^2 + 3s + 2\right] Y_u(s) - 3s - 4 = X_u(s)$$

$$\boxed{X_u(s) = \frac{2}{s}}$$

$$\left[s^2 + 3s + 2\right] Y_u(s) = X_u(s) + 3s + 4$$

$$= \frac{2}{s} + 3s + 4$$

$$Y_u(s) = \frac{3s + 4 + 2/s}{s^2 + 3s + 2} = \frac{3s^2 + 4s + 2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \left.\frac{3s^2 + 4s + 2}{(s+1)(s+2)}\right|_{s=0} = \frac{2}{2} = 1$$

$$B = \left.\frac{3s^2 + 4s + 2}{s(s+2)}\right|_{s=-1} = \frac{3 - 4 + 2}{(-1)(1)} = \frac{1}{-1} = -1$$

$$C = \left.\frac{3s^2 + 4s + 2}{s(s+1)}\right|_{s=-2} = \frac{12 - 8 + 2}{(-2)(-1)} = \frac{6}{2} = 3$$

$$Y_u(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

$$\boxed{y(t) = u(t) - e^{-t}u(t) + 3e^{-2t}u(t), \; t > 0}$$